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The Wavefunction and the Wigner Function of the Universe

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1 Introduction

1.1 Motivation for the quantum cosmology

In cosmology we usually assume that spacetime structure is described by a classical metric in spite of the fact that matter follows quantum mechanics. This is because quantum fluctuations of spacetime structure caused by the quantum nature of matter is not important in most cases. In fact from the classical Einstein equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}, \tag{1.1}$$

it is expected that the quantum fluctuations of metric become important only on scales smaller than the Planck length L_P which is defined as

$$L_P = \sqrt{\hbar G/c^3} \simeq 1.6 \times 10^{-33}\text{cm}. \tag{1.2}$$

In some cases, however, we cannot neglect the quantum nature of spacetime structure. For example, in the Big-Bang model of the Universe, the spacetime curvature of the classical metric describing the Universe becomes larger and larger as we go back in time and becomes divergent within a finite time. Thus in the very early stage of the Universe its curvature radius becomes much smaller than the Planck length. Actually this situation is not specific to special cosmological models since the Singularity Theorem asserts that classical spacetimes always have singularities if a couple of physically natural conditions on matter and spacetime structure are satisfied[1].

Of course the occurrence of spacetime singularities do not necessarily imply that they play important roles in nature because they may be just local phenomena and their influence may be hidden by some mechanism, e.g., by horizons. However, in the case of the initial singularities of the Universe, we cannot neglect them, because the structure near the singularities determines the initial condition for the subsequent classical evolution of the Universe. Thus in order to understand how the present structure of the Universe is formed, we must construct a cosmological model which incorporates the quantum nature of spacetime as well as matter. This kind of theory is called quantum cosmology.

1.2 Difficulties in the basic theory

In studying quantum cosmology, we need a theory which determines the quantum behavior of spacetimes, i.e., a quantum gravity theory. The most natural approach to constructing such a theory is the canonical one which was successful in building the nonrelativistic quantum mechanics and quantum field theories in the flat spacetime. Actually it is an easy task to rewrite the classical theory of general relativity in the canonical form. However, we encounter a difficulty when we translate it to a quantum theory: in constant to ordinary dynamical systems, there appear constraints on the canonical variables due to the general covariance of the theory, and the Hamiltonian itself is a linear combination of the constraints. As a result of this, not only the time evolution is apparently lost, but also the norms of physical state vectors diverge if we adopt the inner product which makes the original canonical variables hermitian. Thus the formal canonical quantization leads to an ill-defined theory within the conventional framework of quantum theory.

This failure of the canonical approach as well as the perturbative unrenormalizability of the general relativity has driven many people to modify the theory or seek new theories of gravity. The most promising in this line is the superstring theories which take strings as the fundamental object in nature[2]. Though superstring theories are shown to have various fascinating features, their development is currently hampered by technical difficulties. Furthermore no string theory is so far constructed which can describe the

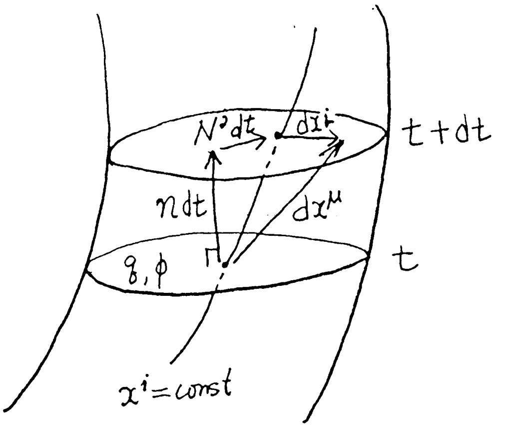

Figure1
(3+ 1)-decomposition of spacetime

global dynamics of spacetimes.

Under this circumstance, some people are trying to save the canonical approach by modifying or extending the framework of quantum mechanics itself[3]. In this talk we review the formulation of the canonical quantum gravity, its difficulties, and attempts to overcome them with special emphasis on their relevance to the WKB theory.

2 Quantum Gravity in the Canonical Approach

2.1 Classical Canonical Theory

The theory of general relativity is originally formulated as a field theory: the fundamental variables are a metric tensor $g_{\mu\nu}(x)$ and a matter field $\Phi(x)$, and their dynamics are determined by the action principle

$$\delta S = 0: \quad S = \int d^4x \left[\sqrt{-g}\frac{R}{2\kappa^2} + \mathcal{L}_m\right] \tag{2.1}$$

where R is the Ricci scalar, $\mathcal{L}_m$ is the lagrangian density for the matter field and $\kappa^2 = 8\pi G$.

