数理解析研究所講究録
第 793 巻 1992 年 149-168

On 6-dimensional S^1 symplectic Hamiltonian manifolds with Euler number 4.

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1. Introduction

Let (M,ω) be a compact connected symplectic manifold. Suppose that a Lie group G acts on M and there exists a moment map $\mu: M \to Lie(G)^*$. Here $Lie(G)^*$ is a dual of a Lie algebra of G. (See [AB].) There has been much interest in the moment map. In 4-dimensional case, diffeo-types of all S^1-symplectic manifolds with moment maps are classified as S^1-almost complex manifolds. (See [AH], [Au1].) If M is an S^1-symplectic manifold with moment map and the action is semi-free, then Hattori [H2] shows that the cohomology ring of M is identified with that of $S^2 \times \cdots \times S^2$. If M is S^1-symplectic and with moment map and M has two components of fixed point set and one of them is isolated, then Delzant [D] shows that M is diffeomorphic to $\mathbf{C}P^n$, a complex projective space. Takakura [T] defines a toral action on a moduli $\mathfrak{M}$ of flat connections on an SU_2 principal bundles over a certain 2-dimensional V-manifold and for a symplectic structure on $\mathfrak{M}$ he calculates its moment map and consider the topology of it.

Suppose that the Lie group G is S^1. If an S^1-symplectic manifold M is simply connected then M has a moment map. Hence the condition that M has a moment map μ might not be so invalid. Moreover if the fixed point set M^{S^1} is discrete then μ is a perfect Morse function. The moment map μ is valid to determine its cohomology ring and its diffeo-type.

In this paper we consider a 6-dimensional S^1 symplectic manifold M with moment map μ and assume that the fixed point set M^{S^1} is isolated and the Euler number $\chi(M)$ is 4. This is one of the simplest cases. From the localization theorem (see [H1]) , if M^{S^1} is isolated then $\chi(M)$ is positive and even. It is easily shown that $\chi(M) \geq 4$. (See Lemma 2.4.) Ahara [Ah] classifies the S^1 actions around their fixed points for S^1-almost complex manifolds (M^6, J) with $\chi(M) = 4$, $c_1^3(M) \neq 0$, and the Todd genus $Todd[M] = 1$. We can apply this theorem and show a classification theorem (see Theorem 2.10) in our case. Hattori pointed out that the Wall's theorem for 6-spin manifolds [Wa] implies that if M is also spin then we can determine the diffeo-type of M. (See Theorem 2.11.)

In 4-dimensional case, diffeo-types of M are classified. We review the methods of the classification. Ahara and Hattori [AH] show that any critical point of μ except the minimum point and the maximum point is isolated and that both Morse index and Morse co-index are 2 at the point and they construct admissible chains. Audin [Au1] notices that the inverse image $\mu^{-1}(a)$ for a general point a in $\mathbf{R}$ is a Seifert 3-manifold if it is not empty. She classifies M's from the classification theorem of Seifert 3-manifolds. In our 6 dimensional case we consider 4-cells determined from fixed points with Morse index 4 and determine their topology and singularities.

There are exactly 4 fixed points $\{P_0, P_1, P_2, P_3\}$ and we can take them such that

$$\text{Morse index of } P_i = 2i.$$

We define a 4-cell F by a closure of a stable manifold $F^s(P_1)$, (which is defined in section 2-2.)

We have the following theorems.

THEOREM 1.

$F = Closure(F^s(P_1)) = F^s(P_1) \sqcup F^s(P_2) \sqcup F^s(P_3)$.

Here we remark that $M = \sqcup_{j=0}^3 F^s(P_j)$ and $F^s(P_j)$ is homeomorphic to D^{6-2j}, $(6-2j)$-disk. This theorem gives a cellular structure of M.

THEOREM 2.

(1) If the action is included in type III of Theorem 2.10, then the singular point set of F is $F^s(P_2) \sqcup F^s(P_3) \approx S^2$.

(2) If F is not singular at P_2 nor at P_3, then F is diffeomorphic to $\mathbf{C}P^2$ and M is diffeomorphic to $\mathbf{C}P^3$.

