Minimax Theorems in Separation Spaces

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1 Introduction and Peliminary Notions

The payoff function appearing in minimax theorems is usually assumed to have two typical properties: continuity and convexity. The purpose of this paper is to investigate general properties of the payoff function related to the convexity which guarantee the minimax equation.

It is well known that convex subsets of a linear topological space are topologically connected, and the intersections of convex subsets are also convex. We shall establish minimax theorems for a payoff function whose level sets are connected in a sense and are closed in the operation of intersection. The connectedness is a generalized version of topologically connectedness, which was introduced by Wallace[11] in a different context from the theory of minimax theorems.

A pair $(S, |)$ of a set S and a binary relation $|$ between two subsets of S is called a *weak separation space* (cf. [11]) if the following axioms hold:

(1) $A|B \Longrightarrow B|A$.

(2) $A|B \Longrightarrow A \cap B = \emptyset$.

(3) $A|B, A_1 \subset A \Longrightarrow A_1|B$.

Wallace listed another axiom for the weak separation space, but it is not important in our discussion.

Two subsets A and B of S are said to be separated provided $A|B$, and a subset A of S is said to be *s-connected* provided that it is not the union of two nonempty separated subsets of S. Note that if S is a topological space and, for $A, B \subset S$, $A|B$ means $(\overline{A} \cap B) \cup (A \cap \overline{B}) = \emptyset$, then the s-connectedness is equivalent to the topological connectedness.

The following proposition is proved in [11]:

If A is s-connected and contained in $B \cup C$ and $B|C$, then $A \subset B$ or $A \subset C$.

Given $f : X \times Y \to R$, where X and Y are sets, a set of the form $\{x \in X : f(x, y) \geq \alpha\}$ for some $y \in Y$ and $\alpha \in R$ is called a *level set* of f in X, and similarly a set of the form $\{y \in Y : f(x, y) \leq \beta\}$ for some $x \in X$ and $\beta \in R$ is called a level set of f in Y. We can regard X and Y as weak separation spaces respectively in the following way:

Let A and B be subsets of X(resp. Y). Then $A|B$ means there are two families $\{L_i\}$ and $\{L'_j\}$ (resp. $\{M_i\}$ and $\{M'_j\}$) of arbitrarily many level sets of f in X (resp. Y) such that

$$A \subset \bigcap_i L_i \qquad B \subset \bigcap_j L'_j,$$

$$(\text{resp. } A \subset \bigcap_i M_i \qquad B \subset \bigcap_j M'_j,)$$

and

$$\bigcap_i L_i \cap \bigcap_j L'_j = \emptyset.$$

$$(\text{resp. } \bigcap_i M_i \cap \bigcap_j M'_j = \emptyset.)$$

2 Main Theorem

According to the previous section, when a function f is given on the product of two sets X and Y, f induces the structures of weak separation spaces to both X and $\dot{Y}$. A subset A of X is called *connected with respect to* f if A is s-connected with respect to the structure of the weak separation space on X. Similarly we can define the connectedness of subsets of Y with respect to f.

We say that f is compact in X (resp. Y) if a family of level sets of f in X (resp. Y) which has the finite intersection property has a nonempty intersection.

Lemma 2.1 *Suppose that any intersection of finitely many level sets of f in X is connected with respect to f and any intersection of arbitrarily many level sets of f in Y is connected with respect to f. Let y_1 and y_2 be two elements of Y and let L be X or a nonempty intersection of finitely many level sets of f in X. Then it follows that*

$$\inf_{y \in Y} \sup_{x \in L} f(x, y) \leq \sup_{x \in L} \min\{f(x, y_1), f(x, y_2)\}.$$

Proof Suppose that

$$\sup_{x \in L} \min\{f(x, y_1), f(x, y_2)\} < \alpha < \inf_{y \in Y} \sup_{x \in L} f(x, y).$$

Let $\beta(x) = \max\{f(x, y_1), f(x, y_2)\}$, $x \in L$ and $W = \bigcap_{x \in L}\{y \in Y : f(x, y) \leq \beta(x)\}$. Then we have $y_1, y_2 \in W$ and W is connected with respect to f. Let $L(w) = \{x \in L : f(x, w) \geq \alpha\}$ and $l(w) = \{x \in L : f(x, w) > \alpha\}$ for any $w \in W$. It is easily seen that $l(w) \neq \emptyset$ and $L(w)$ is connected with respect to f for any $w \in W$. Let $U = \{w \in W : l(w) \subset L(y_1)\}$ and $V = \{w \in W : l(w) \subset L(y_2)\}$. It is easily seen that $y_1 \in U$ and $y_2 \in V$, and $U \cap V = \emptyset$ as $L(y_1) \cap L(y_2) = \emptyset$. Since $L(w) \subset L(y_1) + L(y_2)$ according to the definition of β and W, we have $L(w) \subset L(y_1)$ or $L(w) \subset L(y_2)$ by the proposition mentioned in the previous section. Hence we have $W = U + V$.

