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An infinitesimal analysis in topology

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1° Introduction

Algebraic topology is the study of functors from the category $\mathfrak{Top}$ of topological spaces into some algebraic categories.

Take, for example, functor $H^{\cdot}(\ \ )$ of the singular cohomology in coefficient K a coomutative ring. Given a diagram in $\mathfrak{Top}$

$$\begin{tikzcd} Z & \\ X \arrow[u, "f"] \arrow[r, hook] & Y. \end{tikzcd}$$

If f extends to Y, one will get in the category $\mathfrak{Alg}_K$ of K-algebras a commutative diagram

$$\begin{tikzcd} H^{\cdot}(Z) \arrow[d, "H^{\cdot}(f)"'] \arrow[dr, dashed] \arrow[dr, phantom, "\circlearrowright"] & \\ H^{\cdot}(X) & H^{\cdot}(Y). \arrow[l] \end{tikzcd}$$

Further, if $\tilde{f}$ denotes an extension of f, then $H^{\cdot}(\tilde{f})$ must commute with all the natural transformations, called cohomology operations, from $H^{\cdot}(\ \ )$ into itself, hence if lucky, one can sometimes decide if an extension exists or not by algebraic means.

If K is a field of positive characteristic, we have a well-known set of cohomology operations constituting a skew graded Hopf algebra $\mathcal{A}$, called the Steenrod algebra. One thus wishes to study the algebra $\mathcal{A}$ and the $\mathcal{A}$-module structures of $H^{\cdot}(X)$.

This is a survey to introduce an attempt [KSTY] to throw a new light on the Steenrod algebra using infinitesimal unipotent K-groups.

For simplicity we will fix $K = \mathbf{F}_p$, p odd prime, in what follows.

2° The Steenrod algebra

(2.1) The Bockstein operator β is a natural map

$$H^n(X) \longrightarrow H^{n+1}(X) \quad \forall n \in \mathbf{N},$$

induced by the short exact sequence $0 \to \mathbf{F}_p \to \mathbf{Z}/p^2\mathbf{Z} \xrightarrow{p} \mathbf{F}_p \to 0$ such that

$$\beta^2 = 0 \tag{1}$$

and

$$\beta(xy) = (\beta x)y + (-1)^n x(\beta y) \quad \forall x \in H^n(X), y \in H^m(X), \tag{2}$$

where the multiplication is the cup product $H^{\cdot}(X) \otimes_K H^{\cdot}(X) \to H^{\cdot}(X)$ induced by the diagonal mapping $X \to X \times X$.

Further, one has unique natural maps [SE],(VI.1)

$$\mathfrak{p}^i : H^n(X) \longrightarrow H^{n+2i(p-1)}(X) \quad \forall i, n \in \mathbf{N},$$

called the Steenrod reduced powers, such that

$$\mathfrak{p}^0 = \mathrm{id}, \tag{3}$$

$$\mathfrak{p}^i = \begin{cases} x^p & \text{if } x \in H^{2i}(X) \\ 0 & \text{if } x \in H^j(X) \text{ with } j < 2i, \end{cases} \tag{4}$$

(5) (Cartan formula)

$$\mathfrak{p}^i(xy) = \sum_{j=0}^{i} \mathfrak{p}^j(x)\mathfrak{p}^{i-j}(y) \quad \forall x \in H^n(X), y \in H^m(X)$$

and

(6) (Adem relations)

$$\begin{aligned}
\forall a < pb, \quad \mathfrak{p}^a\mathfrak{p}^b &= \sum_{t=0}^{[\frac{a}{p}]} (-1)^{a+t} \binom{(p-1)(b-t)-1}{a-pt} \mathfrak{p}^{a+b-t}\mathfrak{p}^t; \\
\forall a \le pb, \quad \mathfrak{p}^a\beta\mathfrak{p}^b &= \sum_{t=0}^{[\frac{a}{p}]} (-1)^{a+t} \binom{(p-1)(b-t)}{a-pt} \beta\mathfrak{p}^{a+b-t}\mathfrak{p}^t \\
&\quad + \sum_{t=0}^{[\frac{a-1}{p}]} (-1)^{a+t-1} \binom{(p-1)(b-t)-1}{a-pt-1} \mathfrak{p}^{a+b-t}\beta\mathfrak{p}^t.
\end{aligned}$$