In order to rewrite this field theory in a canonical form, we decompose the spacetime to a family of space-like 3-surfaces on which the time coordinate is constant as shown in Fig.1. Then the spacetime metric is written in terms of the spatial metric q_{jk} on each 3-surface and the the components of the normal vector field $n = (1/N, -N^j/N)$ to them as

$$ds^2 = -N^2dt^2 + q_{jk}(dx^j + N^j dt)(dx^k + N^k dt). \tag{2.2}$$

If we take the quantities $Q(t) = (q_{jk}(x,t), \Phi(x,t))$ as the fundamental variables, the original field theory can be regarded as a dynamical system with infinite degrees of freedom with an action of a structure

$$S = \int dt L(Q, \dot{Q}, N). \tag{2.3}$$

Here $N(t) = (N(x,t), N^j(x,t))$ are not dynamical variables because they correspond to the freedom of specifying the coordinates.

Since it is shown that the lagrangian L is quadratic in $\dot{Q}$ and $d^2L/d\dot{Q}^I d\dot{Q}^J$ is non-degenerate, we can introduce the momentum $P(t) = (p^{jk}(x,t), \Pi(x,t))$ conjugate to Q. In terms of these canonical variables the original action is shown to be equivalent to the following first-order form action[4]:

$$S = \int dt (P \cdot \dot{Q} - H), \tag{2.4}$$

where

$$H = \int d^3x N^\mu \mathcal{H}_\mu(q, p, \Phi, \Pi) \tag{2.5}$$

with

$$\mathcal{H}_0 = \mathcal{H}_0^G + \sqrt{q}T_{nn}; \quad \mathcal{H}_0^G = \frac{2\kappa^2}{\sqrt{q}}[p^{jk}p_{jk} - \frac{1}{2}(p^j_j)^2] - \frac{\sqrt{q}}{2\kappa^2}{}^3R, \tag{2.6}$$

$$\mathcal{H}_j = \mathcal{H}_j^G + \sqrt{q}T_{nj}; \quad \mathcal{H}_j^G = -2D_k p^k_j. \tag{2.7}$$

Here the spatial indices $j, k, \ldots$ are raised and lowered by q^{jk} and by q_{jk}, respectively, D_j is the three-dimensional covariant derivative with respect to q_{jk}, and $T_{nn} = T_{\mu\nu}n^\mu n^\nu$ and $T_{nj} = T_{\mu j}n^\mu$ are the normal components of the energy-momentum tensor $T_{\mu\nu}$ for the matter field.

Variations of the action (2.5) with respect to Q and P yield the canonical equation of motion

$$\dot{F} = \{F, H\}; \quad F = F(Q, P). \tag{2.8}$$

Meanwhile variations with respect to N yield non-dynamical equations

$$\mathcal{H}_\mu = 0, \tag{2.9}$$

which are called constraints.

2.2 Quantization

In an ordinary canonical system we can construct a quantum theory from a classical theory by replacing Poisson brackets by commutation relations among the operators corresponding to the canonical variables as

$$\{F,G\} \to (-i)[\hat{F},\hat{G}]. \tag{2.10}$$

Then the canonical equation of motion is translated to the standard operator equation of motion in the Heisenberg picture

$$\partial_t \hat{F} = i[\hat{H},\hat{F}]. \tag{2.11}$$

In the present case of general relativity, however, this manipulation does not complete the quantization procedure even in the formal level. We must decide how to express the constraints in the quantum theory. They cannot obviously be expressed as operator equations because $\mathcal{H}_\mu$ has non-vanishing Poisson brackets with the canonical variables. So we must look for weaker expressions.

Dirac discussed this problem many years ago, and proposed that the constraints should be expressed as conditions on the physical state vectors[5]. Following his proposal, we obtain quantum constraint equations on a physical state $|\Psi>$,

$$\hat{\mathcal{H}}_\mu(x)|\Psi>= 0. \tag{2.12}$$

This is actually an infinite sets of equations. Hence the consistency of them yields additional equations in general. In the present case, however, we obtain no new equations since the Poisson brackets of $\mathcal{H}_\mu$ close in a weak sense:

$$\{H_\mu(x),H_\nu(y)\} = \int d^3z f^{\lambda}_{\mu\nu}(x,y,z)\mathcal{H}_\lambda(z). \tag{2.13}$$

This results from the fact that the generator of the infinitesimal coordinate transformation $x^\mu \to x^\mu + \delta x^\mu$ is written as a linear combination of $\mathcal{H}_\mu$:

$$\delta F = \{G,F\}; \tag{2.14}$$

$$G = \int d^3x\,[\delta x^0 N^\mu \mathcal{H}_\mu + \delta x^j \mathcal{H}_j]. \tag{2.15}$$

2.3 Difficulties

Though the quantization procedure explained above gives a formal quantum theory for gravity, one finds various defects when one inspects its content in detail. First, since the Hamiltonian is written as a linear combination of the constraints, it vanishes on physical states. Thus the time evolution is lost from the theory:

$$\hat{H}|\Psi>= 0 \quad \to \quad \partial_t < \Psi|\hat{F}|\Psi>= 0. \tag{2.16}$$

Roughly speaking, this is a result of the general covariance of the theory: the general coordinate transformation includes arbitrary time translations, which makes each physical state time-translation invariant because the constraint functions are the generators of the coordinate transformations.

