数理解析研究所講究録
第 796 巻 1992 年 8-21

Some Problems in Formal Language Theory Known as Decidable are Proved EXPTIME Complete

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Abstract

Some problems in formal language theory are considered and shown deterministic exponential time complete. They include the problems for a given context-free language L, a regular set R, a deterministic context-free language L_D, to determine whether $L \subset R$, and to determine whether $L_D \subset R$.

1 INTRODUCTION

A number of complete problems for deterministic exponential time have been presented. Since Chandra and Stockmeyer [1] established the notion of alternation in 1976, many authors have shown complete problems for deterministic exponential time by using of alternation. Most of these problems were related to combionatorial games. [2, 5, 6, 7, 8]

We consider in this paper several problems in the formal language theory and show that the problems are deterministic exponential time complete. They were already known as decidable. Let L be a context-free language, R a regular set, L_D a deterministic context-free language. The problems we consider include the ones to determine whether $L \subset R$, and whether $L_D \subset R$.

In order to prove that the concerned problems are deterministic exponential time-hard, we use the pebble game problem [5], which was already shown complete for deterministic exponential time, and we establish the polynomial-time reduction from the pebble game problem.

We write λ to denote the empty string, and $|x|$ to denote the length of a string x. Let Σ_k denote the set $\{[_1,]_1, [_2,]_2, \cdots, [_k,]_k\}$. See [4] for definitions of *deterministic finite automata* (dfa) $M = (Q, \Sigma, \delta, q_0, F)$ except that the transition function δ is given by a partial function from $Q \times \Sigma$ to Q. See also [4] for definitions of *nondeterministic finite automata* (nfa), *regular set*, *context-free grammar* (cfg), *context-free language* (cfl), *deterministic context-free language* (dcfl), *deterministic pushdown automaton* (dpda), *Turing machine*, *polynomial time*, and *polynomial-time reducibility*.

The *Dyck language* D_k of k balanced parenthesisis the one generated by the cfg $G = (\{S\}, \Sigma_k, P, S)$, where P is the set of productions of the forms

$$S \rightarrow SS \mid \lambda \mid [_i S]_i \ (1 \leq i \leq k).$$

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For a cfg G, let $L(G)$ denote the language generated by G, and for an automaton or a machine M, $L(M)$ denote the language accepted by M. Whenever we say "given a cfl $L, \cdots$", we assume that a cfg G, $L(G) = L$, is given, and in particular when we say "given a dcfl $L, \cdots$", a dpda M, $L(M) = L$ is assumed. When we say "given a regular set $R, \cdots$", it always means that an nfa M, $L(M) = R$ is given.

EXPTIME is the class of sets accepted by 2^{n^k} time bounded deterministic Turing machines for some k. A language L is called EXPTIME *complete* if L is in EXPTIME, and L' is polynomial-time reducible to L for any L' in EXPTIME.

A *pebble game* [5] is a quadruple $\mathcal{G} = (X, R, S, t)$ where:

(1) X is a finite set of *nodes*,
(2) $R \subset \{(x_a, x_b, x_c) \mid x_a, x_b, x_c \in X, x_a \neq x_b, x_b \neq x_c, x_c \neq x_a\}$ is called a set of *rules*,
(3) S is a subset of X, and
(4) $t \in X$ is called the *terminal* node.

At the beginning of a pebble game, pebbles are placed on all nodes of S, and we call the placement the *initial pebble-placement.* A *move* of the game is as follows: if pebbles are placed on x_a, x_b, but not on x_c, and $(x_a, x_b, x_c) \in R$, then a player can move a pebble from x_a to x_c in his turn. The game is played by two players, and each player alternately applies one of the rules of $\mathcal{G}$ to move a pebble. The winner is the player who can first put a pebble on the terminal node, or who can make the other player unable to move.

The first player has a *forced win* (or *winning strategy*) from a pebble-placement in $\mathcal{G}$ if there is a *winning game-tree* for the first player, whose root is labeled with the pebble-placement. The winning game-tree of $\mathcal{G}$ for the first player (*game-tree* for short) is the tree, nodes of which are labeled with pebble-placements, or WIN, where WIN means that the second player is already unable to move, thus the first player wins the game. We sometimes confuse a node of the game-tree with its label. A *level* of a node in the tree is the length of the path from the root to the node. The level of the root is zero. A *depth* of the game-tree is the maximum level among the nodes of the tree. Any node u of the even level in the tree is labeled with a pebble-placement for the first player's turn to move, and has exactly one child v, where v is obtained by an application of a rule of the game to u. Any non-leaf of the odd level is labeled with a pebble-placement for the second player's turn, and has exactly m children, where m is the number of the rules of the game. For $1 \leq j \leq m$, the j-th child of v is labeled with a pebble-placement obtained by an application of the j-th rule r_j of the game to v if r_j is applicable; and with WIN if r_j is not applicable to v. Every leaf of the game-tree is labeled either with WIN or with a pebble-placement in which the first player wins.

The *pebble game problem* is, given a pebble game $\mathcal{G}$, to determine whether there is a winning strategy for the first player from the initial pebble-placement in $\mathcal{G}$.

Theorem 1.1 [5] *The pebble game problem is EXPTIME complete.*

Example 1.1 Consider the following pebble game $\mathcal{G} = (X, R, S, x_5)$, where $X = \{x_1, x_2, x_3, x_4, x_5\}, S = \{x_1, x_2, x_3\}, R = \{r_1, r_2, r_3, r_4\}$, and $r_1 = (x_1, x_2, x_4), r_2 = (x_2, x_1, x_4), r_3 = (x_3, x_4, x_2), r_4 = (x_2, x_4, x_5)$.