With those in mind we define the Steenrod algebra $\mathcal{A}$ to be

$$T_K(M)/(\mathfrak{B}^2, \text{Adem relations}),$$

where $T_K(M)$ is the tensor algebra over K of a K-linear space M with basis $\mathfrak{B}, \mathfrak{P}^i, i \in \mathbf{Z}^+$, corresponding to β, $\mathfrak{p}^i$, respectively. We assign $\mathfrak{B}$ (resp. $\mathfrak{P}^i$) degree 1 (resp. $2i(p-1)$), thus making $\mathcal{A}$ into a graded K-algebra. Put $\mathfrak{P}^0 = 1$.

(2.2) Instead of $\mathcal{A}$ itself we will consider below the graded quotient $\mathcal{S} := \mathcal{A}/(\mathfrak{B})$. If $I = (i_1, \dots, i_k) \in \mathbf{N}^k, k \geq 1$, we set in $\mathcal{A}$

$$\mathfrak{P}^I = \mathfrak{P}^{i_1} \dots \mathfrak{P}^{i_k}.$$

By abuse of notations we will denote the image of $\mathfrak{P}^I$ in $\mathcal{S}$ by the same letter. We say I is admissible iff either $I = 0$ or

$$\forall \nu \in [1, k], \quad i_\nu \geq 1 \text{ and } i_\nu \geq p i_{\nu+1} \text{ with } i_{k+1} = 0$$

in which case we call $\mathfrak{P}^I$ an admissible monomial.

(2.3) THEOREM (Milnor[Mil]). *We have*

(i) *The admissible monomials form a K-linear basis of $\mathcal{S}$. In particular, each homogeneous part $\mathcal{S}_m$ is finite dimensional.*

(ii) *$\mathcal{S}$ is generated as K-algebra by $\mathfrak{P}^{p^i}, i \in \mathbf{N}$.*

(iii) *With comultiplication*

$$\Delta_{\mathcal{S}} : \mathfrak{P}^k \longrightarrow \sum_{i=0}^{k} \mathfrak{P}^i \otimes \mathfrak{P}^{k-i}$$

and the counit $\mathfrak{P}^I \mapsto 0 \ \forall I$ admissible $\neq 0$, $\mathcal{S}$ forms a cocommutative graded bigebra.

(2.4) If $A = \coprod_{i \geq 0} A_i$ is a graded K-bigebra with $A_0 = K$, then A admits a unique antipode σ_A, making A into a graded Hopf algebra, due to R. Thom [MM],(8.7): if $a \in A_n$, one defines $\sigma_A(a)$ inductively on the degree of the elements by

$$\sigma_A(a) = -a - \sum_i \sigma_A(a_i) a_i'$$

if $\Delta_A(a) = 1 \otimes a + a \otimes 1 + \sum_i a_i \otimes a_i'$ with a_i and a_i' homogeneous of less degrees than a. In particular, $\mathcal{S}$ carries a structure of cocommutative graded Hopf algebra.

(2.5) Let $\mathcal{S}^{*gr} = \coprod_{i \geq 0} \mathcal{S}_i^*$ be the graded dual of $\mathcal{S}$. Using the identification $(\mathcal{S} \otimes_K \mathcal{S})^{*gr} \simeq \mathcal{S}^{*gr} \otimes_K \mathcal{S}^{*gr}$ via

$$(f \otimes g)(a \otimes b) = f(a) g(b),$$

$\mathcal{S}^{*gr}$ comes equipped with a structure of commutative graded Hopf algebra.

Let $I_k = (p^{k-1}, p^{k-2}, \dots, p^1, p^0)$, $k \geq 1$, and $\xi_k \in \mathcal{S}^{*gr}$ the dual of $\mathfrak{P}^{I_k}$ with respect to the basis of admissible monomials, so $\deg(\xi_k) = 2(p^k - 1)$.