Second the norm of each physical state diverges with respect to the natural inner product which makes the operators Q and P hermitian. Though this is not proved generally, it is the case at least for minisuperspace models explained in §4. Since the inner product provides the basis for the probability interpretation of quantum theories, this defect is fatal to the theory.

These two difficulties both arises from the presence of constraints. Thus it is expected that they may disappear if one solves the constraints in the classical level to extract true physical degrees of freedom and afterward quantize the theory. Due to the intricate structure of the constraint equations, however, no one has succeeded in solving them explicitly so far except for the asymptotically flat case where the formal nonlocal expression for the solutions were given by Arnowitt, Deser and Misner[6]. Another possible approach to overcome this difficulty is to quantize the gauge-invariant operators. Actually even in the Dirac quantization explained above, not all the hermitian operators but the gauge-invariant subset are observable because only quantities which are invariant under the general coordinate transformation can be measured in the classical theory. Further we can show that knowing all the gauge-invariant quantities, we can recover the original classical dynamics without the equation of motion. Thus restricting to the gauge-invariant quantities, which is equivalent to solving the constraints, is a hopeful approach. However, no trial has been given to this approach in the real canonical formalism so far. Only a partial progress is made in the complex canonical formalism[7,8].

Besides these difficulties a new problem arises when one tries to apply the quantum theory of gravity to cosmology. In the Copenhagen interpretation of quantum mechanics a quantum state vector changes nondynamically to one of common eigenstates of some set of commuting operators by measurement if one tries to assign a state vector to an individual system. This is called a *wave packet reduction.* In ordinary circumstances this gives rise to no practical problem as far as one regards the quantum mechanics as describing the dynamics of an ensemble of identical systems. However, in cosmology, there is only one system. Thus there arises a question whether the state of the Universe is described by a state vector, or more generally whether the Universe has any well-defined state. Actually this problem is out of scope of the conventional quantum mechanics because the physical meaning of measurements or phenomena themselves is not given there. Though many people attacked this problem under the name of *measurement problem,* no essential progress has been made so far. We do not go into this problem in this talk any more.

3 Wavefunction of the Universe and its WKB interpretation

3.1 Wheeler-DeWitt Equation

As explained in the previous section, $\mathcal{H}_j$ is a generator of the spatial coordinate transformation. Hence the constraint $\hat{\mathcal{H}}_j|\Psi>=0$, which is called the *momentum constraint,* implies that the state vector $|\Psi>$ is invariant under the spatial diffeomorphism. Further the formal time evolution equation has no physical meaning. Thus in the canonical theory of quantum gravity with constraints all the dynamical information is contained in the Hamiltonian constraint $\mathcal{H}_0(x)|\Psi>=0$. If we express the Hamiltonian formally as

$$H=\frac{1}{2}\mathcal{G}^{IJ}(Q)P_IP_J+\mathcal{D}^I(Q)P_I+\mathcal{U}(Q), \tag{3.1}$$

this equation is expressed in the representation in which Q is diagonal as a system of sec^nd-order functional differential equations with a structure

$$\left[-\frac{1}{2}\mathcal{G}^{IJ}\partial_I\partial_J-i\mathcal{D}^I\partial_I+\mathcal{U}\right]\Psi[Q]=0, \tag{3.2}$$

where $\partial_I = \partial / \partial_{Q^I}$. This equation is called the *Wheeler-DeWitt equation*[9], and a diffeomorphism invariant solution to it is called a *wavefunction of the Universe* when it is regarded as describing the state of the Universe[10].

3.2 Extraction of Time-Evolution by the WKB Method

For the case where a solution to the Wheeler-DeWitt equation has the WKB form

$$\Psi(Q) = \mathcal{A}(Q) \exp[i\mathcal{S}(Q)] \tag{3.3}$$

with $\mathcal{A}(Q)$ a slowly changing function of Q, one can extract classical solutions from it[11]. In fact from the assumption, $\mathcal{S}$ satisfies the Hamilton-Jacobi equation

$$H(Q, \partial\mathcal{S}) = \frac{1}{2}\mathcal{G}^{IJ}\partial_I\mathcal{S}\partial_J\mathcal{S} + \mathcal{U} \approx 0. \tag{3.4}$$

Here if we define the momentum and the time variable by

$$P_I = \partial_I\mathcal{S}(Q), \tag{3.5}$$

$$\dot{Q}^I = \mathcal{G}^{IJ}P_J, \tag{3.6}$$

it is easily shown that Q and P satisfy the classical canonical equation of motion. Actually this is nothing but the Hamilton-Jacobi theory.