---

begin

1) Let $G, L = L(G)$ be a cfg, and let $M, R = L(M)$ be an nfa.
2) Construct a dfa M' such that $L(M') = \Sigma^* - L(M)$.
3) Construct the cfg G' as in Lemma 2.1 such that $L(G') = L(G) \cap L(M')$.
4) Use polynomial time algorithm to determine whether $L(G') = \phi$.
5) If $L(G') = \phi$ then $L \subset R$ else $L \not\subset R$.

end

Fig. 2.1 Algorithm to determine whether $L \subset R$

If the first player applies r_1 to move a pebble from x_1 to x_4, the second player then applies r_4 to move a pebble from x_2 to x_5 and the second player wins the game. Suppose that the first player first applies r_2 to move a pebble from x_2 to x_4. Then the only rule for the second player to apply is r_3 to move a pebble from x_3 to x_2. Then the first player applies r_4 to move a pebble from x_2 to x_5 and wins the game. Thus the first player has a forced win in $\mathcal{G}$.

2 Complete problem

Lemma 2.1 *For a cfg G and an nfa M, we can construct a cfg G' such that $L(G') = L(G) \cap L(M)$ within polynomial time.*

Proof. Let $G = (V, \Sigma, P, S)$ and $M = (Q, \Sigma', \delta, \{q_0\}, F)$. Without loss of generality, we assume that $\Sigma = \Sigma'$, and that G is in Chomsky normal form. Let $G' = (V', \Sigma, P', S')$, $V' = \{[q, X, p] \mid q, p \in Q, X \in V\} \cup \{S'\}$. P' contains

$$\begin{cases} [q,X,p] \to a \quad \text{if } X \to a \in P \text{ and } p \in \delta(q,a), \qquad \text{for } q,p \in Q, a \in \Sigma, \\ [q,X,p] \to [q,A,q'][q',B,p] \\ \qquad\qquad \text{if } X \to AB \in P \qquad \text{for } q,q',p \in Q, \end{cases}$$

and $S' \to [q_0, S, q_f]$ for $q_f \in F$.

By induction, we can prove that for $q, p \in Q, X \in V, w \in \Sigma^*$

$$[q,X,p] \overset{*}{\underset{G'}{\Rightarrow}} w \text{ if and only if } X \overset{*}{\underset{G}{\Rightarrow}} w \text{ and } p \in \delta(q,w).$$

Thus

$$S' \underset{G'}{\Rightarrow} [q_0, S, q_f] \overset{*}{\underset{G'}{\Rightarrow}} w \text{ if and only if } S \overset{*}{\underset{G}{\Rightarrow}} w \text{ and } q_f \in \delta(q_0, w).$$

The number of productions in G' is polynomial to the length of G and M. Thus the construction of G' can be performed within polynomial time. □

Next we present an algorithm in Fig.2.1 to determine whether $L \subset R$ for a given cfl L and a regular set R.

Lemma 2.2 *Given a cfl L and a regular set R, the algorithm shown in Fig.2.1 determines whether $L \subset R$ within exponential time.*

Proof. In line (2), apply an usual algorithm, for example p.22 of [4], to obtain a dfa M_1 such that $L(M) = L(M_1)$, and exchange the accepting states and the non-accepting states of M_1 to obtain M', which accepts the complement of R. Note that the time for the construction of M' needs exponential time, since the number of states of M_1 is exponential compared with that of M.

In line (4), apply the CYK algorithm [9] for example.

In total, our algorithm runs in exponential time to determine whether $L \subset R$.

□

Consider the following problem P_1:

Given: a cfl L, and a regular set R.
To determine whether: $L \subset R$.

Theorem 2.1 *P_1 is EXPTIME complete.*

Proof. Since EXPTIME is closed under complementation, it is sufficient to show that the problem $P_1{}'$:

Given: a cfl L, and a regular set R.
To determine whether: $L \not\subset R$.

is EXPTIME complete. By Lemma 2.2, $P_1{}'$ is solvable within exponential time.

To show that $P_1{}'$ is EXPTIME hard, we establish that the pebble game problem is polynomial-time reducible to $P_1{}'$. Let $\mathcal{G} = (X, \tilde{R}, S, x_n)$ be a pebble game. We construct a cfg G and an nfa M within polynomial time such that there is a forced win for the first player in $\mathcal{G}$ if and only if $L(G) \not\subset L(M)$.

Prior to the construction of M, we construct dfa's $M_1, M_2, \cdots, M_n$, where n is the number of the nodes of $\mathcal{G}$, such that there is a winning strategy for the first player in $\mathcal{G}$ if and only if $L(G) \cap \bigcap_{i=1}^{n} L(M_i) \neq \phi$. Then we construct an nfa M, which accepts the complement of $\bigcap_{i=1}^{n} L(M_i)$. Thus $L(G) \cap \bigcap_{i=1}^{n} L(M_i) \neq \phi$ is equivalent to $L(G) \not\subset L(M)$.

We will explain briefly how the simulation of $\mathcal{G}$ works in G and M_i's. The derivation of G guesses a game-tree of $\mathcal{G}$, that is, what rules of $\mathcal{G}$ the first player applies in order to win the game. For the first player's turn to move in the game-tree, a derivation of G guesses a rule which the first player applies to the pebble-placement, while for the second player's turn, derivations in G guess for each rule whether the rule is applicable to the coressponding pebble-placement. The purpose of M_i's is to examine whether the above guesses by G are correct, and whether the derivation is the one for the first player to win the game.

Assume that $X = \{x_1, x_2, \cdots, x_n\}$, and that $\tilde{R} = \{r_1, r_2, \cdots, r_m\}$. We write $\Sigma_{4m} = \{r_j, \overline{r_j}, a_j, \overline{a_j}, b_j, \overline{b_j}, c_j, \overline{c_j} \mid 1 \leq j \leq m\}$, where a symbol without bar and the symbol with bar are intended to form a pair of balanced parenthesis in Σ_{4m}. Let $G = (\{U, W, V_1, V_2, \cdots, V_m\}, \Sigma_{4m}, P, U)$, where P contains

(1) $W \to V_1 V_2 \cdots V_m$,

and for each rule $r_j = (x_{j1}, x_{j2}, x_{j3}), 1 \leq j \leq m$ of $\tilde{R}$,

$$(2) \begin{cases} U \to r_j W \overline{r_j} & (j3 \neq n) \\ U \to r_j \overline{r_j} & (j3 = n), \end{cases}$$

(3) $V_j \to a_j \overline{a_j} \mid b_j \overline{b_j} \mid c_j \overline{c_j}$, and

(4) $V_j \to r_j U \overline{r_j} \quad (j3 \neq n)$.