Finally the authors are very grateful to Akio Hattori and Yukio Matsumoto and Nariya Kawazumi for several useful comments and constant encouragement.

2. Classification of the S^1 actions

2-1. S^1 symplectic manifold with moment map.

DEFINITION 2.1. *A quadruple $(M, \omega, \varphi, \mu)$ is an S^1 symplectic manifold with moment map if*

(1) M is a $2n$ dimensional compact connected smooth manifold,

(2) ω is a symplectic form on M, that is, ω is a closed 2-form and $\omega^n \neq 0$ everywhere,

(3) φ: $S^1 \times M \to M$ is an effective S^1 action which preserves the symplectic structure ω, and

(4) μ: $M \to \mathbf{R}$ is a moment map, that is, $d\mu = i(X)\omega$, where X is a vector field on M determined from tangents of S^1 orbits, and $i(\cdot)$ is an inner product.

The following proposition gives a primitive character of moment maps.

PROPOSITION 2.1.

(1) The critical point set of μ coincides with the fixed point set of the S^1 action.

(2) The moment map μ is a non-degenerate function in the sense of Bott. (See [B].) In particular if the fixed point set M^{S^1} is isolated then μ is a perfect Morse function.

(3) Suppose that J, $\langle \cdot, \cdot \rangle$ are an almost complex structure and a metric compatible with ω, that is, J is an automorphism of $\mathfrak{X}(M)$ such that $J^2 = -1$ and $\omega(u, Jv) = \langle u, v \rangle$ for any tangent vectors u, v. Then $grad\, \mu = JX$.

Next we define a system of weights. Let P be a fixed point. From the equivariant Darboux's theorem, we can take a complex coordinate $(z_0, \cdots, z_{n-1})$ around P such that

(a) $\omega = \dfrac{\sqrt{-1}}{2} \sum_{j=0}^{n-1} dz_j \wedge d\bar{z}_j$,

(b) There exist integers $m_0, \cdots, m_{n-1}$ and they satisfies

$$g \cdot (z_0, \cdots, z_{n-1}) = (g^{m_0} z_0, \cdots, g^{m_{n-1}} z_{n-1})$$

for $g \in S^1 \subset \mathbf{C}$.

We call the integers $(m_0, \cdots, m_{n-1})$ the weights at P. The system of weights determine the Morse indices at the fixed points. In fact,

PROPOSITION 2.3.

(1) Around P,

$$\mu(z_0, \cdots, z_n) = \mu(p) - \sum_{j=0}^{n-1} m_j |z_j|^2.$$

(2) If P is an isolated fixed point, then $m_j \neq 0$ and

$$\text{Morse index}(P) = 2\#\{m_j \mid m_j > 0\}.$$

We can take an S^1-invariant almost complex structure J and an S^1 invariant metric $\langle \cdot, \cdot \rangle$ which are compatible with ω. Considering indices of twisted Dirac operators, Hattori [Ha1;Proposition 2.6] gives relations of weights of S^1-almost complex manifolds. From this proposition we have

LEMMA 2.4. *If $\dim M = 6$ and the fixed point set is isolated, then $\chi(M)$ is even and $\chi(M) \geq 4$.*

(Proof) Hopf's theorem $\chi(M) = \# M^{S^1}$ implies that $\chi(M)$ is non-negative. μ has a maximum point and it follows that $\chi(M)$ is positive. From [Ha1; Proposition 2.6] we have $\dfrac{3\chi(M)}{2} \in \mathbf{Z}$. Hence $\chi(M)$ is even. Assume that $\chi(M)$ is exactly 2. The moment map μ always have a minimum point and a maximum point and these two points are all of the fixed points. At the minimum point (resp. a maximum point), the weights are all negative (resp. positive.) But such system of weights does not satisfy Hattori's relation. This completes the proof.

Following the previous lemma, we assume the next conditions.

ASSUMPTION 2.5. *(M,ω,φ,μ) satisfies*

(1) $\dim M = 6$,

(2) M^{S^1} is isolated and $\chi(M) = 4$.