Thus with the aid of the Hamilton-Jacobi theory, a WKB-type solution to the Wheeler-DeWitt equation is interpreted as describing an ensemble of classical universes.

Of course the wavefunction of the Universe is not generally expected to be well approximated by a WKB solution. To treat such a general case, we decompose the variables to semi-classical ones (X, P) and quantum ones, and write the total Hamiltonian as

$$H = H_C(X, P) + H_Q. \tag{3.7}$$

Then if we expand the state vector $|\Psi>$ by the eigenstates of X, $|X>$, and express it as an integral over X of tensor products of $|X>$ and some state vectors belonging to a Hilbert space for the quantum variables as

$$|\Psi>= \int dX\, |X> \otimes |\Psi(X)>, \tag{3.8}$$

the Wheeler-DeWitt equation is written as

$$\left[-\frac{1}{2}\mathcal{G}^{IJ}\partial_I\partial_J + \mathcal{U}(X) + H_Q(X)\right]|\Psi(X)>= 0, \tag{3.9}$$

where the first two terms on the left-hand side come from H_C, and $\partial_J = \partial/\partial_{X^I}$.

Here let us assume that $|\Psi(X)>$ is written as

$$|\Psi(X)>= e^{i\mathcal{S}(X)}|\Phi(X)>; \tag{3.10}$$

$$H_C(X, \partial\mathcal{S}) = 0, \tag{3.11}$$

where $|\Psi(X)>$ varies much slowly in X compared with $\mathcal{S}$, which is an extension to general state vectors of the WKB approximation discussed above. Then by introducing a time variable t along each classical solution $X(t)$ determined from $\mathcal{S}$ as

$$\mathcal{G}^{IJ}\partial_I\mathcal{S}\partial_J = \dot{X}^I\partial_I \quad \rightarrow \quad \partial_t, \tag{3.12}$$

and neglecting the second-order derivative of $\mathcal{S}$ and $|\Phi(X)>$, the Wheeler-DeWitt equation reduces to

$$i\partial_t|\Phi>\simeq H_Q|\Phi>. \tag{3.13}$$

Thus we can derive the Schrödinger equation for the quantum freedom along each WKB trajectories of X from the Wheeler-DeWitt equation.

4 Path-Integral and WKB approximation

4.1 Path-Integral Expression for the Wavefunction of the Universe

As is well-known, each solution to the Schrödinger equation

$$i\partial_t|\Phi>= \hat{H}|\Phi>; \quad H = \frac{p^2}{2m} + V(x) \tag{4.1}$$

in the ordinary quantum mechanics can be expressed in terms of the path-integral

$$\begin{aligned}\Phi(x,t) &= \int DxDp\exp\left[i\int_{t_0}^{t} dt'(\dot{x}p - H)\right]\Phi(x_0,t_0),\\ &= \int Dx\exp\left[i\int_{t_0}^{t} dt' L(t')\right]\Phi(x_0,t_0).\end{aligned} \tag{4.2}$$

We can formally derive similar expressions for solutions to the Wheeler-DeWitt equation. First let us consider a Schrödinger equation

$$i\partial_t\Phi(q,\phi;t,N) = \hat{H}\Phi(q,\phi;t,N); \quad \hat{H} = \int d^x N^\mu \mathcal{H}_\mu. \tag{4.3}$$

Here $N = (N, N^j)$ is not considered as variables but is assumed to be some fixed functions. This is the reason why the argument of Φ contains N. Now we define Ψ as a functional integration of Φ over N:

$$\Psi(q,\phi;t) = \int DN\,\Phi(q,\phi;t,N). \tag{4.4}$$

Then Ψ becomes time-independent because $\int dN(t)N^\mu = 0$, and expressed by the path-integral

$$\begin{aligned}\Psi(q,\phi) &= \int DNDqDp\cdots\exp\left[i\int_{t_0}^{t} dt'(\dot{q}p + \dot{\phi}\pi - H)\right]\Phi_0(q_0,\phi_0)\\ &= \int DqDp\cdots\prod\delta(\mathcal{H}_\mu)\exp[\cdots]\cdots\\ &= \int Dg_{\mu\nu}D\phi\exp\left[i\int_{\Sigma_0}^{\Sigma} d^4x\mathcal{L}\right]\Phi_0(q_0,\phi_0;\Sigma_0).\end{aligned} \tag{4.5}$$

Since the second line of this equation contains $\delta(\mathcal{H}_\mu)$ in the integrand, Ψ is shown to satisfy the Wheeler-DeWitt equation (in the formal sense). This shows that at least some solutions to the Wheeler-DeWitt equation an be expressed by the path-integral.