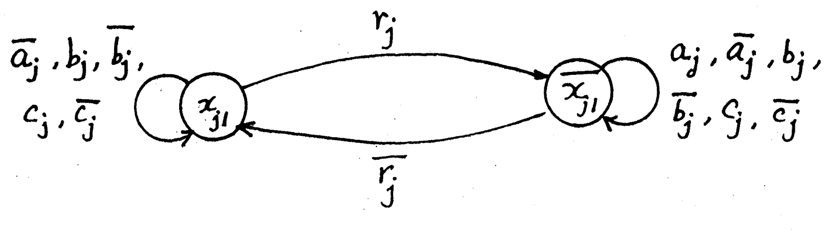


Fig. 2.2 transition $\delta_{j1}(p_{j1}, \sigma_j)$

Fig. 2.3 transition $\delta_{j2}(p_{j2}, \sigma_j)$

The nonterminal U is associated with a pebble-placement for the first player's turn to move, while W is for the second player. $V_j, 1 \le j \le m$, means in the simulation to guess an application of a rule $r_j \in \tilde{R}$ to the pebble-placement associated with W. The production rules in (2) are for the simulation of the first player in $\mathcal{G}$ to select r_j to move a pebble from x_{j1} to x_{j3}. The production $U \to r_j W \overline{r_j}$ is the one to denote that the first player applies r_j and the next turn is the second player, while $U \to r_j \overline{r_j}$ denotes for the first player to apply r_j and wins to put a pebble on x_n. The productions (1),(3),(4) are for the second player's move. (1) is to try every rule $r_1, r_2, \cdots, r_m$ as the second player's move. (3) is to indicate that r_j is not a proper rule to make: if a pebble is not on x_{j1} (is not on x_{j2}, is on x_{j3}), then $V_j \to a_j \overline{a_j}$ ($V_j \to b_j \overline{b_j}$, $V_j \to c_j \overline{c_j}$, respectively) can be applied. (4) is to select r_j to move. $V_j \to r_j U \overline{r_j}$ is to apply r_j and the next turn is the first player.

For $1 \le i \le n$, M_i keeps track of the existence of a pebble on x_i in $\mathcal{G}$. If the state of M_i is in x_i ($\overline{x_i}$) then it means that there is (there is not, respectively) a pebble on x_i in $\mathcal{G}$. Let $M_i = (\{x_i, \overline{x_i}\}, \Sigma_{4m}, \delta_i, q_i, \{q_i\})$, and $q_i = x_i$ for $x_i \in S$, and $q_i = \overline{x_i}$ for $x_i \notin S$. For each i $(1 \le i \le n)$ and j $(1 \le j \le m)$, let $\delta_i(p_i, \sigma_j)$, $p_i \in \{x_i, \overline{x_i}\}$, $\sigma_j \in \{r_j, \overline{r_j}, a_j, \overline{a_j}, b_j, \overline{b_j}, c_j, \overline{c_j}\}$, be the following transition. Assume that $r_j = (x_{j1}, x_{j2}, x_{j3})$ is a rule in $\tilde{R}$.

If $i = j1$ then $\delta_i(p_i, \sigma_j)$ is the transitions shown in Fig.2.2. If $i = j2$ then it is shown in Fig.2.3, and if $i = j3$ then it is in Fig.2.4. If $i \notin \{j1, j2, j3\}$ then $\delta_i(p_i, \sigma_j) = p_i$ for each $p_i \in \{x_i, \overline{x_i}\}$, and $\sigma_j \in \{r_j, \overline{r_j}, a_j, \overline{a_j}, b_j, \overline{b_j}, c_j, \overline{c_j}\}$. Note that $\delta_{j1}(x_{j1}, a_j)$, $\delta_{j2}(x_{j2}, b_j)$, and $\delta_{j3}(\overline{x_{j3}}, c_j)$ are undefined. (See Fig's 2.2, 2.3, and 2.4.) The object of the construction of $M_1, M_2, \cdots, M_n$ is to define a "product dfa" N of $M_1, M_2, \cdots, M_n$, which is defined below. We consider N as a tool for the proof of the theorem, and we do not actually construct N in the simulation.

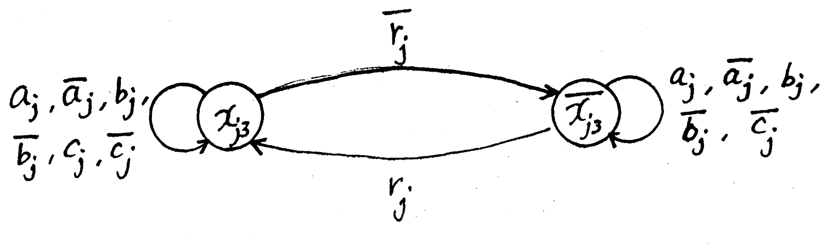


Fig. 2.4 transition $\delta_{j3}(p_{j3}, \sigma_j)$

Now we define $N = (Q, \Sigma_{4m}, \delta, S, \{S\})$, where

$$
\begin{aligned}
&Q = \{x_1, \overline{x_1}\} \times \{x_2, \overline{x_2}\} \times \cdots \times \{x_n, \overline{x_n}\},\\
&S = (q_1, q_2, \cdots, q_n),\\
&\delta((p_1, p_2, \cdots, p_n), \sigma) = (\delta_1(p_1, \sigma), \delta_2(p_2, \sigma), \cdots, \delta_n(p_n, \sigma)), p_i \in \{x_i, \overline{x_i}\},
\end{aligned}
$$

and $\delta((p_1, p_2, \cdots, p_n), \sigma)$ is undefined if $\delta_i(p_i, \sigma)$ is undefined for some i.

We use a state $(p_1, p_2, \cdots, p_n)$ of N and a pebble-placement P of the game-tree in the same meaning: for each $i\,(1 \le i \le n)$, $p_i = x_i$ if and only if there is a pebble on x_n in P, and $p_i = \overline{x_i}$ if and only if there is not a pebble on x_n in P.