(2.6) THEOREM (Milnor[Mil]). *$\mathcal{S}^{*gr}$ is the polynomial algebra $K[\xi_1, \xi_2, \ldots]$ in indeterminates ξ_i, $i \geq 1$, with the comultiplicatication*

$$\xi_k \longmapsto \sum_{i=0}^{k} \xi_{k-i}^{p^i} \otimes \xi_i$$

and the counit annihilating all ξ_i.

(2.7) Let $\mathcal{I}_k$, $k \geq 0$, be the ideal of $\mathcal{S}^{*gr}$ generated by

$$\xi_1^{p^k}, \xi_2^{p^{k-1}}, \ldots, \xi_k^p, \xi_{k+j}, \quad j \geq 1.$$

Then $\mathcal{I}_k$ is a Hopf ideal, hence $\mathcal{S}^{*gr}/\mathcal{I}_k \simeq K[\xi_1, \ldots, \xi_k]/(\xi_1^{p^k}, \ldots, \xi_k^p)$ is a finite dimensional graded Hopf algebra. In turn, its dual $\mathcal{S}(k)$ is a Hopf subalgebra of $\mathcal{S}$.

One can show that $\mathcal{S}(k)$ is generated as K-algebra by

$$\mathfrak{P}^{p^i}, \qquad 0 \leq i \leq k-1,$$

hence $\mathcal{S} = \cup_{k \geq 1} \mathcal{S}(k)$ by (2.3). In [KSTY] $\mathcal{S}(k)$ is denoted by $P(k-1)$.

3° INFINITESIMAL UNIPOTENT GROUPS

(3.1) An (affine) K-group (scheme) $\mathfrak{G}$ is a representable functor from the category $K\mathfrak{Alg}$ of commutative K-algebras into the category $\mathfrak{Grp}$ of groups: there is commutative Hopf K-algebra $K[\mathfrak{G}]$ such that

$$\mathfrak{G}(\ ) = K\mathfrak{Alg}(K[\mathfrak{G}], \_). \tag{1}$$

If $m_{\mathfrak{G}}$ (resp. $\Delta_{\mathfrak{G}}$, $\epsilon_{\mathfrak{G}}$, $\sigma_{\mathfrak{G}}$) is the multiplication (resp. comultiplication, counit, antipode) of $K[\mathfrak{G}]$, then for each $R \in K\mathfrak{Alg}$, $\mathfrak{G}(R)$ is a group under the multiplication

$$\begin{tikzcd}
\mathfrak{G}(R) \times \mathfrak{G}(R) \arrow[r, dashed] \arrow[d, "\wr"'] \arrow[dr, phantom, "\circlearrowright"] & \mathfrak{G}(R) \arrow[d, equal] \\
K\mathfrak{Alg}(K[\mathfrak{G}] \otimes_K K[\mathfrak{G}], R) \arrow[r, "K\mathfrak{Alg}(\Delta_{\mathfrak{G}}, R)"'] & K\mathfrak{Alg}(K[\mathfrak{G}], R)
\end{tikzcd} \tag{2}$$

and inversion

$$\begin{tikzcd}
\mathfrak{G}(R) \arrow[r, dashed] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & \mathfrak{G}(R) \arrow[d, equal] \\
K\mathfrak{Alg}(K[\mathfrak{G}], R) \arrow[r, "K\mathfrak{Alg}(\sigma_{\mathfrak{G}}, R)"'] & K\mathfrak{Alg}(K[\mathfrak{G}], R)
\end{tikzcd} \tag{3}$$

with the identity element defined by

$$
\begin{tikzcd}
e_K(R) \arrow[r, dashed] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & \mathfrak{G}(R) \arrow[d, equal] \\
K\mathfrak{Alg}(K,R) \arrow[r, "K\mathfrak{Alg}(\epsilon_{\mathfrak{G}},R)"'] & K\mathfrak{Alg}(K[\mathfrak{G}],R).
\end{tikzcd} \tag{4}
$$