4.2 Hartle-Hawking Proposal

Motivated by the result in the previous subsection, Hartle and Hawking proposed an Ansatz that the wavefunction of Our Universe is given by the path-integral whose integration paths correspond to spacetimes which have no boundary other than the one on which the observation is made[10,12]. To be precise, the path-integral here is the Euclidean one which is obtained by the analytic continuation of the original expression to the imaginary time(Wick rotation). Thus the spacetimes corresponding to the integration paths are actually not Lorentzian manifolds but Riemannian manifolds. The motivation of this Euclideanization is two-fold. First in the quantum field theories in the Minkowski spacetime the path-integral becomes a mathematically well-defined objects only by the Wick rotation. Second a Lorentzian spacetime with a single boundary has time-like closed curves violating the causality or singularities.

Though the Hartle-Hawking proposal is a natural one in the path-integral approach and it is shown that we can derive some interesting consequences from it in some simple models, it was plagued from the start by a serious defect of the Euclidean path-integral expression: the integrand does not have a lower bound in contrast to the usual field theories.

To see this, let us consider a quantum theory of spatially homogeneous and isotropic spacetime with a positive spatial curvature and a spatially homogeneous scalar field ϕ on it. Since the metric of a spatially homogeneous and isotropic spacetime is expressed for the choice $N^j = 0$ as

$$ds^2 = -N^2 dt^2 + a^2 d\Omega_3^2, \tag{4.6}$$

where $d\Omega_3^2$ is the metric of the unit Euclidean 3-sphere, this system has only two degrees of freedom: the scale factor a and the scalar field ϕ. This type of systems with finite degrees of freedom obtained from the general system by imposing some symmetries are called *minisuperspace models.*

The action is written for the present minisuperspace model as

$$S = \int d^4x\, \mathcal{L} = S_G + S_m; \tag{4.7}$$

$$S_G = \frac{1}{2}\int dtN \left[-a\frac{\dot{a}^2}{N^2} + a - \frac{1}{3}\lambda a^3\right], \tag{4.8}$$

$$S_m = \frac{1}{2}\int dtN a^3 \left[\frac{1}{N^2}\dot{\phi}^2 - 2V(\phi)\right], \tag{4.9}$$

where λ is the cosmological constant. As is expected from the form of the metric, the Euclidean action I corresponding to this system is obtained by replacing N by $-iN$:

$$I = -iS = I_G + I_m; \tag{4.10}$$

$$I_G = \frac{1}{2}\int dt N\left[-a\frac{\dot{a}^2}{N^2} - a + \frac{1}{3}\lambda a^3\right], \tag{4.11}$$

$$I_m = \frac{1}{2}\int dt N a^3\left[\frac{1}{N^2}\dot{\phi}^2 + 2V(\phi)\right]. \tag{4.12}$$

From these equations one easily see that the Euclidean action for matter I_m is positive definite but that for gravity I_G has no definite sign. Since the Euclidean action appears in the path-integral as e^{-I}, this means that the integrand of tl.e Euclidean path-integral has no lower bound.

This signature structure is reflected into the structure of the Wheeler-DeWitt equation. For example, in the present model, the Wheeler-DeWitt equation is written as

$$\left[-\frac{1}{2}\frac{\partial^2}{\partial a^2} + \frac{1}{2a^2}\frac{\partial^2}{\partial \phi^2} + \left(-\frac{1}{6}\lambda a^4 + \frac{1}{2}a^4 - a^4 V(\phi)\right)\right]\Psi(a,\phi) = 0, \tag{4.13}$$

which has the hyperbolic structure. This hyperbolic structure of the Wheeler-DeWitt equation is generic.