Then by the definition of N, we have the following lemmas 2.3 and 2.4:

Lemma 2.3 *Let P be a pebble-placement and let r_j be a rule of $\mathcal{G}$. If r_j is applicable to P and if P' is the resultant pebble-placement then*

$$\delta(P, r_j) = P' \text{ and } \delta(P', \overline{r_j}) = P.$$

If r_j is not applicable to P, then $\delta(P, r_j)$ is undefined.

Proof. Let $P = (p_1, p_2, \cdots, p_n)$ and let $r_j = (x_{j1}, x_{j2}, x_{j3})$. Suppose that r_j is not applicable to P. Then either $p_{j1} = \overline{x_{j1}}$ (there is not a pebble on x_{j1}), $p_{j2} = \overline{x_{j2}}$ (a pebble is not on x_{j2}), or $p_{j3} = x_{j3}$ (a pebble is on x_{j3}) holds. If $p_{j1} = \overline{x_{j1}}$ then $\delta_{j1}(p_{j1}, r_j)$ is undefined (see Fig.2.2), if $p_{j2} = \overline{x_{j2}}$ then $\delta_{j2}(p_{j2}, r_j)$ is undefined (see Fig.2.3), and if $p_{j3} = x_{j3}$ then $\delta_{j3}(p_{j3}, r_j)$ is undefined (see Fig.2.4). Thus $\delta(P, r_j)$ is undefined.

Suppose that r_j is applicable to P. Then $p_{j1} = x_{j1}$, $p_{j2} = x_{j2}$, and $p_{j3} = \overline{x_{j3}}$. Thus

$$
\begin{aligned}
&\delta(P, r_j) = (p_1', p_2', \cdots, p_n'),\\
&p_{j1}' = \overline{x_{j1}},\ p_{j2}' = \overline{x_{j2}},\ p_{j3}' = \overline{x_{j3}},\ \text{and } p_i' = p_i, i \notin \{j1, j2, j3\}.
\end{aligned}
$$

Further we have $\delta((p_1', p_2', \cdots, p_n'), \overline{r_j}) = P$. □

Lemma 2.4 *For any pebble-placement P and any symbol $\sigma \in \{a_j, \overline{a_j}, b_j, \overline{b_j}, c_j, \overline{c_j} \mid 1 \le j \le m\}$,*

$$\delta(P, \sigma) = P \text{ or it is undefined.}$$

Further, r_j is not applicable to P if and only if there is $w_j \in \{a_j\overline{a_j}, b_j\overline{b_j}, c_j\overline{c_j}\}$ such that $\delta(P, w_j) = P$.

Proof. For any $p_i \in \{x_i, \overline{x_i}\}, 1 \le i \le n$, and $\sigma \in \{\overline{a_j}, \overline{b_j}, \overline{c_j}\}, 1 \le j \le m$, we have $\delta_i(p_i, \sigma) = p_i$. (See Fig's.2.2, 2.3, and 2.4.) For any $\sigma \in \{a_j, b_j, c_j\}$, either $\delta_i(p_i, \sigma) = p_i$ or $\delta_i(p_i, \sigma)$ is undefined.

The necessary and sufficient condition that $\delta_i(p_i, a_j)$ is undefined is that $i = j1$ and $p_i = x_{j1}$, that is, there is a pebble on x_{j1} in P. Likewise, the necessary and sufficient condition for $\delta_i(p_i, b_j)$ to be undefined is that $i = j2$ and $p_i = x_{j2}$, that is, a pebble is on x_{j2} in P, and the necessary and sufficient condition for $\delta_i(p_i, c_j)$ to be undefined is that $i = j3$ and $p_i = x_{j3}$, that is, a pebble is not on x_{j3} in P. Thus, r_j is applicable to P if and only if none of $\delta(P, a_j)$, $\delta(P, b_j)$, nor $\delta(P, c_j)$ are defined.

□

Note that $L(G)$ is a subset of D_{4m}. Further we can obtain the following lemma:

Lemma 2.5 *For any $\alpha \in D_{4m}$ and a pebble-placement P,*

$$\delta(P, \alpha) = P \textit{ or it is undefined.}$$

Proof. We can show the lemma by induction on $|\alpha|$. □

Lemma 2.6 *The first player has a winning strategy from a pebble-placement P if and only if there is $w \in \Sigma_{4m}{}^*$ such that*

$$U \overset{*}{\Rightarrow} w \textit{ and } \delta(P, w) = P.$$

Example 2.1 Before we prove the lemma, consider the pebble game $\mathcal{G}$ of Example 1.1. The cfg G guesses the following derivation:

$$\begin{aligned} U &\Rightarrow r_2 W \overline{r_2} \Rightarrow r_2 V_1 V_2 V_3 V_4 \overline{r_2} \\ &\overset{*}{\Rightarrow} r_2 b_1 \overline{b_1} a_2 \overline{a_2} r_3 U \overline{r_3} a_4 \overline{a_4}\, \overline{r_2} \\ &\Rightarrow r_2 b_1 \overline{b_1} a_2 \overline{a_2} r_3 r_4 \overline{r_4}\, \overline{r_3} a_4 \overline{a_4}\, \overline{r_2}. \end{aligned}$$

Let $P_0 = (x_1, x_2, x_3, \overline{x_4}, \overline{x_5})$. P_0 is the initial pebble-placement of $\mathcal{G}$. Then

$$\delta(P_0, r_2) = (x_1, \overline{x_2}, x_3, x_4, \overline{x_5}) = P_1.$$

P_1 is the resultant pebble-placement after an application of r_2 to P_0.