A $\mathfrak{G}$-module is a K-linear space M together with a map $\Delta_M : M \to M \otimes_K K[\mathfrak{G}]$, called a $K[\mathfrak{G}]$-comodule map, such that for each $R \in K\mathfrak{Alg}$, the map $\mathfrak{G}(R) \times (M \otimes_K R) \longrightarrow M \otimes_K R$ via

$$(x, m \otimes r) \longmapsto ((M \otimes_K x) \circ \Delta_M(m))r = \sum_i m_i \otimes rx(a_i) \tag{5}$$

if $\Delta_M(m) = \sum_i m_i \otimes a_i$, makes $M \otimes_K R$ into a $\mathfrak{G}(R)$-module over R.

(3.2) We say a K-group $\mathfrak{G}$ is algebraic iff the algebra $K[\mathfrak{G}]$ is of finite type over K. In this note we will consider only algebraic K-groups.

Let $\mathfrak{I}_{\mathfrak{G}} = \ker(\epsilon_{\mathfrak{G}})$, called the augmentation ideal of the Hopf algebra $K[\mathfrak{G}]$, and set

$$\mathrm{Dist}_m(\mathfrak{G}) = \{\mu \in K[\mathfrak{G}]^* \mid \mu({\mathfrak{I}_{\mathfrak{G}}}^{m+1}) = 0\}, \quad m \in \mathbf{N}. \tag{1}$$

Then $\mathrm{Dist}(\mathfrak{G}) := \cup_{m \in \mathbf{N}} \mathrm{Dist}_m(\mathfrak{G})$ carries a structure of cocommutative Hopf algebra, called the algebra of distributions of $\mathfrak{G}$, with the multiplication given by

$$(\mu\nu)(a) = (\mu \bar{\otimes} \nu) \circ \Delta_{\mathfrak{G}}(a) = \sum_i \mu(a_i)\nu(a_i') \quad \text{if} \quad \Delta_{\mathfrak{G}}(a) = \sum_i a_i \otimes a_i',$$

comultiplication $\Delta'_{\mathfrak{G}}$ such that $\Delta'_{\mathfrak{G}}(\mu)(a \otimes b) = \mu(ab)$, counit $\epsilon'_{\mathfrak{G}}$ such that $\epsilon'_{\mathfrak{G}}(\mu) = \mu(1)$, and the antipode $\sigma'_{\mathfrak{G}}$ such that $\sigma'_{\mathfrak{G}}(\mu) = \mu \circ \sigma_{\mathfrak{G}}$, using natural isomorphisms [J],(I.7.4)(2)

$$
\begin{tikzcd}
\mathrm{Dist}(\mathfrak{G}) \otimes_K \mathrm{Dist}(\mathfrak{G}) \arrow[r, "\sim"] & \mathrm{Dist}(\mathfrak{G} \times \mathfrak{G}) \\
\sum_{i=0}^{m} \mathrm{Dist}_i(\mathfrak{G}) \otimes_K \mathrm{Dist}_{m-i}(\mathfrak{G}) \arrow[r, "\sim"] \arrow[u, hook] & \mathrm{Dist}_m(\mathfrak{G} \times \mathfrak{G}). \arrow[u, hook]
\end{tikzcd} \tag{2}
$$

In particular,

$$\mathrm{Dist}_1^+(\mathfrak{G}) := \{\mu \in \mathrm{Dist}_1(\mathfrak{G}) \mid \mu(1) = 0\} \tag{3}$$

forms a Lie algebra over K, called the Lie algebra of $\mathfrak{G}$, with $[\mu, \nu] := \mu\nu - \nu\mu$.