On the other hand, we have

$$U = \bigcap_{x \in L \setminus L(y_1)} \{w \in W : f(x, w) \leq \alpha\} \quad \text{and} \quad V = \bigcap_{x \in L \setminus L(y_2)} \{w \in W : f(x, w) \leq \alpha\}.$$

This contradicts the assertions $W = U + V$, $y_1 \in U$, and $y_2 \in V$, since W is connected with respect to f.

Lemma 2.2 *Suppose that any intersection of finitely many level sets of f in X is connected with respect to f and any intersection of arbitrarily many level sets of f in Y is connected with respect to f. Let $y_1, \ldots, y_n$ be finitely many elements of Y. Then it follows that*

$$\inf_{y \in Y} \sup_{x \in X} f(x, y) \leq \sup_{x \in X} \min\{f(x, y_1), \ldots, f(x, y_n)\}.$$

Proof The assertion is trivial for $n = 1$, and for $n = 2$ it is nothing more than Lemma 2.1. We assume $n \geq 3$ and the assertion holds for $n - 1$. Take any α with $\alpha < \inf_{y \in Y} \sup_{x \in X} f(x, y)$. Then we have

$$\alpha < \sup_{x \in X} \min\{f(x, y_1), \ldots, f(x, y_{n-2}), f(x, y)\}$$

for any $y \in Y$ by the assumption. Define a nonempty connected set L with respect to f by

$$L = \bigcap_{i=1}^{n-2} \{x \in X : f(x, y_i) \geq \alpha\}.$$

Then we have $\alpha < \sup_{x \in L} f(x, y)$ for any $y \in Y$, and hence

$$\alpha \leq \inf_{y \in Y} \sup_{x \in L} f(x, y).$$

By Lemma 2.1, $\alpha \leq \sup_{x \in L} \min\{f(x, y_{n-1}), f(x, y_n)\}$, and hence

$$\alpha \leq \sup_{x \in X} \min\{f(x, y_1), \ldots, f(x, y_n)\}.$$

Theorem 2.1 *Suppose that any intersection of finitely many level sets of f in X is connected with respect to f, any intersection of arbitrarily many level sets of f in Y is connected with respect to f and f is compact in X. Then it follows that*

$$\inf_{y \in Y} \max_{x \in X} f(x, y) = \max_{x \in X} \inf_{y \in Y} f(x, y).$$

Moreover if any intersection of arbitrarily many level sets of f in X is connected with respect to f and f is compact in Y, then f has a saddle point and it follows that

$$\min_{y \in Y} \max_{x \in X} f(x, y) = \max_{x \in X} \min_{y \in Y} f(x, y).$$

Proof Note that the maximum $\max_{x \in X} f(x, y)$ is attained for each $y \in Y$ from the comapctness of f in X and set $\alpha = \inf_{y \in Y} \max_{x \in X} f(x, y)$. We put $X_y = \{x \in X : f(x, y) \geq \alpha\}$ for any $y \in Y$. Let $y_1, \ldots, y_n$ be any finitely many elements of Y and let

$$A_m = \{x \in X : \min\{f(x, y_1), \ldots, f(x, y_n)\} \geq \alpha - \frac{1}{m}\}$$

for any positive integer m. Then we have

$$A_m = \bigcap_{i=1}^{n} \{x \in X : f(x, y_i) \geq \alpha - \frac{1}{m}\},$$

and $\{A_m\}_{m=1}^\infty$ has the finite intersection property by Lemma 2.2. Hence $\{A_m\}_{m=1}^\infty$ has a common point $\bar{x} \in X$ by the compactness of f in X, and we have

$$\alpha \le \min\{f(\bar{x}, y_1), \dots, f(\bar{x}, y_n)\}.$$

Thus the family $\{X_y\}_{y\in Y}$ has the finite intersection property and hence there exists x_0 in X with $x_0 \in \bigcap_{y\in Y} X_y$, which means

$$\alpha \le \inf_{y\in Y} f(x_0, y) \le \sup_{x\in X} \inf_{y\in Y} f(x, y) \le \alpha.$$

Hence we have $\alpha = \max_{x\in X} \inf_{y\in Y} f(x, y) = \inf_{y\in Y} f(x_0, y)$.

