数理解析研究所講究録
第789巻 1992年 179-191

Differential Inclusions Defined by Nonconvex-valued Correspondences : An Exposition

by

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I. Introduction

We consider a differential inclusion

$$\dot{x} \in \Gamma(t,x) \ , \ x(0)=a \qquad (*)$$

defined by a correspondence(=multivalued mapping) $\Gamma:[0,T]\times \mathbb{R}^{\ell} \twoheadrightarrow \mathbb{R}^{\ell}$, where a is a fixed vector in $\mathbb{R}^{\ell}$. Throughout this paper, $\Gamma(t,x)$ is assumed to be nonempty for all $(t,x)\in[0,T]\times\mathbb{R}^{\ell}$. A function $x:[0,T]\to\mathbb{R}^{\ell}$ is said to be a *solution* of (*) if (i) it is absolutely continuous, (ii) $\dot{x}(t)\in\Gamma(t,x(t))$ a.e., and (iii) $x(0)=a$. The set of all the solutions of (*) is denoted by $\triangle(a)$.

A lot of works have been devoted to finding out sufficient conditions which guarantee the existence of solutions for (*) as well as to examining the structure of solution set. We have reached at more or less satisfactory results through transparent reasonings in the case Γ is convex-valued (that is, $\Gamma(t,x)$ is convex for all $(t,x)\in[0,T]\times\mathbb{R}^{\ell}$). However the assumption of convex-valuedness seems to be quite restrictive because we encounter with abundant important situations in which this requirement is not necessarily satisfied. The primary concern of this expository article is to illuminate various difficulties which may arise in the case $\Gamma(t,x)$ is nonconvex.

II. Convex versus Nonconvex

The following standard existence theorem is due to Castaing [3]. (For the related results, see Filippov [7] [8], Lasota=Opial [11], and Maruyama [12].)

THEOREM 1 *Assume that the correspondence* $\Gamma:[0,T]\times\mathbb{R}^{\ell}\twoheadrightarrow\mathbb{R}^{\ell}$ *satisfies the following four conditions.*

(i) *The set* $\Gamma(t,x)$ *is nonempty, compact, and convex for every* $(t,x)\in[0,T]\times\mathbb{R}^{\ell}$.

(ii) *The correspondence* $x\mapsto\Gamma(t,x)$ *is upper hemi-continuous for each fixed* $t\in[0,T]$.

(iii) *The correspondence* $t\mapsto\Gamma(t,x)$ *is measurable for each fixed* $x\in\mathbb{R}^{\ell}$.

(iv) Γ *is* L^1*-integrably bounded ; i.e. there exists some function* $\psi\in L^1([0,T],\mathbb{R}_+)$ *such that*

$$\Gamma(t,x)\subset S_{\psi(t)} \qquad \text{for all } (t,x)\in[0,T]\times\mathbb{R}^{\ell},$$

where $S_{\psi(t)}$ *denotes the closed ball in* $\mathbb{R}^{\ell}$ *with center* 0 *and radius* $\psi(t)$.

Then the solution set $\Delta(a)$ *of* (*) *is nonempty for each* $a\in\mathbb{R}^{\ell}$. *And the correspondence* $\Delta:\mathbb{R}^{\ell}\twoheadrightarrow C([0,T],\mathbb{R}^{\ell})$ *defined by* $\Delta: a\mapsto\Delta(a)$ *is compact-valued and upper hemi-continuous with respect to the sup-norm topology.*

However the requirements listed in the above theorem are not necessarily admitted in some typical situations in which differential inclusions of the form (*) play an active part. Among them, the following two problems seem particularly serious.

1. To replace $\mathbb{R}^\ell$ by a Banach space of infinite dimension.

2. To remove the assumption that Γ is convex-valued.

For the first probrem, see Castaing=Valadier [4], De Blasi=Pianigiani[5], Maruyama [13], Pianigiani [15], and Tolstonogov [17].

In this paper, we are concerned with the second obstacle. Let me exhibit a couple of typical situations in which the convexity of $\Gamma(t,x)$ is not satisfied.