4.3 Complex-Contour Path-Integral and WKB approximation

Some people tried to eliminate this defect and make the Ansatz well-posed by extending the path-integral to complex paths[13,14]. To see its basic idea, let us consider the minisuperspace model described in the previous subsection. For this simple system, it is exactly shown that the path-integral expression for the wavefunction $\Psi(a,\phi)$ is reduced to the following two-dimensional integral[15]

$$\Psi(a,\phi) = \int_\Gamma dT \int d\phi_0\, G(a,\phi,T;0,\phi_0,0), \tag{4.14}$$

where Γ is some path in a complex plane, and G is the Green function of the Schrödinger equation for the $a-\phi$ system

$$i\partial_t G = \hat{H}G. \tag{4.15}$$

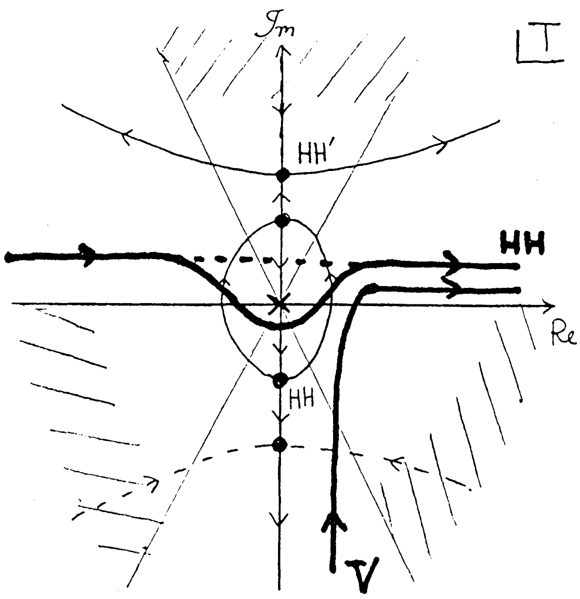

Figure2
Convergent contours in
T-plane:$a^2\lambda \ll 1$

The problem now is to find paths Γ for which the integration converges.

Unfortunately we cannot discuss this problem with generality even for this simple system. So let us omit the freedom of the scalar field and consider a system with one degree of freedom[13]. Then by rewriting the metric as

$$ds^2 = -\frac{N^2}{q}dt^2 + qd\Omega_3^2 \quad (q = a^2). \tag{4.16}$$

the wavefunction $\Psi(q)$ is expressed as

$$\Psi(q) = \int_\Gamma \frac{dT}{T^{1/2}} \exp[iS_0(q,T)], \tag{4.17}$$

where

$$S_0 = \frac{\lambda^2}{24}T^3 - \left(\frac{\lambda}{4}q - \frac{1}{2}\right)T - \frac{q^2}{8T}. \tag{4.18}$$

The contours in the complex T-plane for which the integration converges are shown in Fig.2 for $a^2\lambda \ll 1$ and in Fig.3 for $a^2\lambda \gg 1$. The saddle points of $\exp(iS_0)$ are depicted by filled circles. $\exp(iS_0)$ grows exponentially in the shaded regions as T approaches infinity.

From these figures one sees that there are two classes of convergent paths. The first is the paths which run along the real axis, and the second is those which start from $-i\infty$, go up along the imaginary axis, and then approaches infinity along the real axis.

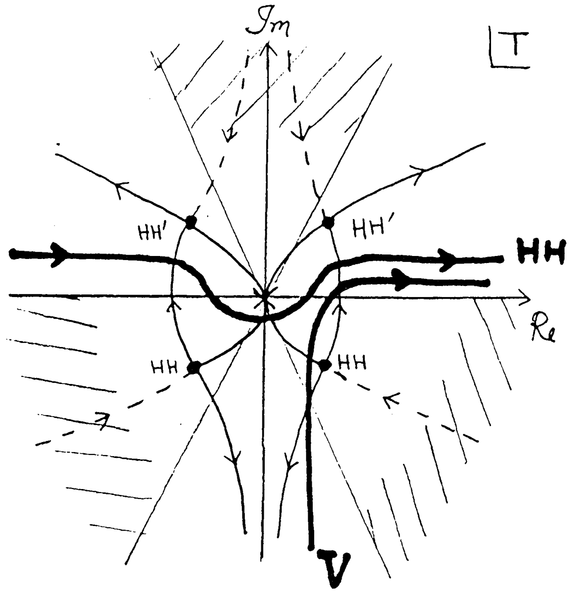

Figure3
Convergent contours in T-plane:$a^2\lambda \gg 1$

The integral along the first paths is dominated by the contribution from the saddle point(s) HH' and approximately given by

$$\Psi_{\mathrm{HH}} \sim \begin{cases} \exp[-\frac{1}{3\lambda}(1-\lambda a^2)^{3/2}] & ;a^2\lambda \ll 1, \\ \cos[\frac{1}{3\lambda}(\lambda a^2-1)^{3/2}-\frac{\pi}{4}] & ;a^2\lambda \gg 1. \end{cases} \tag{4.19}$$

Actually this corresponds to the wavefunction specified by the Hawking-Hartle Ansatz. On the other hand the integral along the second paths are approximately given by

$$\Psi_{\mathrm{V}} \sim \begin{cases} \exp[+\frac{1}{3\lambda}(1-\lambda a^2)^{3/2}] & ;a^2\lambda \ll 1, \\ \exp[i\frac{1}{3\lambda}(\lambda a^2-1)^{3/2}] & ;a^2\lambda \gg 1, \end{cases} \tag{4.20}$$

which corresponds to the solution proposed by Vilenkin[16,17].