Since there is not a pebble on the second component x_2 of r_1, r_1 is not applicable to P_1, and $\delta(P_1, b_1\overline{b_1}) = P_1$. Similarly, r_2 and r_4 are not applicable to P_1, since there is not a pebble on the first component x_2 of r_2 and r_4. Thus $\delta(P_1, a_2\overline{a_2}) = P_1$, and $\delta(P_1, a_4\overline{a_4}) = P_1$. Further

$$\begin{aligned} \delta(P_1, r_3) &= (x_1, x_2, \overline{x_3}, x_4, \overline{x_5}) = P_2, \text{ and} \\ \delta(P_2, r_4) &= (x_1, \overline{x_2}, \overline{x_3}, x_4, x_5) = P_3. \end{aligned}$$

P_2 is the pebble-placement after the second player applies r_3 to P_1, and P_3 is the pebble-placement after the first player applies r_4 to P_2. The symbols $\overline{r_4}, \overline{r_3}, \overline{r_2}$ are for backtracking procedures. Thus we have

$$\delta(P_3, \overline{r_4}) = P_2, \ \delta(P_2, \overline{r_3}) = P_1, \text{ and } \delta(P_1, \overline{r_2}) = P_0.$$

Therefore, there is $w \in \Sigma_{4m}{}^*$ such that $U \overset{*}{\Rightarrow} w$, and $\delta(P_0, w) = P_0$.

Proof. (Only if): There is a game-tree, the root of which is P. We will prove the "only if" part by induction on the depth of the game-tree. Assume that the depth of the tree is one. That is, the first player applies $r_j = (x_{j1}, x_{j2}, x_{j3})$ to put a pebble on x_n, and $j3 = n$. Then $U \Rightarrow r_j\overline{r_j}$, and if P' is the resultant pebble-placement after the application of r_j to P, then

$$\delta(P, r_j\overline{r_j}) = \delta(P', \overline{r_j}) = P$$

by Lemma 2.3. Thus the "only if" part holds for the basis of the induction.

Assume that the depth of the tree is greater than one, that $r_j = (x_{j1}, x_{j2}, x_{j3})$ is the first player's rule to apply to P and that P' is the resultant pebble-placement. Prior to show the inductive step, we will show that

for each $j\,(1 \leq j \leq m)$, there is $w_j \in D_{4m}$ such that

$$(*) \qquad V_j \stackrel{*}{\Rightarrow} w_j, \delta(P', w_j) = P'.$$

If r_j is not applicable to P' then there is $w_j \in \{a_j\overline{a_j}, b_j\overline{b_j}, c_j\overline{c_j}\}$ which satisfies (*) by Lemma 2.4.

Suppose that r_j is applicable to P', and that $P_j{}'$ is the pebble-placement after the application of r_j to P'. Since the first player has a winning strategy from $P_j{}'$, there is $v_j \in \Sigma_{4m}{}^*$ such that

$$U \stackrel{*}{\Rightarrow} v_j, \delta(P_j{}', v_j) = P_j{}'$$

by the inductive hypothesis. If we put $w_j = r_j v_j \overline{r_j}$ then

$$\begin{aligned} &V_j \Rightarrow r_j U \overline{r_j} \stackrel{*}{\Rightarrow} r_j v_j \overline{r_j} = w_j, \\ &\delta(P', w_j) = \delta(P_j{}', v_j\overline{r_j}) = \delta(P_j{}', \overline{r_j}) = P'. \end{aligned}$$

Thus (*) holds in the inductive step. We have shown (*).

Therefore we have

$$\begin{aligned} &U \Rightarrow r_j W \overline{r_j} \Rightarrow r_j V_1 \cdots V_m \overline{r_j} \stackrel{*}{\Rightarrow} r_j w_1 \cdots w_m \overline{r_j}, \text{ and} \\ &\delta(P, r_j w_1 \cdots w_m \overline{r_j}) = \delta(P', w_1 \cdots w_m \overline{r_j}) = \delta(P', \overline{r_j}) = P. \end{aligned}$$

(If): We use induction on the number of steps of the derivation $U \stackrel{*}{\Rightarrow} w$. Assume that the number of the steps is one, that is, $U \Rightarrow r_j\overline{r_j} = w$. Obviously the first player has a winning strategy from P.

Assume that

$$\begin{aligned} &U \Rightarrow r_j W \overline{r_j} \Rightarrow r_j V_1 \cdots V_m \overline{r_j} \stackrel{*}{\Rightarrow} r_j w_1 \cdots w_m \overline{r_j} = w, \\ &V_j \stackrel{*}{\Rightarrow} w_j, (1 \leq j \leq m). \end{aligned}$$

Since $\delta(P, w) = P$, $\delta(P, r_j)$ is defined. If $\delta(P, r_j) = P'$, then P' is the pebble-placement after the application of r_j to P, and $\delta(P', \overline{r_j}) = P$. By Lemma 2.5 and by $\delta(P', w_1 \cdots w_n) = P'$, we have

$$\delta(P', w_j) = P'$$

for every $j\,(1 \leq j \leq m)$. If $w_j \in \{a_j\overline{a_j}, b_j\overline{b_j}, c_j\overline{c_j}\}$, then r_j is not applicable to P' by Lemma 2.4. If $w_j \notin \{a_j\overline{a_j}, b_j\overline{b_j}, c_j\overline{c_j}\}$, then r_j is applicable to P' and w_j is of the form $r_j v_j \overline{r_j}$, $v_j \in D_{4m}$. Thus

$$V_j \Rightarrow r_j U \overline{r_j} \stackrel{*}{\Rightarrow} r_j v_j \overline{r_j} = w_j, \text{ and } U \stackrel{*}{\Rightarrow} v_j.$$

If $\delta(P', r_j) = P_j'$ then P_j' is the pebble-placement after the application of r_j to P', and $\delta(P_j', v_j) = P_j'$. By the inductive hypothesis, $U \stackrel{*}{\Rightarrow} v_j$ and $\delta(P_j', v_j) = P_j'$ imply that the first player has a winning strategy from P_j'. Thus the first player can win the game no matter what rule r_j the second player may apply to P'.

Therefore the lemma is proved. □

By Lemma 2.6, the necessary and sufficient condition for the first player to have a winning strategy from the initial pebble-placement in $\mathcal{G}$ is that there is $w \in \Sigma_{4m}{}^*$ such that $w \in L(G) \cap L(N)$, and the condition is also that $L(G) \cap \bigcap_{i=1}^{n} L(M_i) \neq \phi$.