2-2. $\mathbf{C}^\times$ action and stable submanifold.

For a general (M,ω,φ,μ), we can define a $\mathbf{C}^\times = \mathbf{C} - \{0\}$ action on M. In fact, for $p \in M$, $g \in S^1$, $z \in \mathbf{R}_+ = \{z \in \mathbf{R} \mid z > 0\}$,

$$(zg)p = g \cdot \exp_p(\log z)(-grad\,\mu).$$

To show this definition is well-defined, it is sufficient to prove the following lemma.

LEMMA 2.6. *$[grad\,\mu, X] = 0$,where X is a vector field determined by the S^1 action.*

It is easy to show that the symplectic structure ω is preserved by this $\mathbf{C}^\times$ action. The following lemma is important to investigate the cellular structure of M.

LEMMA 2.7. $\lim_{z\to 0}(zg)p \in M^{S^1}$, $\lim_{z\to\infty}(zg)p \in M^{S^1}$.

We call the former the north pole and the latter the south pole of the orbit. This lemma implies that the closure of any $\mathbf{C}^\times$ orbit is a point or S^2 topologically. For a fixed point P, we defines a stable submanifold $F^s(P)$ (resp. an unstable submanifold $F^u(P)$) as follows.

$$F^s(P) = \{p \in M \mid \lim_{z\to\infty}(zg)p = P\}$$
$$F^u(P) = \{p \in M \mid \lim_{z\to 0}(zg)p = P\}.$$

The following proposition is primitive.

PROPOSITION 2.8.

(1) $F^s(P)$, $F^u(P)$ are $\mathbf{C}^\times$ invariant smooth submanifold.

(2) $F^s(P) \approx D^d$, $F^u(P) \approx D^{6-d}$, where d =Morse index(P).

2-3. Classification of weights and Wall's theorem.

From [Hal], The Todd genus $Todd[M]$ is given by the number of fixed points with all weights positive. In our case we can show that $Todd[M] = 1$. (Because if there are two local maximum points then there would be a critical point with index $(2n-1)$.) Since $H^2(M;\mathbf{R}) = \mathbf{R}$ and $\omega^3 \neq 0$, we have $c_1(M)^3 \neq 0$. Under this conditions we apply Ahara's theorem [Ah;Theorem 1.2]. First we can take fixed point set $M^{S^1} = \{P_0, P_1, P_2, P_3\}$ such that Morse index$(P_j) = 2j$.

LEMMA 2.9.

If the weights at P_j are (m_{j0}, m_{j1}, m_{j2}) and $m_{20} < 0$ then we have

$$(m_{30} + m_{31} + m_{32}) - (m_{20} + m_{21} + m_{22}) = -m_{20}N > 0,$$

where N is the largest positive integer dividing $c_1(M)$ in $H^2(M;\mathbf{Z})$.

(Proof) The stable submanifold $F^s(P_2)$ of P_2 gives a 2-cycle of a generator of $H_2(M;\mathbf{Z})$ because μ is a perfect Morse function. If x in $H^2(M;\mathbf{Z})$ denotes a dual of this, then we have

$$c_1(M) = \pm N x.$$

Consider a complex line bundle $\wedge^3 TM$ on M. It is clear that $c_1(\wedge^3 TM)$ $= c_1(M)$. If we identify $R(S^1)$ with $\mathbf{Z}[t]$ then

$$\wedge^3 TM|_{P_j} = t^{m_{j0}+m_{j1}+m_{j2}} \qquad (j = 0, 1, 2.)$$

On the other hand, let ζ be a complex line bundle over M such that $c_1(\zeta) = \frac{1}{2\pi}[\omega]$. Here we assume that $[\omega]$ is an integral class and $\frac{1}{2\pi}[\omega] = kx$ for some integer k. It is known that if M has a moment map μ then $[\omega] \in \mathrm{Im}(H^2_{S^1}(M) \to H^2(M))$ (see [AB],) and hence any S^1-action on M is lifted to ζ (see [HY].) If integers a_j is defined by $\zeta|_{P_j} = t^{a_j}$ then

$$\frac{(m_{30}+m_{31}+m_{32})-(m_{20}+m_{21}+m_{22})}{\pm N} = \frac{a_3 - a_2}{k} = -m_{20}.$$