[1] (Optimal Control) A nonempty subset U of $\mathbb{R}^\ell$, a point $a \in \mathbb{R}^\ell$, and a function $f:[0,T]\times\mathbb{R}^\ell\times\mathbb{R}^\ell \longrightarrow \mathbb{R}$ are assumed to be given. Let us consider the problem to find a couple of a differentiable (in some sense) function $x:[0,T] \longrightarrow \mathbb{R}^\ell$ and a measurable function $u:[0,T] \longrightarrow \mathbb{R}^\ell$ such that

$$\dot{x}(t) = f(t,x(t),u(t)),$$

$$x(0)=a \ , \ u(t) \in U.$$

In order to find a solution for this problem, we define the correspondence $\Gamma:[0,T]\times\mathbb{R}^\ell \twoheadrightarrow \mathbb{R}^\ell$ by

$$\Gamma(t,x)=\{f(t,x,u) \mid u \in U\}.$$

If there exists a solution $x^*:[0,T] \longrightarrow \mathbb{R}^\ell$ for the differential inclusion

$$\dot{x} \in \Gamma(t,x) \ , \ x(0)=a,$$

then we can also find a suitable measurable function $u^*:[0,T]$

$\longrightarrow \mathbb{R}^{\ell}$ such that $(x^*(\cdot),u^*(\cdot))$ forms a solution, by making use of Filippov's measurable implicit function theorem. However the correspondence Γ can not be assumed to be convex-valued in this case.

[2] (Implicit Differential Equation) Let $f:[0,T]\times\mathbb{R}^{\ell}\times\mathbb{R}^{\ell} \longrightarrow \mathbb{R}$ be a given function. Then the implicit differential equation

$$f(t,x,\dot{x})=0 \ , \ x(0)=a$$

can be reduced to a differential inclusion of the form (*) if we define the correspondence $\Gamma:[0,T]\twoheadrightarrow\mathbb{R}^{\ell}$ by

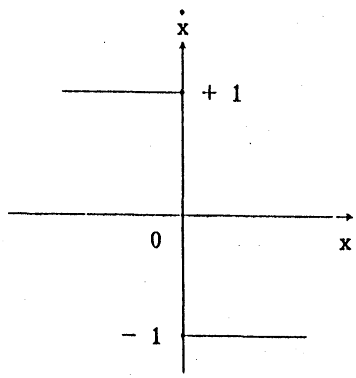
$$\Gamma(t,x)=\{y\in\mathbb{R}^{\ell} \mid f(t,x,y)=0\}.$$

But the correspondence Γ is not necessarily convex-valued in this case too.

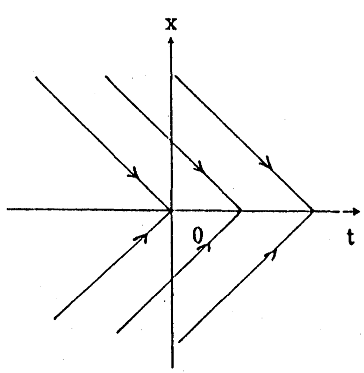
Ⅲ. Examples

[A. Existence] Define the correspondence $\Gamma:\mathbb{R}\twoheadrightarrow\mathbb{R}$ by

$$\Gamma(x)=\begin{cases} -\text{sgn } x & \text{for } x\neq 0 \\ \{-1,+1\} & \text{for } x=0. \end{cases}$$



⟨Fig.1⟩



⟨Fig.2⟩

Then Γ is compact-valued, measurable in t, upper hemi-continuous in x, and L^1-integrably bounded. But it is not convex-valued.

In this case, the differential inclusion

$$\dot{x} \in \Gamma(x) \ , \ x(0)=0$$

does not have a solution. On the other hand, the "relaxed" differential inclusion

$$\dot{x} \in co\Gamma(x) \ , \ x(0)=0$$

has a solution, say $x(t)\equiv 0$.

[B. Continuous dependence of $\triangle(\cdot)$ on initial conditions] (Pianigiani [15], Pliś [16]) Let $X \subset \mathbb{R}^2$ be a nonempty bounded set. Define the correspondnce $\Gamma : X \twoheadrightarrow \mathbb{R}^2$ by

$$\Gamma(x,y)=\{(+1,|x|+Max(sgn(y)\cdot\sqrt{|y|} \ , 0)),$$
$$(-1,|x|+Max(sgn(y)\cdot\sqrt{|y|} \ , 0))\}.$$