To complete the proof of the theorem, we have to construct M. It is clear that we can easily construct the dfa M_i' from M_i which accepts $\Sigma_{4m}{}^* - L(M_i)$, the complement of $L(M_i)$. Now we consider an nfa M such that M accepts the complement of $\bigcap_{i=1}^{n} L(M_i)$. Since

$$\Sigma_{4m}{}^* - \bigcap_{i=1}^{n} L(M_i) = \bigcup_{i=1}^{n}(\Sigma_{4m}{}^* - L(M_i)) = \bigcup_{i=1}^{n} L(M_i') = L(M),$$

we can construct an nfa M as the collection of $M_1', M_2', \cdots, M_n'$ together with the initial state q_0 of M by simply adding λ-moves from q_0 to each initial state of $M_1', M_2', \cdots, M_n'$. The set of the accepting states of M is the union of the ones of $M_1', M_2', \cdots, M_n'$.

Therefore, there is a winning strategy for the first player from the initial pebble-placemene in $\mathcal{G}$ if and only if $L(G) \not\subset L(M)$. The constructions of G and M can be performed within polynomial time. We note that M can be constructed within polynomial time since M is nondeterministic. Thus both P_1' and P_1 are complete for EXPTIME. □

3 Problems on DCFL's

We consider in this section some problems concerning dcfl's.

Theorem 3.1 *The problem P_2:*

> *Given: a regular set $R \subset \Sigma_2{}^*$.*
> *To determine whether: $D_2 \subset R$.*

is EXPTIME complete.

Proof. To prove the theorem, it suffices to show that the following P_2':

> Given: a regular set $R \subset \Sigma_2{}^*$.
> To determine whether: $D_2 \not\subset R$.

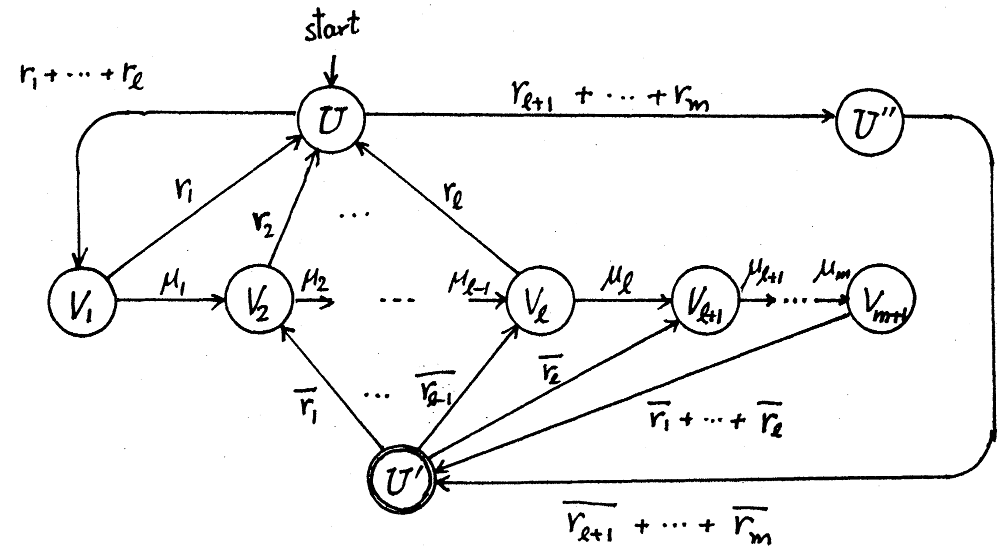


Fig. 3.1 dfa M_0

is EXPTIME complete. By Lemma 2.2, $P_2{}'$ is solvable within exponential time. We show that the pebble game problem is polynomial time reducible to $P_2{}'$. The proof proceeds similarly as in the one of Theorem 2.1.

Let $\mathcal{G} = (X, \tilde{R}, S, x_n)$ be a pebble game, $X = \{x_1, x_2, \cdots, x_n\}, |\tilde{R}| = m$. Let G be the cfg, let $M_1, M_2, \cdots, M_n$ be the dfa's, and let M be the nfa constructed in the proof of Theorem 2.1. We have shown in the preceeding proof that the necessary and sufficient condition for the first player having a forced win from the initial pebble-placement in $\mathcal{G}$ is $L(G) \not\subset L(M)$, hence $L(G) \cap \bigcap_{i=1}^{n} L(M_i) \neq \phi$. We will construct a dfa M_0 such that $L(G) = D_{4m} \cap L(M_0)$.

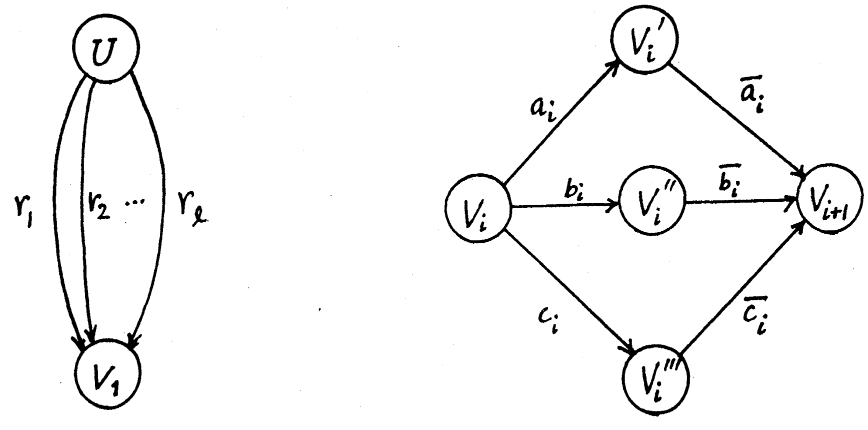
Lemma 3.1 *There exist a dfa M_0 such that $L(G) = D_{4m} \cap L(M_0)$.*

Proof. Assume that R_1 is the set of rules of $\mathcal{G}$ to put a pebble not on x_n, i.e., $R_1 = \{r_j | r_j = (x_{j1}, x_{j2}, x_{j3}), j3 \neq n\}$, and that R_2 is the set of rules to put a pebble on x_n, $R_2 = \{r_j | r_j = (x_{j1}, x_{j2}, x_{j3}), j3 = n\}$. Without loss of generality, we may assume that $R_1 = \{r_1, \cdots, r_\ell\}$ and $R_2 = \{r_{\ell+1}, \cdots, r_m\}$. We construct M_0, which is shown in Fig.3.1, where the transition $r_1 + \cdots + r_\ell$ from U to V_1 stands for ℓ transitions by $r_1, \cdots, r_\ell$ from U to V_1. (See Fig.3.2(a).) Transitions by $\overline{r_1} + \cdots + \overline{r_\ell}$, $r_{\ell+1} + \cdots + r_m$ and $\overline{r_{\ell+1}} + \cdots + \overline{r_m}$ in Fig.3.1 are similar abbreviations. For $1 \leq j \leq m$, let $\mu_j = a_j\overline{a_j} + b_j\overline{b_j} + c_j\overline{c_j}$. The transition by μ_j from V_j to V_{j+1} implies that either $a_j\overline{a_j}$, $b_j\overline{b_j}$, or $c_j\overline{c_j}$ causes the transition from V_j to V_{j+1}. (See Fig.3.2(b).)