Next we prove that k is positive. In fact,

$$k = \langle \frac{1}{2\pi}[\omega], [F^s(P_2)] \rangle = \frac{1}{2\pi} \int_{F^s(P_2)} \omega$$

$$= \frac{1}{-m_{20}} (\mu(P_3) - \mu(P_2)) > 0.$$

Consider a generic point p such that $Closure(\mathbf{C}^\times(p)) = \mathbf{C}^\times(p) \cup P_0 \cup P_3$. Then

$$\frac{(m_{30}+m_{31}+m_{32})-(m_{00}+m_{01}+m_{02})}{\pm N} = \langle x, [\mathbf{C}^\times(p)] \rangle$$

$$= \langle \frac{1}{2k\pi}[\omega], [\mathbf{C}^\times(p)] \rangle = \frac{1}{2k\pi} \int_{\mathbf{C}^\times(p)} \omega = \frac{1}{k}(\mu(P_3) - \mu(P_0)) > 0$$

This implies that $c_1(M) = Nx$ and completes the proof.

From this lemma 2.9 and [Ah; Theorem 1.2], we have a classification of weights and N.

THEOREM 2.10.

If $(M, \omega, \varphi, \mu)$ satisfies Assumption 2.5 then the system of weights and N are one of the following types.

type I

$$P_3 : (a, b, c)$$
$$P_2 : (-a, b-a, c-a)$$
$$P_1 : (-b, a-b, c-b)$$
$$P_0 : (-c, a-c, b-c)$$

where $0 < a < b < c$, *G.C.D.*$(a, b, c) = 1$, *and* $N = 4$.

type II

$$P_3 : (a+b, b-a, b)$$
$$P_2 : (a+b, a-b, a)$$
$$P_1 : (-a-b, b-a, -a)$$
$$P_0 : (-a-b, a-b, -b)$$

where $0 < a < b$, *G.C.D.*$(a, b) = 1$, *and* $N = 3$.

type III

$$P_3 : (1, 2, 3)$$
$$P_2 : (1, a, -1)$$
$$P_1 : (1, -a, -1)$$
$$P_0 : (-1, -2, -3)$$

where $a = 4$ *or* $a = 5$. *If* $a = 4$ *then* $N = 2$. *If* $a = 5$ *then* $N = 1$.

Wall [Wa] shows that diffeo-types of 6 dimensional simply connected spin manifolds with torsion-free homology are determined by the cohomology ring and the Pontryagin class. We apply this theorem in our case we have

THEOREM 2.11.

(1) If $(M, \omega, \varphi, \mu)$ satisfies Assumption 2.5 and its system of weights is of type I then M is diffeomorphic to $\mathbf{C}P^3$.

(2) If $(M, \omega, \varphi, \mu)$ satisfies Assumption 2.5 and its system of weights is of type III with $a = 4$ then M is diffeomorphic to $\mathbf{V}_5$, a Fano 3-Fold. (About $\mathbf{V}_5$, see [Ah],[I],[MU].)

(Remark) Each $\mathbf{C}P^3$, $\mathbf{V}_5$ has an S^1 symplectic structure and has moment map. In the above theorem, we don't know if M is S^1-diffeomorphic to $\mathbf{C}P^3$ or $\mathbf{V}_5$. $\mathbf{C}P^3$ and $\mathbf{V}_5$ are obtained by a surgery of S^6 via a certain embedding $g\colon S^3 \times D^3 \to S^6$. If we could make and S^1-surgery of S^6 then we would solve this problem.

3. Singularity of a 4-cell F

3-1. Results

As mentioned in the introduction, to determine the diffeo-type of M we consider a stable submanifold $F^s(P_1)$. If 4 cell F is defined by $F =$ Closure$(F^s(P_1))$ then we have the following theorem.