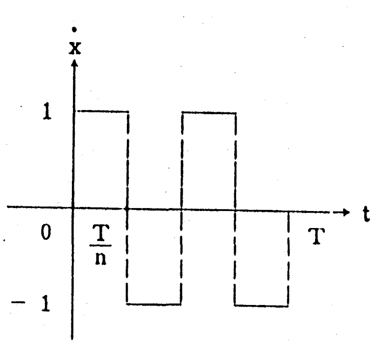
Consider the initial value problem:

(P_0) $\quad (\dot{x},\dot{y}) \in \Gamma(x,y) \ , \ x(0)=y(0)=0,$

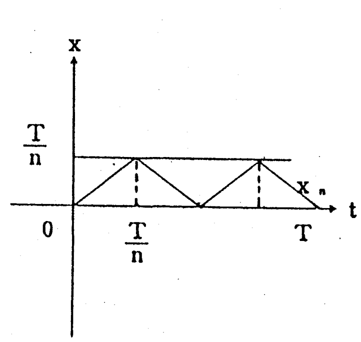
that is

$$\dot{x} \in \{-1,+1\} \ , \ x(0)=0, \qquad (1)$$

$$\dot{y}=|x|+Max(sgn(y)\cdot\sqrt{|y|} \ , 0) \ , \ y(0)=0. \qquad (2)$$



⟨Fig.3⟩



⟨Fig.4⟩

⟨Fig.4⟩ depicts a particular solution of (1) which satisfies

$$\dot{x}(t)=\begin{cases} +1 & \text{if} \quad \frac{2i}{n}T \leq t < \frac{2i+1}{n}T \\ -1 & \text{if} \quad \frac{2i+1}{n}T \leq t < \frac{2i+2}{n}T \end{cases} \qquad (3)$$

$$(i=0,1,2,\cdots\cdots)$$

We must note that $\sup_{t\in[0,T]} |x(t)|$ can be arbitrarily small as $n \to \infty$.

On the other hand, a solution $y(t)$ of (2) is nonnegative for all $t\in[0,T]$ since

$$|x| + \mathrm{Max}(\mathrm{sgn}(y)\cdot\sqrt{|y|}\ ,\ 0) \geq 0 \quad \text{and} \quad y(0)=0.$$

Hence we obtain a simpler relation

$$\dot{y} = |x| + \sqrt{y} \geq \sqrt{y} \qquad (4)$$

instead of (2). Since the solution of the initial value problem : $\dot{y}=\sqrt{y}$, $y(0)=0$ is given by $y(t)=t^2/4$, it follows from (4) that

$$y(t) \geq \frac{t^2}{4} \qquad \text{for all } t\in[0,T]. \qquad (5)$$

Now let $\{\varepsilon_n\}$ be a sequence of positive numbers which tends to 0. And consider the perturbed initial value problem :

$$(P_n) \qquad (\dot{x},\dot{y}) \in \Gamma(x,y)\ ,\ x(0)=0\ ,\ y(0)=-\varepsilon_n.$$

The change of the problem from (P_0) to (P_n) does not influence the behavior of $x(t)$. If we choose n in (3) so large that $0 < \frac{T}{n} < \frac{\varepsilon_n}{2T}$, then we have

$$|x_n(t)| < \frac{\varepsilon_n}{2T} \qquad \text{for all } t \in [0,T]. \qquad (6)$$

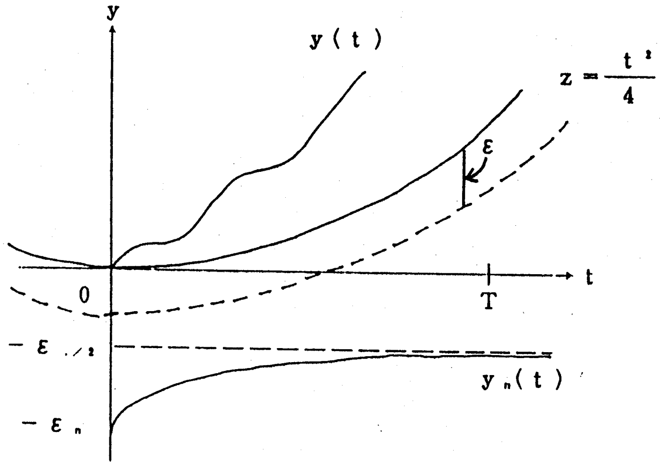
Since $y_n(0) = -\varepsilon_n < 0$, the set $V = \{t \in [0,T] \mid y_n(t) < 0\}$ is a neighborhood of 0 in $[0,T]$. Taking account of the relation

$$\dot{y}_n(t) = |x_n(t)| + \mathrm{Max}(\mathrm{sgn}(y_n(t)) \cdot \sqrt{|y_n(t)|}, 0) = |x_n(t)| \qquad \text{on } V,$$

we obtain

$$y_n(t) = -\varepsilon_n + \int_0^t |x_n(s)|ds \leq -\varepsilon_n + \frac{\varepsilon_n}{2T} \cdot t \leq -\frac{\varepsilon_n}{2} \quad \text{on } V. \quad (7)$$