Thus the convergence condition of the path-integral is not sufficient to pick up the solution proposed by Hartle-Hawking. Furthermore, though the integration along the HH-path for the case $a^2\lambda \ll 1$ is dominated by the saddle point on the imaginary axis which corresponds to the Euclidean path-integral, it is different from the saddle point adopted by Hartle-Hawking: the latter is the point HH. Deviation from the original Hartle-Hawking proposal becomes large for the case $a^2\lambda \gg 1$: the real parts of the saddle points get larger and larger as a increases.

5 Wigner Function of the Universe

The WKB interpretation explained in §3 has various defects. First it is generally difficult to find the semi-classical variables. Second the wavefunction is in general a superposition of many WKB-type solutions as was shown in the example in the previous section. It is difficult to separate each WKB solution. One natural method to eliminate these defects is to use Wigner function.

5.1 Definition and Fundamental Properties

First we briefly review the definition of the Wigner function and its fundamental properties[18].

The original motivation for the introduction of the Wigner function is find a formulation of quantum mechanics which has a similar structure to the classical mechanics. In quantum mechanics a state of a system is described by a density operator $\hat{\rho}$, and the expectation value of an observable $\hat{A}$ is given by $\mathrm{Tr}\hat{\rho}\hat{A}$. If we define functions on the classical phase space (Q, P) from $\hat{\rho}$ and $\hat{A}$ by

$$W(Q,P) = \int d^n q < Q - q/2|\hat{\rho}|Q + q/2 > e^{ip\cdot q}, \tag{5.1}$$

$$A(Q,P) = \int d^n q < Q - q/2|\hat{A}|Q + q/2 > e^{ip\cdot q}, \tag{5.2}$$

(5.3)

this expectation value is expressed as

$$\mathrm{Tr}\hat{\rho}\hat{A} = \int \frac{d^n Q d^n P}{(2\pi)^n} W(Q,P)A(Q,P). \tag{5.4}$$

Thus quantum mechanics apparently gets a formulation similar to the classical statistical mechanics. W is called the Wigner function. Of course the Wigner function cannot be regarded as the classical distribution function because it is not positive definite and behaves oscillatorily in general.

In terms of the Wigner function the evolution equation for $\hat{\rho}$

$$i\partial_t\hat{\rho} = [\hat{H}, \hat{\rho}]; \quad H = P^2/2m + V(Q) \tag{5.5}$$

is written in a form similar to the classical Liouville equation:

$$\partial_t W + \frac{P}{m}\frac{\partial W}{\partial Q} - V'(Q)\frac{\partial W}{\partial P} = \sum_{n=1} \frac{(-1)^n}{2^{2n}(2n+1)!} V^{(2n+1)}(Q)\frac{\partial^{2n+1} W}{\partial P^{2n+1}}. \quad (5.6)$$

In particular for the harmonic oscillator the right-hand side of this equation vanishes. Thus the right-hand side represents the higher-order quantum fluctuations.

5.2 Relation to the WKB Approximation

In the case where the wavefunction has the WKB form

$$\hat{\rho} = |\Phi><\Phi|; \quad \Phi(Q) = A(Q)e^{iS(Q)}, \quad (5.7)$$

the Wigner function is estimated in the lowest order as

$$\begin{aligned} W(Q,P) &= \int dq A(Q-q/2)\bar{A}(Q+q/2)\exp[i(S(Q-q/2)-S(Q+q/2)+P\cdot q)] \\ &= \int dq |A(Q)|^2 \exp[i(P-S'(Q))\cdot q + \cdots] \\ &\sim \delta(P-S'(Q)). \end{aligned} \quad (5.8)$$

Thus it is sharply peaked around the WKB trajectories $P = S'(Q)$.

To be precise, the behavior of the Wigner function around the peak is much more intricate[19]. For example, if the WKB approximation of a solution to the energy eigenvalue problem

$$\hat{H}\Phi(Q,E) = E\Phi(Q,E) \quad (5.9)$$

is good, its Wigner function is approximated around the peak as

$$W(Q,P) \simeq \sigma Ai(\sigma(H(Q,P)-E)), \quad (5.10)$$

where

$$\sigma(Q,E) = 2^{5/3}\left(\frac{3S}{2\hbar}\right)^{2/3} / \left(\frac{\hbar k}{m}\right)^2, \quad (5.11)$$

$$S(Q,E)/\hbar = \int_Q^{Q_t(E)} k(x,E)dx, \quad (5.12)$$

$$k(Q,E) = \frac{1}{\hbar}\sqrt{2m(E-V(Q))}, \quad (5.13)$$

$$k(Q_t,E) \equiv 0. \quad (5.14)$$

By inspecting these expression, one sees that the Wigner function falls off exponentially in the side $H > E$ while it damps oscillatorily in the side $H < E$.