THEOREM 3.1.

$F = F^s(P_1) \sqcup F^s(P_2) \sqcup F^s(P_3)$.

We postpone the proof of this theorem. (See the section 3-3.) If NF is a tubular neighborhood of F then we have $M \approx NF \cup D^6$, where $D^6 = F^s(P_0)$ is a 6 disk. To investigate NF, we consider a singular point set of F. We have the following theorem.

THEOREM 3.2.

(1) If the S^1 action is of type III, then the singular point set of F is $F^s(P_2) \sqcup F^s(P_3)$.

(2) If F is non-singular at P_2 and at P_3, then F is S^1-diffeomorphic to $\mathbf{C}P^2$ and M is diffeomorphic to $\mathbf{C}P^3$.

For the proof, we need preliminaries of Seifert manifolds.

3-2. Weighted homogeneous polynomials and Seifert invariant.

Let a coordinate (z_0, z_1, z_2) around P_3 be fixed. Let (m_0, m_1, m_2) be weights at P_3.

LEMMA 3.3.

(1) If D_ϵ is a small ball with center P_3, then $F^s(P_1) \cap D_\epsilon$ is not empty and it is a complex submanifold of D_ϵ.

(2) $F \cap D_\epsilon$ is an algebraic subvariety in D_ϵ.

From the proof of Proposition 3.9(2), $F^s(P_1) \cap D_\epsilon$ is not empty. $F^s(P_1)$ is a $\mathbf{C}^\times$ invariant almost complex submanifold and we consider the following situation.

$\mathbf{C}^3 = \{(z_0, z_1, z_2)\}$, $\mathbf{C}^\times$ acts on $\mathbf{C}^3$ with weights m_0, m_1, m_2.

E : a real 4-dimensional smooth submanifold of $\mathbf{C}^3$ such that it is $\mathbf{C}^\times$ invariant and it is an almost complex submanifold.

Let $\mathbf{P}(m_0, m_1, m_2)$ be $\mathbf{C}^3/\mathbf{C}^\times$, a weighted projective space. $E' = E/\mathbf{C}^\times - \{\text{singular points}\}$ is an almost complex submanifold of $\mathbf{P}(m_0, m_1, m_2)$. E' is 2-dimensional and we can show that E' is complex submanifold of $\mathbf{P}(m_0, m_1, m_2)$. It follows that E is a complex submanifold of $\mathbf{C}^3$. If $F^s(P_1) \cap D_\epsilon$ is represented by

$$\{(z_0, z_1, z_2) \mid f(z_0, z_1, z_2) = \sum \alpha_{ijk} z_0^i z_1^j z_2^k = 0\},$$

then the defining function f has a finite degree. In fact, f is S^1-equivariant, that is,

$$(3.4) \qquad f(g^{m_0}z_0, g^{m_1}z_1, g^{m_2}z_2) = \lambda f(z_0, z_1, z_2) \quad \text{for some } \lambda \in \mathbf{C}^\times,$$

and it follows that f has only finite non-zero coefficients α_{ijk}. Hence $F \cap D_\epsilon = \mathrm{Closure}(F^s(P_1)) \cap D_\epsilon$ is an algebraic variety. This completes the proof.

$f(z_0, z_1, z_2)$ is a weighted homogeneous polynomial if and only if f is a finite polynomial satisfying (3.4). For a weighted homogeneous polynomial $f(z_0, z_1, z_2) = \sum \alpha_{ijk} z_0^i z_1^j z_2^k$, $d = m_0 i + m_1 j + m_2 k$ is called a weighted degree of f.

(Remark) Usually the weights of f is defined by $(\frac{d}{m_0}, \frac{d}{m_1}, \frac{d}{m_2})$. But avoiding any confusion, we do not use this term in this paper.

Orlik and Wagreich [OW] classify algebraic varieties in $\mathbf{C}^3$ with one isolated singular point $0 \in \mathbf{C}^3$.