This implies that $y_n(t) < 0$ for all $t \in [0,T]$.



⟨Fig.5⟩

Thus we conclude that the correspondence $\triangle(\cdot,\cdot)$ is not upper hemi-continuous because $\triangle((0, -\varepsilon_n))$ is not contained in a sufficiently small neighborhood of $\triangle((0,0))$.

[C. Closedness of $\triangle(a)$] Define the correspondence $\Gamma : [0,T] \times \mathbb{R} \twoheadrightarrow \mathbb{R}$ by

$$\Gamma(t,x) = \{-1, +1\}, \quad x(0) = 0. \qquad (8)$$

Consider the following initial value problem :

$$\dot{x} \in \Gamma(t,x), \quad x(0) = 0. \qquad (9)$$

⟨fig.4⟩ in [B] can be regarded as depicting a particular solution $x_n(\cdot)$ of (9) which satisfies (3). Then the sequence $\{x_n\}$ uniformly converges to the function $x(\cdot)\equiv 0$ as $n\to\infty$. However the limit function $x(\cdot)\equiv 0$ is not a solution of (8).

[D. Closedness of reachable sets] (Filippov [6]) Consider the differential equation :

$$(\dot{x},\dot{y})=(-y^2+u^2,u) \tag{10}$$

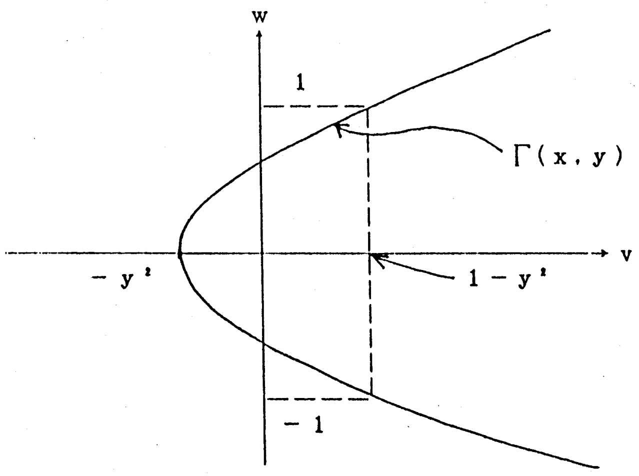
$$-1\le u(t)\le 1 \ ,\ x(0)=y(0)=0.$$

If we define the correspondence $\Gamma:\mathbb{R}^2\twoheadrightarrow\mathbb{R}^2$ by

$$\Gamma(x,y)=\{(v,w)\in\mathbb{R}^2 \mid v=w^2-y^2,-1\le w\le 1\}, \tag{11}$$

then the problem (10) can be trasformed into

$$(\dot{x},\dot{y})\in\Gamma(x,y)\ ,\ x(0)=y(0)=0. \tag{12}$$



⟨Fig.6⟩

Let us examine the reschable set R(1) of (12) at t=1 :

$$R(1)=\{(x(1),y(1))\in\mathbb{R}^2 \mid (x,y)\in\triangle((0,0))\}. \tag{13}$$

If $y(t)\equiv 0$, then $u(t)=0$ a.e., and hence $\dot{x}(t)=-y^2(t)+u^2(t)=0$ a.e. Consequently it follows that $x(t)\equiv 0$.

Assume that $y(t)\not\equiv 0$. Then we must have

$$\dot{x}(t)=-y^2(t)+u^2(t)\le 1 \quad \text{for all } t\in[0,1] \tag{14}$$

and

$$\dot{x}(t) < 1 \quad \text{on some interval in } [0,1]. \qquad (15)$$

Hence $(1,0) \notin R(1)$.