Any $\mathfrak{G}$-module M carries a structure of $\mathrm{Dist}(\mathfrak{G})$-module such that

$$\mu m = (M \bar{\otimes}_K \mu) \circ \Delta_M(m) = \sum_i \mu(a_i) m_i \tag{4}$$

$$\forall \mu \in \mathrm{Dist}(\mathfrak{G}) \text{ and } m \in M \text{ if } \Delta_M(m) = \sum_i m_i \otimes a_i.$$

(3.3) We say a K-group $\mathfrak{G}$ is infinitesimal iff $K[\mathfrak{G}]$ is finite dimensional over K with the nilpotent augmentation ideal $\mathfrak{I}_{\mathfrak{G}}$.

If $\mathfrak{G}$ is infinitesimal, then $\mathfrak{G}(R)$ is a singleton for any integral domain R. Also $\mathrm{Dist}(\mathfrak{G}) = K[\mathfrak{G}]^*$.

Let $\mathfrak{G}$ be an arbitrary K-group again. The map

$$(1) \qquad \phi : K[\mathfrak{G}] \longrightarrow K[\mathfrak{G}] \quad \text{via} \quad a \longmapsto a^p$$

is a homomorphism of Hopf algebras, inducing a morphism of K-groups $F_{\mathfrak{G}} := K\mathfrak{Alg}(\phi, \_) : \mathfrak{G} \longrightarrow \mathfrak{G}$, called the Frobenius morphism of $\mathfrak{G}$. Then $\mathfrak{G}^1 := \ker(F_{\mathfrak{G}}) = \mathfrak{G} \times_{\mathfrak{G}} \mathfrak{e}_K$ is a normal subgroup of $\mathfrak{G}$, with

$$(2) \qquad K[\mathfrak{G}^1] \simeq K[\mathfrak{G}] \otimes_{K[\mathfrak{G}]} (K[\mathfrak{G}]/\mathfrak{I}_{\mathfrak{G}}) \simeq K[\mathfrak{G}]/(a^p \mid a \in \mathfrak{I}_{\mathfrak{G}}),$$

hence $\mathfrak{G}^1$ is infinitesimal. More generally, $\mathfrak{G}^r := \ker(F^r_{\mathfrak{G}})$, $r \in \mathbf{Z}^+$, is an infinitesimal normal subgroup of G with $K[\mathfrak{G}^r] \simeq K[\mathfrak{G}]/(a^{p^r} \mid a \in \mathfrak{I}_{\mathfrak{G}})$, called the r-th Frobenius kernel of $\mathfrak{G}$.

(3.4) We now focus on the unipotent K-group $\mathfrak{U}_n$ such that $K[\mathfrak{U}_n] = K[x_{ij}]_{1 \leq j < i \leq n}$ polynimial algebra in indeterminates $x_{ij}, 1 \leq j < i \leq n$, with the comultiplication

$$(1) \qquad x_{ij} \longmapsto \sum_{k=j}^{i} x_{ik} \otimes x_{kj}$$

and the counit $x_{ij} \mapsto 0 \ \forall i,j$, where we agree that $x_{ii} = 1 \ \forall i$. If $R \in K\mathfrak{Alg}$, $\mathfrak{U}_n(R)$ is isomorphic to the group of $n \times n$ lower triangular unipotent matrices with the entries in R. Also if each x_{ij} is assigned degree 1, then

$$(2) \qquad \mathrm{Dist}(\mathfrak{U}_n) \simeq K[\mathfrak{U}_n]^{*gr} \qquad \text{as } K\text{-linear spaces.}$$

Let $\mathfrak{U}_{ij}$ be a subgroup of $\mathfrak{U}_n$ with $K[\mathfrak{U}_{ij}] = K[\mathfrak{U}_n]/(x_{st})_{(s,t)\neq(i,j)} \simeq K[x_{ij}]$, hence $\mathfrak{U}_{ij}(R)$ consists of the (i,j)-th elementary unipotent matrices with the entries in R. With $\deg(x_{ij}) = 1$, $K[\mathfrak{U}_{ij}]$ is a graded commutative cocommutative Hopf algebra, and

$$(4) \qquad \mathrm{Dist}(\mathfrak{U}_{ij}) \simeq K[\mathfrak{U}_{ij}]^{*gr} \simeq S_K(x_{ij})^{*gr}$$