LEMMA 3.5 [OW]. *If $V = \{(z_0, z_1, z_2) \mid f(z_0, z_1, z_2) = 0\}$ has one isolated singular point $0 \in \mathbf{C}^3$, then the defining function f is analytically isomorphic to one of the following functions.*

(I) $z_0^a + z_1^b + z_2^c$ *(II)* $z_0^a + z_1^b + z_1 z_2^c$ $(b > 1)$

(III) $z_0^a + z_1^b z_2 + z_1 z_2^c$ $(b > 1, c > 1)$ *(IV)* $z_0^a + z_0 z_1^b + z_1 z_2^c$ $(a > 1)$

(V) $z_0^a z_1 + z_1^b z_2 + z_2^c z_0$

Let a 5-sphere $S^5_{P_3}$ around P_3 be defined by

$$\{(z_0, z_1, z_2) \mid |z_0|^2 + |z_1|^2 + |z_3|^2 = \epsilon\}$$

for small $\epsilon > 0$. Orlik and Wagreich [OW] calculate the Seifert invariants

$$\{b; (O_1, g); (\alpha_1, \beta_1), \cdots, (\alpha_r, \beta_r)\}$$

for a Seifert 3-manifold $K = V \cap S^5_{P_3}$. Let three irreducible ratios $\frac{u_j}{v_j}$ $(j = 0, 1, 2)$ be given by $\frac{u_j}{v_j} = \frac{d}{m_j}$. And we define

$$
\begin{aligned}
&C_{012} = (u_0, u_1, u_2),\\
&\quad C_0 = (u_1, u_2)/C_{012}, \quad C_1 = (u_2, u_0)/C_{012}, \quad C_2 = (u_0, u_1)/C_{012},\\
&\quad C_{12} = u_0/C_{012}C_1C_2, \quad C_{20} = u_1/C_{012}C_2C_0, \quad C_{01} = u_2/C_{012}C_0C_1,
\end{aligned}
$$

where $(\cdot, \cdot)$ denotes G.C.D..

The indices α_j of the singular orbits and the numbers n_j of singular orbits with indices α_j are given as follows

	α_0	n_0	α_1	n_1	α_2	n_2
I	C_{12}	$C_{012}C_0$	C_{02}	$C_{012}C_1$	C_{01}	$C_{012}C_2$
II	C_{12}	$\frac{C_{012}C_0 - 1}{v_2}$	v_2C_{12}	1	C_{01}	C_{012}
III	C_{12}	$\frac{C_{012}C_0 - v_1 - v_2}{v_1v_2}$	v_2C_{12}	1	v_1C_{12}	1
IV	C_{01}	$\frac{C_{012} - 1}{v_1}$	v_2	1	v_1C_{01}	1
V	v_0	1	v_1	1	v_2	1

β_j are given by $\beta_j\nu_j \equiv 1$ $(mod.\, \alpha_j)$ $(0 \le \beta_j < \alpha_j)$, where ν_j are given by followings.

$$
(\nu_0, \nu_1, \nu_2) = \begin{cases}
(m_0, m_1, m_2) & \text{type I}\\
(m_0, m_0, m_2) & \text{type II}\\
(m_0, m_0, m_0) & \text{type III}\\
(m_2, m_0, m_2) & \text{type IV}\\
(m_2, m_0, m_1) & \text{type V.}
\end{cases}
$$

The invariants b, g are given by

$$b = \frac{d}{m_0 m_1 m_2} - \sum \frac{\beta_j}{\alpha_j} \tag{3.6}$$

$$\begin{aligned} 2g = {} & \frac{d^2}{m_0 m_1 m_2} - \frac{d(m_0, m_1)}{m_0 m_1} - \frac{d(m_1, m_2)}{m_1 m_2} - \frac{d(m_2, m_0)}{m_2 m_0} \\ & + \frac{(d, m_0)}{m_0} + \frac{(d, m_1)}{m_1} + \frac{(d, m_2)}{m_2} - 1 \end{aligned} \tag{3.7}$$

We introduce a lemma to determine whether K is homeomorphic to S^3 or not.