Let δ be the transition function of M_0. Recall that D_{4m} is generated by $G' = (\{S\}, \Sigma_{4m}, P, S)$, where P contains $S \to SS \,|\, \lambda \,|\, [_i\, S]_i$ for $1 \leq i \leq 4m$. It is clear that $L(G) \subset D_{4m}$ since any derivation in G can be "mapped into" a derivation in G' by replacing $U, W, V_1, \cdots, V_m$ by S.

Thus in order to prove the lemma it suffices to show that for $\alpha \in D_{4m}$

$$U \overset{*}{\underset{G}{\Rightarrow}} \alpha \text{ if and only if } \delta(U, \alpha) = U',$$



(a) (b)

Fig. 3.2 abbreviations in Fig.3.1

for each j $(1 \leq j \leq m)$, $V_j \overset{*}{\underset{G}{\Rightarrow}} \alpha$ if and only if $\delta(V_j, \alpha) = V_{j+1}$,

$W \overset{*}{\underset{G}{\Rightarrow}} \alpha$ if and only if $\delta(V_1, \alpha) = V_{m+1}$.

(Only if): Let us use induction on $|\alpha|$. If $|\alpha| \leq 2$, the cases are trivial. Consider $\alpha, |\alpha| = k > 2$, assuming that the "only if" part holds for each $\beta \in D_{4m}$, $|\beta| < k$. Suppose $U \overset{*}{\underset{G}{\Rightarrow}} \alpha$. Then the first step of the derivation should be $U \underset{G}{\Rightarrow} r_j W \overline{r_j}$ for some j $(1 \leq j \leq m)$, and $W \overset{*}{\underset{G}{\Rightarrow}} \beta \in D_{4m}$, $\alpha = r_j \beta \overline{r_j}$, $|\beta| < k$. By the inductive hypothesis, we have $\delta(V_1, \beta) = V_{m+1}$. Thus $\delta(U, \alpha) = \delta(U, r_j \beta \overline{r_j}) = \delta(V_1, \beta \overline{r_j}) = \delta(V_{m+1}, \overline{r_j}) = U'$. The cases that $V_j \overset{*}{\underset{G}{\Rightarrow}} \alpha$ and $W \overset{*}{\underset{G}{\Rightarrow}} \alpha$ can be similarly proved.

(If): By simple induction on $|\beta|, \beta \in D_{4m} - \{\lambda\}$, we can show that

(i) $\delta(U, \beta) = U'$ or it is undefined, and

(ii) for each j $(1 \leq j \leq m)$, $\delta(V_j, \beta) \in \{V_{j+1}, \cdots, V_{m+1}\}$ or it is undefined.

Again we will use induction on $|\alpha|$ to show the "if" part. If $|\alpha| \leq 2$ the proof is obvious. Consider α, $|\alpha| = k > 2$, and assume that the "if" part holds for each β, $|\beta| < k$.

Suppose $\delta(U, \alpha) = U'$. If $\alpha = \alpha_1 \alpha_2$ and if $\alpha_1, \alpha_2 \in D_{4m} - \{\lambda\}$, then $\delta(U, \alpha_1) = U'$ by (i). The transition from U' is made only by one of $\overline{r_1}, \cdots, \overline{r_\ell}$ and $\delta(U', \alpha_2)$ is undefined. Thus M_0 does not accept $\alpha_1 \alpha_2$. So $\alpha = r_j \beta \overline{r_j}$ for some j $(1 \leq j \leq \ell)$ and $\beta \in (D_{4m} - \{\lambda\})$. Since $\delta(U, r_j) = V_1$ and $\delta(V_1, \beta \overline{r_j}) = U'$, we obtain $\delta(V_1, \beta) = V_{m+1}$. By the inductive hypothesis we have $W \overset{*}{\underset{G}{\Rightarrow}} \beta$. Thus

$$U \underset{G}{\Rightarrow} r_j W \overline{r_j} \overset{*}{\underset{G}{\Rightarrow}} r_j \beta \overline{r_j} = \alpha.$$

The cases $\delta(V_i, \alpha) = V_{i+1}$ and $\delta(V_1, \alpha) = V_{m+1}$ can be similarly proved. □

We define a homomorphism $h : \Sigma_{4m}{}^* \to \Sigma_2{}^*$ as follows:

$$\left.\begin{array}{l} h([_i) = [_1 [_2{}^i \\ h(]_i) =]_2{}^i]_1 \end{array}\right\} \quad (1 \leq i \leq 4m)$$

Assume that $\Delta = \{h([_i), h(]_i) \mid 1 \leq i \leq 4m\}$. Then the following lemma holds.

Lemma 3.2 $h(D_{4m}) = D_2 \cap \Delta^*$.