the graded dual of the symmetric algebra in x_{ij} [B],(III.11), with the dual monomial basis $X_{ijk} : X_{ijk}(x_{ij}^{\ell}) = \delta_{k\ell} \quad \forall k, \ell \in \mathrm{N}$. Hence

$$(5) \qquad X_{ijk}X_{ij\ell} = \binom{k+\ell}{k} X_{ijk+\ell},$$

and

$$(6) \qquad \Delta'_{\mathfrak{U}_{ij}}(X_{ijk}) = \sum_{\ell=0}^{k} X_{ij\ell} \otimes X_{ij\ell-k}.$$

Under the multiplication one has a natural bijection

$$\prod_{1\le j<i\le n} \mathfrak{U}_{ij}(R) \xrightarrow{\sim} \mathfrak{U}_n(R) \quad \forall R \in K\mathfrak{Alg}, \tag{7}$$

where the product is taken in any order, hence also a K-linear isomorphism

$$\bigotimes_{i,j} \mathrm{Dist}(\mathfrak{U}_{ij}) \xrightarrow{\sim} \mathrm{Dist}(\mathfrak{U}_n), \tag{8}$$

which is, however, not an isomorphism of Hopf algebras if $n \geq 3$.

We now arrange the $\mathfrak{U}_{ij}$ in (7) and (8) in the increasing order such that

$$(i,j) \succ (s,t) \quad \text{iff} \quad i > s \text{ or } i = s \text{ with } j < t, \tag{9}$$

and fix the arrangement in taking the product once and for all. Then under (8)

$$\Big(\prod_{i,j} X_{ijk}\Big)_{k \in \mathbf{N}^{\frac{n(n-1)}{2}}} \quad \text{with the product taken in the order of (9)}$$

forms a K-linear basis of $\mathrm{Dist}(\mathfrak{U}_n)$ dual to the monomial basis $\prod_{i,j} x_{ij}^{k}$ of $K[\mathfrak{U}_n]$ in the sense of (2).

4° The Steenrod algebra revisited

(4.1) In 1988 Tezuka M. found a homomorphism of bigebras

$$\psi : K[\mathfrak{U}_n] = K[x_{ij}]_{1\le j<i\le n} \longrightarrow K[\xi_1, \xi_2, \dots] = \mathcal{S}^{*gr} \quad \text{via} \quad x_{ij} \longmapsto \xi_{i-j}^{p^{j-1}}. \tag{1}$$

Further, by assigning x_{ij} degree $2(p^{i-j}-1)p^{j-1}$ we can make $K[\mathfrak{U}_n]$ into a graded bigebra. Then by the unicity of the antipode on graded bigebras (2.4), ψ is actually a homomorphism of graded Hopf algebras.

Now $\mathrm{im}\psi = K[\xi_1, \dots, \xi_n]$ is a Hopf subalgebra of $\mathcal{S}^{*gr}$, so let us write ψ again for the homomorphism of Hopf algebras $K[\mathfrak{U}_n] \to K[\xi_1, \dots, \xi_n]$ induced by ψ. If

$$\pi_1 : K[\mathfrak{U}_n] \longrightarrow K[x_{ij}]_{i,j}/(x_{ij}^{p^n})_{i,j} = K[\mathfrak{U}_n^n]$$

and

$$\pi_2 : K[\xi_1, \dots, \xi_{n-1}] \longrightarrow K[\xi_1, \dots, \xi_{n-1}]/(\xi_1^{p^{n-1}}, \xi_2^{p^{n-2}}, \dots, \xi_{n-1}^{p}) = \mathcal{S}(n-1)^*$$

are the natural maps, we get a commutative diagram of surjective homomorphisms of Hopf algebras

$$\begin{tikzcd} k[\mathfrak{U}_n] \arrow[r, "\psi"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\circlearrowright"] & K[\xi_1, \dots, \xi_n] \arrow[d, "\pi_2"] \\ K[\mathfrak{U}_n^n] \arrow[r, "\psi_n"'] & \mathcal{S}(n-1)^*. \end{tikzcd} \tag{2}$$

Dualizing ψ_n,

(4.2) THEOREM [KSTY], (3.3). *We have an imbedding of Hopf algebras $\mathcal{S}(n-1)$ into* $\mathrm{Dist}(\mathfrak{U}_n^n)$.