On the other hand, define the sequence $\{u_n(t)\}$ of functions exactly in the same manner as the RHS of (3), for the case $T=1$. Then the corresponding sequence $\{(x_n, y_n)\}$ of solutions of (10) satisfies

$$0 \leq y_n(t) \leq \frac{1}{n}$$

$$\dot{x}_n(t) \geq 1-\frac{1}{n^2} \Rightarrow x_n(1) \geq 1-\frac{1}{n^2}.$$

Therefore the sequence $\{(x_n(1), y_n(1))\}$ in $R(1)$ converges to $(1,0)$ $\notin R(1)$ as $n \to \infty$. This shows that $R(1)$ is not closed.

Ⅳ. Basic Theorem for Nonconvex Case

The above examples illuminate the various difficulties which arise in the case Γ is not convex-valued. In this case, we are forced to impose much restristive assumptions on Γ in order to obtain a similar result as Theorem 1. One of the basic theorems for differential inclusions (*) in the nonconvex case is as follows.

THEOREM 2 (Filippov [7], Kaczyński=Olech [10]) *Assume that the correspondence* $\Gamma : [0,T] \times \mathbb{R}^{\ell} \twoheadrightarrow \mathbb{R}^{\ell}$ *satisfies the following four conditions.*

(i) *The set* $\Gamma(t,x)$ *is nonempty and compact for every*

$(t,x) \in [0,T] \times \mathbb{R}$.

(ii) *The correspondence* $x \mapsto\!\!\!\rightarrow \Gamma(t,x)$ *is continuous in the Hausdorff metric for each fixed* $t \in [0,T]$.

(iii) *The correspondence* $t \mapsto\!\!\!\rightarrow \Gamma(t,x)$ *is measurable for each fixed* $x \in \mathbb{R}^{\ell}$.

(iv) Γ *is* L^1*-integrably bounded by some* $\psi \in L_1([0,T], \mathbb{R}_+)$.

Then $\triangle(a)$ *is nonempty for each* $a \in \mathbb{R}^{\ell}$.

Furthermore if we replace (ii) *by the stronger conditoin* (ii ') *stated below, then* $\triangle(a)$ *is dense (in the sup-norm topology) in the solution set of*

$$\dot{x} \in \mathrm{co}\,\Gamma(t,x) \quad , \quad x(0)=a.$$

(ii ') *The function* $\psi \in L^1([0,T], \mathbb{R}_+)$ *appearing in* (iv) *also satisfies the relation*

$$h(\Gamma(t,x), \Gamma(t,y)) \leq \psi(t) \, \| x-y \| \qquad \textit{for all } x,y \in \mathbb{R}^{\ell}$$

where $h(\cdot,\cdot)$ *stands for the Hausdorff metric.*

The significance of condition (ii ') is illuminated by the following example.

EXAMPLE (Pliś [16]) Define the correspondence $\Gamma : \mathbb{R}^2 \rightarrow\!\!\!\rightarrow \mathbb{R}^2$ by

$$\Gamma(x,y) = \{ (+1, |x| + \sqrt{|y|}) \ , \ (-1, |x| + \sqrt{|y|}) \}.$$

And consider a couple of initial value problems as follows :

(P_1) $\quad (\dot{x},\dot{y}) \in \Gamma(x,y) \ , \ x(0)=y(0)=0.$

(P_2) $\quad (\dot{x},\dot{y}) \in \mathrm{co}\,\Gamma(x,y) \ , \ x(0), \ y(0)=0.$

It is clear that $(x_0(t), y_0(t)) \equiv (0,0)$ is a solution of (P_2), but it is not a solution of (P_1).

Let $\{(x(\cdot),y(\cdot))\}$ be any solution of (P_1). Then $x(\cdot)$ can not be identically 0 on any interval. Hence there exists a sequence $\{t_k\}$ in $[0,T]$ such that

$$t_k \to 0 \quad \text{as} \quad k \to \infty \ ; \ |x(t_k)| \neq 0 \quad \text{for all } k.$$

It is not hard to show that

$$y(t) \geq \frac{1}{4}(t-t_k)^2 \qquad \text{for } t \geq t_k.$$

Since $t_k \to 0$, we can conclude that

$$y(t) \geq \frac{1}{4} t^2 \qquad \text{for } t \geq 0.$$

Hence there .xists no sequence of solutions of (P_1) which uniformly converges to (x_0,y_0).

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