If, in addition, any intersection of arbitrarily many level sets of f in X is connected with respect to f and f is compact in Y, then the similar argument leads to the existence of $y_0 \in Y$ such that $\alpha = \max_{x\in X} f(x, y_0)$. It is easily seen that (x_0, y_0) is a saddle point of f and

$$\max_{x\in X} \min_{y\in Y} f(x, y) = \min_{y\in Y} \max_{x\in X} f(x, y) = f(x_0, y_0).$$

Example 2.1 Consider the sets

$$X = \{x_1, x_2, x_3, x_4\}, \qquad Y = \{y_1, y_2, y_3, y_4\}.$$

A function f on $X \times Y$ defined by the diagram

	y_1	y_2	y_3	y_4
x_1	3	1	2	2
x_2	2	[2]	2	2
x_3	2	1	3	3
x_4	2	1	2	3

satisfies the hypothesis of Theorem 2.1 and actually has a saddle point (x_2, y_2). Note that this example cannot be covered by the minimax theorem of Kindler and Trost for interval spaces (cf. [2]).

Example 2.2 Let $X = [0, 1]$ and $Y = [0, 2\pi]$, and let $f(x, y) = x \sin y$ for $x \in X$ and $y \in Y$. Then the minimax equation

$$\min_{y\in Y} \max_{x\in X} f(x, y) = 0 = \max_{x\in X} \min_{y\in Y} f(x, y)$$

holds. This example satisfies the hypothesis of Theorem 2.1, but level sets of f in Y are not necessarily topologically connected.

Corollary 2.1 *Let X be a compact topological space, Y a topological space and $f :$ $X \times Y \to R$ upper semicontinuous on X and lower semicontinuous on Y. Suppose that any intersection of finitely many level sets of f in X is topologically connected and any intersection of infinitely many level sets of f in Y is topologically connected. Then it follows that*

$$\inf_{y\in Y} \max_{x\in X} f(x, y) = \max_{x\in X} \inf_{y\in Y} f(x, y).$$

Moreover if any intersection of arbitrarily many level sets of f in X is topologically connected and Y is compact, then f has a saddle point and it follows that

$$\min_{y\in Y} \max_{x\in X} f(x, y) = \max_{x\in X} \min_{y\in Y} f(x, y).$$

3 Connectedness of Intersection of Infinitely Many Level Sets

In the hypothesis of Theorem 2.1, it is not known whether the connectedness with respect to f of interesections of infinitely many level sets in Y can be replaced by the connectedness with respect to f of intersections of finitely many level sets. The following is the partial answer to this question.

Theorem 3.1 *Let X and Y be two compact topological spaces and let $f : X \times Y \to R$ be jointly continuous. Suppose that any intersection of finitely many level sets of f in X are connected with respect to f and that any intersection of finitely many level sets of f in Y are connected with respect to f. Then it follows that*

$$\min_{y \in Y} \max_{x \in X} f(x, y) = \max_{x \in X} \min_{y \in Y} f(x, y).$$

We need some lemmas to prove Theorem 3.1.

Lemma 3.1 *Let X and Y be topological spaces and let $f : X \times Y \to R$ be jointly continuous. Then for any $x \in X$, any compact subset K of Y and any open interval $]a, b[$ with $f(x, K) \subset]a, b[$, there exists a neighborhood M of x with $f(M, K) \subset]a, b[$.*

Proof For any $y \in K$, take a neighborhood M_y of x and a neighborhood N_y of y with $f(M_y, N_y) \subset]a, b[$. Since $K = \bigcup_{y \in K} N_y$ and K is compact, there are $y_1, \ldots, y_n \in K$ with $K = \bigcup_{i=1}^{n} N_{y_i}$. The set $M = \bigcap_{i=1}^{n} M_{y_i}$ is the desired one. In fact, for any $(x, y) \in M \times K$, there is i with $(x, y) \in M_{y_i} \times N_{y_i}$, and hence $f(x, y) \in]a, b[$.

Lemma 3.2 *Let X and Y be compact topological spaces and let $f : X \times Y \to R$ be jointly continuous. Then for any $\epsilon > 0$, there is a finite subset F of X such that for any $x \in X$ there is $x' \in F$ such that $|f(x, y) - f(x', y)| < \epsilon$ for all $y \in Y$.*

Proof Fix a point $x \in X$. For any $y \in Y$, let

$$Y_y^x = \{z \in Y : |f(x, z) - f(x, y)| < \epsilon\}.$$

Then $Y = \bigcup_{y \in Y} Y_y^x$ and Y is compact, hence there are $y_1, \ldots, y_n \in Y$ with $Y = \bigcup_{i=1}^{n} Y_{y_i}^x$. Since $\overline{Y_{y_i}^x}$ is compact, and

$$|f(x, z) - f(x, y_i)| < 2\epsilon, \qquad z \in \overline{Y_{y_i}^x},$$

there is a neighborhood M_i^x of x such that

$$|f(w, z) - f(x, y_i)| < 2\epsilon, \qquad w \in M_i^x, \quad z \in Y_{y_i}^x,$$

by Lemma 3.1. Set $M^x = \bigcap_{i=1}^{n} M_i^x$, then

$$|f(w, z) - f(x, y_i)| < 2\epsilon, \qquad w \in M^x, \quad z \in Y_{y_i}^x, \quad i = 1, 2, \ldots, n.$$