(4.3) Hence any $\mathfrak{GL}_n$-module carries a structure of $\mathcal{S}(n-1)$-module upon restriction, enabling one to exploit the representation theory of $\mathfrak{GL}_n$ in the study of $\mathcal{S}(n-1)$-modules, where $\mathfrak{GL}_n$ is the K-group such that $\mathfrak{GL}_n(R)$ is the group of $n \times n$ invertible matrices with the entries in R, $R \in K\mathfrak{Alg}$.

(4.4) To illustrate an application, let us first recall some representation theory of $\mathfrak{GL}_n$. Let E be the natural n-dimensional $\mathfrak{GL}_n$-module of basis $e_1, \dots, e_n$. If $\lambda = (\lambda_1, \dots, \lambda_{n-1})$ is a partition of $r = \sum_{i=1}^{n-1} \lambda_i$, let $(\lambda'_1, \dots, \lambda'_m)$ be the transposed partition of λ, and put

$$(1) \qquad \Phi_\lambda = \Big(\sum_{\sigma \in \mathfrak{S}_{\lambda'_1}} \mathrm{sgn}(\sigma) e_{\sigma(1)} \otimes \dots \otimes e_{\sigma(\lambda'_1)}\Big) \otimes \dots \otimes \Big(\sum_{\sigma \in \mathfrak{S}_{\lambda'_m}} \mathrm{sgn}(\sigma) e_{\sigma(1)} \otimes \dots \otimes e_{\sigma(\lambda'_m)}\Big)$$

in $E^{\otimes r}$. After R. Carter and G. Lusztig [CL] we call $\mathrm{Dist}(\mathfrak{GL}_n)\Phi_\lambda$ in $E^{\otimes r}$ the Weyl module of highest weight λ, and denote it by $V(\lambda)$.

In case λ is column p-regular, i.e.,

$$(2) \qquad 0 \leq \lambda_i - \lambda_{i+1} \leq p-1 \quad \forall i \in [1, n-1] \quad \text{with} \quad \lambda_n = 0,$$

one can show [KSTY],(3.7)

$$(3) \qquad V(\lambda) = \mathcal{S}(n-1)\Phi_\lambda.$$

(4.5) Let Y be the complex $(p^n - 1)$-projective space. Then

$$(1) \qquad H^{\cdot}(Y) \simeq K[x]/(x^{p^n}) \qquad \text{as graded } K\text{-algebras},$$

where x is an indeterminate of degree 2. Hence $H^{\cdot}(Y)$ admits a structure of $\mathcal{S}$-module. Explicitly,

$$(2) \qquad \mathfrak{P}^i(x^j) \equiv \binom{j}{i} x^{j+i(p-1)} \mod x^{p^n}.$$

If V is a K-linear span of $x, x^p, \dots, x^{p^{n-1}}$ in $H^{\cdot}(Y)$, V is stable under the action of $\mathcal{S}(n-1)$. Further, there is an isomorphism

$$(3) \qquad \theta : E \xrightarrow{\sim} V \quad \text{via} \quad e_i \longmapsto x^{p^{i-1}}, \quad i \in [1, n]$$

of $\mathcal{S}(n-1)$-modules [KSTY],(4.3), which induces by the Künneth formula or by the Cartan formula an imbedding of $\mathcal{S}(n-1)$-modules

$$(4) \qquad \theta^{\otimes r} : E^{\otimes r} \hookrightarrow H^{\cdot}(Y^r) \quad \text{via} \quad e_{i_1} \otimes \dots e_{i_r} \longmapsto x^{p^{i_1 - 1}} \otimes \dots x^{p^{i_r - 1}} \quad \forall r \in \mathbf{Z}^+.$$

In particular, the Weyl module $V(\lambda)$ with $r = \sum_{i=1}^{n-1} \lambda_i$ imbeds in $H^{\cdot}(Y^r)$ as an $\mathcal{S}(n-1)$-submodule.