LEMMA 3.8. *A Seifert 3-manifold K is homeomorphic to S^3 if and only if the Seifert invariants of K are one of followings.*

$\{\pm 1; (O_1, 0)\}$, $\{-1; (O_1, 0); (\alpha, \alpha - 1)\}$,

$\{0; (O_1, 0); (\alpha, 1)\}$, $\{-1; (O_1, 0); (\alpha_1, \beta_1), (\alpha_2, \beta_2)\}$,

where $-\alpha_1\alpha_2 + \alpha_1\beta_2 + \alpha_2\beta_1 = \pm 1$.

3-3. Proof of Theorem 3.1

We prove Theorem 3.1. It is sufficient to show the following proposition.

PROPOSITION 3.9.

(1) If F =Closure($F^s(P_1)$), then one of the followings occurs.

(a) $F = F^s(P_1) \sqcup F^s(P_2)$

(b) $F = F^s(P_1) \sqcup F^s(P_3)$

(c) $F = F^s(P_1) \sqcup F^s(P_2) \sqcup F^s(P_3)$.

(2) The case (a) and (b) cannot happen.

(Proof) The north pole of a $\mathbf{C}^\times$-orbit with south pole P_1 is P_2 or P_3. Hence (1) is trivial.

(2) We show first that (a) cannot happen. Assume that $F = F^s(P_1) \sqcup F^s(P_2)$. The north pole of any $\mathbf{C}^\times$-orbit with south pole P_1 is only P_2. In the other hand it is easy to show that the south pole of any $\mathbf{C}^\times$-orbit with north pole P_2 is P_1. Hence we obtain

$$\text{Closure}(F^s(P_1)) = \text{Closure}(F^u(P_2)) = F \approx S^4.$$

F is a smooth symplectic submanifold of M. But S^4 is not symplectic and this is a contradiction.

Next we show that (b) cannot occur when the weights are of type III. We consider a fixed point set $M^{Z/aZ}$ of a subgroup $Z/aZ \subset S^1$. $M^{Z/aZ} \approx 2\,points \sqcup S^2$ and $M^{Z/aZ} - M^{S^1}$ consists of one $\mathbf{C}^\times$- orbit with south pole P_1 and with north pole P_2. This contradicts (b).

In the case the weights are of type II the proof is more complicated. Assume that $F = F^s(P_1) \sqcup F^s(P_2)$.

First suppose that $a \neq 1$. In this case F contains a $\mathbf{C}^\times$-orbit with isotropy a and hence P_3 has a weight a. $0 < a < b$ follows that $a = b - a$. But this contradicts to the condition $G.C.D.(a, b) = 1$.

Hence $a = 1$. Next suppose that $b \neq 2$. F contains a $\mathbf{C}^\times$-orbit with isotropy $b + 1$ but does not contain any orbits with isotropy b or $b - 1$. If we take a coordinate (z_0, z_1, z_2) around P_3 by

$$g(z_0, z_1, z_2) = (g^{b-1}z_0, g^b z_1, g^{b+1} z_2),$$

then F around P_3 can be represented by the following equations.

$$z_0^{bk} + z_1^{(b-1)k} + z_1 z_2^p = 0, \text{ or}$$
$$z_0^{bk} + z_1^{(b-1)k} + z_0 z_2^p = 0$$

for some integers k, p. Here the weighted degree $d = b(b-1)k$. Since $K = S^5_{P_3} \cap F \approx S^3$, the Seifert invariant g of K equals 0. From (3.7),

$$\frac{b(b-1)k^2}{b+1} - k - \frac{(b-1)k}{b+1} - \frac{(b,2)bk}{b+1} + \frac{(b(b-1)k, b+1)}{b+1} + 1 = 0,$$

(3.10) $b(b-1)k^2 - ((b,2)+2)bk + (b(b-1)k, b+1) + b + 1 = 0.$

From this equation, b divides $(b(b-1)k, b+1) + 1$. On the other hand $(b(b-1)k, b+1) \leq b+1$. This implies that $(b(b+1)k, b+1) = b-1$, $b-1$ divides $b+1$, and $b = 3$. We solve (3.10) and we have $k = 1$. It follows that the definition equation of F around P_3 is

$$z_0^3 + z_1^2 + z_0 z_2 = 0,$$

and Seifert invariants of K is $\{-1; (O_1, 0); (2,1), (4,3)\}$ and we have $K \approx L(2,1)$. This is a contradiction. In the similar way when $b = 2$ we can prove that $K \approx L(2,1)$

In the case of type I, the proof is much more complicated and we omit it.