Since $X = \bigcup_{x \in X} M^x$, there are $x_1, \ldots, x_m \in X$ with $X = \bigcup_{j=1}^{m} M^{x_j}$. The finite set $F = \{x_1, \ldots, x_m\}$ is the desired one. In fact, for any $x \in X$, there is x_j with $x \in M^{x_j}$. For any $y \in Y$, there is y_i with $y \in Y_{y_i}^{x_j}$. Hence we have

$$|f(x,y) - f(x_j, y_i)| < 2\epsilon, \qquad |f(x_j, y) - f(x_j, y_i)| < 2\epsilon.$$

Therefore we have

$$|f(x,y) - f(x_j, y)| < 4\epsilon, \qquad y \in Y.$$

It is easily seen from the discussion in Section 2 that we merely need the following Lemma 3.3 in order to prove Theorem 3.1.

Lemma 3.3 *Let X and Y be two compact topological spaces and let $f : X \times Y \to R$ be jointly continuous. Suppose that any intersection of finitely many level sets of f in X are connected with respect to f and that any intersection of finitely many level sets of f in Y are connected with respect to f. Let y_1 and y_2 be any two elements of Y and let L be X or a nonempty intersection of finitely many level sets of f in X. Then it follows that*

$$\min_{y \in Y} \max_{x \in L} f(x,y) \leq \max_{x \in L} \min\{f(x,y_1), f(x,y_2)\}.$$

Proof Suppose that

$$\max_{x \in L} \min\{f(x,y_1), f(x,y_2)\} < \alpha - 2\epsilon < \alpha < \alpha + \epsilon < \min_{y \in Y} \max_{x \in L} f(x,y).$$

Let $\beta(x) = \max\{f(x,y_1), f(x,y_2)\}$, $x \in L$. Then by Lemma 3.2 there is a finite subset F of L such that for any $x \in L$ there is $x' \in F$ such that

$$|f(x,y) - f(x',y)| < \epsilon \quad y \in Y; \qquad |\beta(x) - \beta(x')| < \epsilon.$$

Hence we have $\alpha < \min_{y \in Y} \max_{x' \in F} f(x',y)$. Let $W = \bigcap_{x' \in F} \{y \in Y : f(x',y) \leq \beta(x')\}$. Then we have $y_1, y_2 \in W$ and W is connected with respect to f. Let $\tilde{L}(y_i) = \{x \in L : f(x,y_i) \geq \alpha - 2\epsilon\}$, for $i = 1, 2$, and $L(w) = \{x \in L : f(x,w) \geq \alpha\}$ and $l(w) = \{x' \in F : f(x',w) > \alpha\}$ for any $w \in W$. It is easily seen that $l(w) \neq \emptyset$ and $L(w)$ is connected with respect to f for any $w \in W$. Moreover we have $L(w) \subset \tilde{L}(y_1) + \tilde{L}(y_2)$. Indeed the disjointness of $\tilde{L}(y_1)$ and $\tilde{L}(y_2)$ is obvious. If there is a point $x \in L(w)$ with $x \notin \tilde{L}(y_1)$ and $x \notin \tilde{L}(y_2)$, then there is $x' \in F$ such that

$$\alpha \leq f(x,w) < f(x',w) + \epsilon \leq \beta(x') + \epsilon$$

$$< \beta(x) + 2\epsilon < \alpha - 2\epsilon + 2\epsilon = \alpha,$$

which is a contradiction. Therefore we have $L(w) \subset \tilde{L}(y_1)$ or $L(w) \subset \tilde{L}(y_2)$.

Hence setting $U = \{w \in W : l(w) \subset \tilde{L}(y_1)\}$ and $V = \{w \in W : l(w) \subset \tilde{L}(y_2)\}$, we have $y_1 \in U$ and $y_2 \in V$, and $U + V = W$.

On the other hand, we have

$$U = \bigcap_{F \backslash \tilde{L}(y_1)} \{w \in W : f(x,w) \leq \alpha\} \quad \text{and} \quad V = \bigcap_{F \backslash \tilde{L}(y_2)} \{w \in W : f(x,w) \leq \alpha\}.$$

This contradicts the assertions $W = U + V$, $y_1 \in U$, and $y_2 \in V$, since W is connected with respect to f.

Corollary 3.1 *Let X and Y be two compact topological spaces and let $f : X \times Y \to R$ be jointly continuous. Suppose that any intersection of finitely many level sets of f in X are topologically connected and that any intersection of finitely many level sets of f in Y are topologically connected. Then it follows that*

$$\min_{y \in Y} \max_{x \in X} f(x, y) = \max_{x \in X} \min_{y \in Y} f(x, y).$$

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