(4.6) Fix a column p-regular partition $\lambda = (\lambda_1, \ldots, \lambda_{n-1})$ of $r = \sum_{i=1}^{n-1} \lambda_i$ with its transpose $(\lambda'_1, \ldots, \lambda'_m)$.

Composed several times with the cup product, $\theta^{\otimes r}$ of (4.4)(4) yields an $\mathcal{S}(n-1)$-homomorphism

(1) $$\theta' : E^{\otimes r} \longrightarrow H^{\cdot}(Y^{\lambda'_1}) \quad \text{such that} \quad e_{i_1} \otimes \ldots \otimes e_{i_r} \longmapsto \bigotimes_{j=1}^{\lambda'_1} x_j^{e(j)},$$

where $e(j) = p^{i_j - 1} + p^{i_{j+\lambda'_1} - 1} + \ldots + p^{i_{j+\lambda'_1+\ldots+\lambda'_{k(j)}} - 1}$ with $k(j) = \max\{i \mid 1 \le i \le m, \lambda'_i \ge j\} - 1$. Set $\theta_\lambda = \theta'|_{V(\lambda)}$. One finds

(2) $$\theta_\lambda(\Phi_\lambda) \neq 0.$$

(4.7) Theorem (Smith, Mitchell[Mit], [KSTY],(4.10)). *If* $\lambda = ((n-1)(p-1), (n-2)(p-1), \ldots, p-1)$, *then* θ_λ *imbeds* $V(\lambda)$, *called the Steinberg module that is free over* $S(n-1)$, *into* $H^{\cdot}(Y^{n-1})$ *as* $\mathcal{S}(n-1)$*modules.*

(4.8) Further, we have a curious

Proposition [KSTY],(4.10). *If* $\lambda_1 \le p-1$ *and if* $V(\lambda)$ *is* $\mathfrak{G}\mathfrak{L}_n$*-simple, then* θ_λ *imbeds* $V(\lambda)$ *into* $H^{\cdot}(Y^{\lambda'_1})$ *as a* $\mathcal{S}(n-1)$*-submodule.*

(4.9) To further our speculation, Mabuchi [Ma] has verified that in case $n \ge 3$ and $\lambda = (p, 1, \ldots, 1)$ with 1 appearing $(n-2)$-times,

(1) $$\theta_\lambda \text{ is injective iff } V(\lambda) \text{ is } \mathfrak{G}\mathfrak{L}_n - \text{simple}.$$

Computer work by his fellow student Takeno S. has also checked (1) for all column p-regular λ in case $n = 3$ and $p \le 7$.

References

[B] Bourbaki, N., Algèbre Chap.III, Hermann, Paris 1971

[CL] Carter, R.W. and Lusztig, G., *On the modular representations of the general linear and symmetric groups*, Math. Z. 136 (1974), 193-242

[J] Jantzen, J.C., The representations of algebraic groups, Academic Press, Orlando 1987

[KSTY] Kaneda M., Shimada N., Tezuka M. and Yagita N., *Representations of the Steenrod algebra*, to appear in J. Alg.

[Ma] Mabuchi M., *On some imbedding of Weyl modules*, Master's thesis (in Japanese), Niigata Univ. (1990)

[Mil] Milnor, J.W., *The Steenrod algebra and its dual*, Ann. Math. 67 (1958), 150-171

[MM] Milnor, J.W. and Moore, J.C., *On the structure of Hopf algebras*, Ann. Math. 81 (1965), 211-264

[Mit] Mitchell, S.A., *On the Steinberg module, representations of the symmetric groups, and the Steenrod algebra*, J. Pure Appl. Alg. 39 (1986), 275-281

[SE] Steenrod, N.E. and Epstein, D.B.A., Cohomology operations, Ann. Math. Stud. 50 Princeton Univ. Press 1962