5.3 Relation to the Husimi Function

Though the Wigner function is not positive definite as noted above, it becomes positive definite if it is averaged over a volume larger than the minimum quantum volume in the phase space. The easiest as well as physically meaningful way to see this is to utilize the coherent state expansion.

The coherent states are a set of states parametrized by the phase-space coordinate (Q,P) and expressed in the q-representation as

$$< q|Q,P >= (\pi\sigma^2)^{-n/4} \exp\left[iP\cdot(q-Q/2)-(Q-q)^2/(2\sigma^2)\right], \quad (5.15)$$

where σ is a complex constant expressed in terms of the dispersions the state in Q and P, σ_Q and σ_P, as

$$\sigma^2 = (1+i\sqrt{\sigma_Q^2\sigma_P^2-1})/\sigma_P^2. \quad (5.16)$$

The coherent states $|Q,P>$ form a non-orthogonal overcomplete basis. The unit operator is expanded by them as

$$1 = \int \frac{d^nQd^nP}{(2\pi)^n}|Q,P>< Q,P|. \quad (5.17)$$

Hence if we define the amplitude of a state vector $|\Phi>$ with respect to these coherent states by

$$\Phi(Q,P) =< Q,P|\Phi>, \quad (5.18)$$

$|\Phi>$ is expanded as

$$|\Phi>= \int \frac{d^nQd^nP}{(2\pi)^n}\Phi(Q,P)|Q,P>. \quad (5.19)$$

In terms of the coherent amplitude we define the Husimi function of the state $|\Phi>$ by

$$F(Q,P) = |\Phi(Q,P)|^2. \quad (5.20)$$

Then after a short calculation we find that it is expressed in terms of the Wigner function as

$$F(Q,P) = (4\pi\sigma_Q\sigma_P)^{n/2}\int d^nq d^np W(Q+q,P+p) \times \exp[-(\sigma_P^2 q^2 + \sigma_Q^2 p^2 + 2\sqrt{\sigma_Q^2\sigma_P^2 - 1}qp)]. \quad (5.21)$$

Thus the Husimi function is obtained from the Wigner function by averaging it over a region of size $|\sigma|$ around each point in the phase space. Since the Husimi function is non-negative definite from its definition, this prove the statement at the beginning of this subsection.

5.4 Application to Quantum Cosmology

The result in the previous subsection and the behavior of the Wigner function for WKB solutions suggest that the Husimi function can be used as a good tool to judge for which variables the WKB approximation is good as well as to give a probabilistic interpretation to deviation from the WKB approximation.

Historicallv the Wigner function is introduced into the study of quantum cosmology in order to understand the transition of the Universe in a quantum era to that in a classical one[20,21]. Here we consider the simple system taken up in the previous section to illustrate how the Wigner function and Husimi function is used to analyze such problem.

The Wheeler-DeWitt equation for this system

$$\left(4\frac{d^2}{dq^2} - 1 + \lambda q\right)\Psi(q) = 0 \quad (5.22)$$

is exactly soluble, and the Hartle-Hawking wavefunction is explicitly expressed in terms of the Airy function as

$$\Psi_{\rm HH}(q) = Ai[(1-\lambda q)/(2\lambda)^{2/3}]. \quad (5.23)$$

The Wigner function of this wavefunction is also exactly calculable and given by[22]

$$W(q,p) = \frac{1}{\pi(2\lambda)^{2/3}} Ai[-2H(q,p)/(2\lambda)^{2/3}]; \quad (5.24)$$

$$H(q,p) = (-4p^2 + \lambda q - 1)/2. \quad (5.25)$$

This is a special case where the approximate formula in the previous subsection becomes exact. The behavior of the Husimi function is obtained by smoothing the oscillatory part on the one side of the WKB peak.

From this expression one finds that, though the peak of the Wigner function or the Husimi function has a broad with of order $\lambda^{1/3}$ along p direction, it becomes narrower and narrower as $q = a^2$ increases. This behavior can be interpreted that the classical approximation of the Universe becomes better and better as the universe expands. Actually the classical approximation becomes good much faster since the cosmic expansion rate $\dot{a}/a$ is proportional to p/a^2.

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