3-4. Proof of Theorem 3.2.

We prove Theorem 3.2

LEMMA 3.11.

Assume that the action is of type III.

(1) Let d be the homogeneous degree of the defining function of F around P_3. Then 6 divided d and P_3 is a singular point of F.

(2) $K = S^5_{P_3} \cap F$ is a Seifert manifold obtained from S^3 by Dehn surgery on a trivial knot. Hence K is homeomorphic to a lens space.

(3) P_3 is not an isolated singular point.

(Proof) (1) We take a coordinate (z_0, z_1, z_2) around P_3 such that

$$g(z_0, z_1, z_2) = (g^2 z_0, g^3 z_1, g z_2).$$

Since F does not contain any $\mathbf{C}^\times$-orbits with isotropy 2 or 3, the defining function f of F around P_3 is given by

$$f(z_0, z_1, z_2) = z_0^{3k} + z_1^{2k} + (\text{other terms}),$$

where k is a positive integer. Hence $d = 6k$ and

$$\frac{\partial f}{\partial z_j}(0,0,0) = 0. \quad (\text{for } j = 0, 1, 2.)$$

This implies that P_3 is a singular point.

(2) Let a be $\mu(P_2)$ and δ be a small positive constant. Let

$$F_- = \mu^{-1}(a - \delta) \cap F \approx S^3, \quad F_+ = \mu^{-1}(a + \delta) \cap F \approx K.$$

Suppose that $\{s_1, \cdots, s_r\}$ are S^1-orbits in F_- such that their north poles are P_2. It is clear that

$$\begin{aligned} F_- \setminus \{s_1, \cdots, s_r\} &\approx F^s(P_1) \cap \mu^{-1}(a + \delta) \\ &\approx K \setminus \{\text{one } S^1\text{-orbit}\}. \end{aligned}$$

These are Seifert manifolds and their base spaces are given by

$$S^1 \to F_- \setminus \{s_1, \cdots, s_r\} \to S^2 \setminus \{r \text{ points}\}$$

$$S^1 \to K \setminus \{\text{one } S^1\text{-orbit}\} \to \Sigma_{g(K)} \setminus \{\text{a point}\},$$

where $g(K)$ is a Seifert invariant g of K. This follows that $g(K) = 0$ and $r = 1$.

(3) We apply (3.7) in this case and we have

$$0 = \frac{(6k)^2}{6} - \frac{6k}{6} - \frac{6k}{3} - \frac{6k}{2} + 2$$
$$3k^2 - 3k + 1 = 0.$$

This equation does not have an integral solution. Hence P_3 is not an isolated singular point. It implies that the singular point set of F is $F^s(P_2) \sqcup F^s(P_3)$. This completes the proof.

PROPOSITION 3.12. *If F is not singular at P_2 nor at P_3, then F is S^1-diffeomorphic to $\mathbf{C}P^2$ and M is diffeomorphic to $\mathbf{C}P^3$.*

(Proof) If F is not singular at P_2 nor at P_3 then F is smooth submanifold of M. Hence F is a 4-dimensional S^1-symplectic manifold with moment map $\mu_F = \mu|F$. From [AH], F is S^1-diffeomorphic to $\mathbf{C}P^2$ since $\chi(F) = 3$. The sphere bundle SF of the normal bundle NF over F is given by

$$S^1 \to S^5 \to F.$$

This follows that $M \approx D^6 \cup_{S^5} NF$ is homeomorphic to $\mathbf{C}P^3$. If two spin manifolds with torsion-free homology are homeomorphic then they are diffeomorphic (See [Wa].) It follows that M is diffeomorphic to $\mathbf{C}P^3$.

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