Proof. By the definition of h and D_{4m}, $h(D_{4m})$ is the language, which can be generated by the cfg $(\{S\}, \Sigma_2, P, S)$, where P contains $S \to SS \mid \lambda \mid [_1 [_2{}^i S]_2{}^i]_1$ for $1 \leq i \leq 4m$. Thus the lemma follows. □

We will complete the proof of Theorem 3.1. By the definition of h, for languages $L, L' \subset \Sigma_{4m}{}^*$, we have that $L = \phi$ if and only if $h(L) = \phi$, and that $h(L \cap L') = h(L) \cap h(L')$. Thus

$$L(G) \not\subset L(M) \quad \text{if and only if} \quad D_{4m} \cap \bigcap_{i=0}^{n} L(M_i) \neq \phi$$
$$\text{if and only if} \quad h(D_{4m}) \cap \bigcap_{i=0}^{n} h(L(M_i)) \neq \phi.$$

It is easy to construct a dfa $\widehat{M_i}$ such that $h(L(M_i)) = L(\widehat{M_i})$ for $0 \leq i \leq n$. Let $\widehat{M_{n+1}}$ be the dfa, which accepts Δ^*. Then,

$$L(G) \not\subset L(M) \quad \text{if and only if} \quad D_2 \cap \Delta^* \cap \bigcap_{i=0}^{n} L(\widehat{M_i}) \neq \phi$$
$$\text{if and only if} \quad D_2 \cap \bigcap_{i=0}^{n+1} L(\widehat{M_i}) \neq \phi.$$

We can construct an nfa $\widehat{M}$ which accepts the complement of $\bigcap_{i=0}^{n+1} L(\widehat{M_i})$ as in the proof of Theorem 2.1, since $\widehat{M_0}, \widehat{M_1}, \cdots, \widehat{M_{n+1}}$ are deterministic. Thus,

$$L(G) \not\subset L(M) \quad \text{if and only if} \quad D_2 \not\subset L(\widehat{M}).$$

The construction of $\widehat{M}$ can be performed within polynomial time. Therefore the proof of the theorem is completed. □

Corollary 3.1 *For a given regular set R and for each $k \geq 2$, the problem to determine whether $D_k \subset R$ is EXPTIME complete.*

Proof. The problem can be solved within EXPTIME. Let R be a regular set. We prove that

$$D_2 \subset R \quad \text{if and only if} \quad D_k \subset R \cup (\Sigma_k{}^* - \Sigma_2{}^*).$$

Assume that $D_2 \subset R$, and that $w \in D_k$. If $w \in \Sigma_2{}^*$ then $w \in D_2$. If $w \notin \Sigma_2{}^*$ then $w \in \Sigma_k{}^* - \Sigma_2{}^*$. Thus $w \in R \cup (\Sigma_k{}^* - \Sigma_2{}^*)$ and we obtain that $D_k \subset R \cup (\Sigma_k{}^* - \Sigma_2{}^*)$.

Assume that $D_k \subset R \cup (\Sigma_k{}^* - \Sigma_2{}^*)$, and $w \in D_2$. Since $w \in R \cup (\Sigma_k{}^* - \Sigma_2{}^*)$ and $w \notin \Sigma_k{}^* - \Sigma_2{}^*$, we obtain that $w \in R$. Thus $D_2 \subset R$.

As we can construct the nfa accepting $R \cup (\Sigma_k{}^* - \Sigma_2{}^*)$ within polynomial time, the corollary is proved. □

Open problem 1 The complexity of the problem to determine whether $D_1 \subset R$ for a given regular set R is remained open.

Since we can construct a dpda M to accept D_2, we obtain the following corollary.

Corollary 3.2 *The problem P_3:*

> *Given: a dcfl L, and a regular set R.*
> *To determine whether: $L \subset R$.*

is EXPTIME complete.

Corollary 3.3 *The problem P_4:*

> *Given: a dcfl $L \subset \Sigma^*$, and a regular set $R \subset \Sigma^*$.*
> *To determine whether: $L \cup R = \Sigma^*$.*

is EXPTIME complete.

Proof. Let M be a dpda which accepts L. Since M is deterministic, we can construct a dpda M' such that M' accepts $\Sigma^* - L$. (See [4],p.238, for example.) Then we can construct a cfg G, which satisfies $L(G) = L(M')$.

Since $L \cup R = \Sigma^*$ is equivalent to $L(G) \subset R$, and G can be constructed within polynomial time, P_4 is EXPTIME complete by Corollary 3.2. □

Remark The problem to determine whether $R \subset L$ for a given regular set R and a dcfl L is solvable within polynomial time by constructing a cfg G generating the complement of L and by applying the algorithm of Fig.2.1 to determine whether $R \cap L(G) = \phi$, which is equivalent to $R \subset L$.

Open problem 2 Let L be a dcfl and R be a regular set. The following problems are in EXPTIME, however, their complexities are open.

(1) $R = L$?
(2) $L \subsetneq R$?
(3) $R \subsetneq L$?

References

[1] A. K. Chandra and L. J. Stockmeyer, *Alternation*, Proceedings 17th Ann. IEEE Symp. on Found. of Comput. Sci. (1976), pp.151–174.

[2] A. S. Fraenkel, and D. Lichtenstein, *Computing a perfect strategy for n × n chess requires time exponential in n*, J. Combinatorial Theory, 31(1981), pp.199–214.

[3] H. B. Hunt III, D. J. Rosenkrantz, and T. G. Szymanski, *On the equivalence, containment, and covering problems for the regular and context-free languages*, J. Comput. System Sci., 12(1976), pp.222–268.

[4] J. E. Hopcroft, and J. D. Ullman, *Introduction to Automata Theory, Languages, and Computation*, Addison-Wesley, Reading Mass., 1979.

[5] T. Kasai, A. Adachi, and S. Iwata, *Classes of pebble games and complete problems*, SIAM J. Comput., 8(1979), pp.578–586.

[6] J. M. Robson, *The complexity of GO*, Proceedings, IFIP 1983(1983), pp.413–417.

[7] J. M. Robson, *N by N checkers is EXPTIME complete,* SIAM J. Comput., 13(1984), pp.252–267.

[8] L. J. Stockmeyer, and A. K. Chandra, *Provably difficult combinatorial games,* SIAM J. Comput., 8(1979), pp.151–174.

[9] D. H. Younger, *Recognition and parsing of context-free languages in time* n^3, Inform. Contr., 10(1967), pp.189–208.

謝辞 1

岐阜べんなんて忘れてしまったであかんわ．町田さん来年も開くまわししてちょ．

謝辞 2

くに荘のおばさん，おっかなかったよ．町田さん，研究会しらいてくれてあんがと．またひてね．