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Max-Flow Problem of Strang's Type

椋 毅 (Takeshi MUKU)　山崎 稀嗣 (Maretsugu YAMASAKI)
島根大学理学部 (Shimane Univ.)

1. Introduction

The celebrated duality theorem called max-flow min-cut theorem on a finite network due to Ford and Fulkerson [1] has been generalized to many directions. Among them, we shall be interested in Strang's work [4]. Strang's results were further generalized by Nozawa [3] in the continuous case. Strang gave a max-flow min-cut theorem on a finite network as a motivation of his theory. Here we shall be concerned with the Strang's max-flow problem on an infinite network. Related to this max-flow problem, we shall discuss several mathematical programming problems as in [5].

More precisely, let X be the countable set of nodes, Y be the countable set of arcs and K be the node-arc incidence matrix. We always assume that the graph $G = \{X, Y, K\}$ is connected and locally finite and has no self-loop. For a strictly positive real function r on Y, the pair $N = \{G, r\}$ is called an infinite (discrete) network in this paper. In case $r = 1$, we can identify G with $N = \{G, 1\}$, and we may call G an infinite network.

Denote by $L(X)$ the set of real valued functions on X. For $u \in L(X)$, let Su be its support, i.e.,

$$Su = \{x \in X; u(x) \neq 0\},$$

and let $L_0(X)$ be the set of $u \in L(X)$ such that Su is empty or a finite set. For notation and terminolgy, we mainly follow [5] and [6].

For a given $f \in L(X)$, we call $w \in L(Y)$ a f-flow if there exists a number t which satisfies the condition

$$\sum_{y \in Y} K(x, y)w(y) = tf(x) \quad \text{on} \quad X.$$

Denote by $\mathbf{F}(f)$ the set of all f-flows. In case $f \neq 0$, the number t in the above definition is uniquely determined by w, so we call it the strength of w and denote it by $I(w)$.

Given a non-negative real function C on Y which is called a capacity, we consider the following max-flow problem which was studied by Strang in the case where G is a finite network:

(1.1) Find $M(\mathbf{F}(f); C) = \sup\{I(w); w \in \mathbf{F}(f), |w(y)| \leq C(y) \text{ on } Y\}$.

For a subset A of X, denote by φ_A the characteristic function of A, i.e., $\varphi_A(x) = 1$ for $x \in A$ and $\varphi_A(x) = 0$ for $x \in X - A$. Let a, b two distinct nodes and consider the special case where $f = \varphi_{\{b\}} - \varphi_{\{a\}}$. Then $w \in \mathbf{F}(f)$ implies

$$\sum_{y \in Y} K(x, y)w(y) = 0 \quad \text{on} \quad X - \{a, b\},$$

$$I(w) = -\sum_{y\in Y} K(a,y)w(y) = \sum_{y\in Y} K(b,y)w(y).$$

Namely every f-flow is a usual flow from the source a to the sink b and Problem (1.1) is the usual max-flow problem.

To state a dual problem of Problem (1.1), let us recall the definition of a cut. For mutually disjoint nonempty subsets A and B of X, denote by $A \ominus B$ the set of all arcs which connect directly A with B. A subset Q of Y is a cut if there exists a nonempty proper subset A of X such that $Q = A \ominus (X - A)$.

Let us define a quasi-norm $\|u\|_C$ of $u \in L(X)$ by

$$\|u\|_C = \sum_{y\in Y} C(y) \mid \sum_{x\in X} K(x,y)u(x) \mid .$$

For $Q = A \ominus (X - A)$, we have

$$\|\varphi_A\|_C = \|1 - \varphi_A\|_C = \sum_{y\in Q} C(y).$$

Let us define an inner product $< u, v >$ of $u, v \in L(X)$ by

$$< u, v >= \sum_{x\in X} u(x)v(x)$$

whenever the sum is well-defined.

Let $\mathbf{U}(X)$ be the set of all functions $u \in L(X)$ taking values only 0 and 1, i.e., the range $u(X)$ of u is equal to $\{0, 1\}$. Notice that for every cut $Q = A \ominus (X - A)$, both φ_A and $1 - \varphi_A$ belong to $\mathbf{U}(X)$.

Now we consider the general case where f satisfies the condition

(1.2) $\qquad f \neq 0, \quad <|f|, 1>< \infty \quad$ and $\quad < f, 1 >= 0$.

This condition holds if G is a finite network and $\mathbf{F}(f)$ contains w such that $I(w) \neq 0$.

Strang introduced the following min-cut problem:

(1.3) Find $M^*(\mathbf{U}(f); C) = \inf\{\|\varphi\|_C / |< f, \varphi >|; \varphi \in \mathbf{U}(f)\}$,

where $\mathbf{U}(f) = \{\varphi \in \mathbf{U}(X); < \varphi, f > \neq 0\}$.

In the special case where $f = \varphi_{\{b\}} - \varphi_{\{a\}}$ as above, it is easily seen that Problem (1.3) is reduced to the usual min-cut problem.

Strang stated the following duality theorem [4; p.128]:

THEOREM 1.1. *Let G be a finite network. Then $M(\mathbf{F}(f); C) = M^*(\mathbf{U}(f); C)$ holds and both Problems (1.1) and (1.3) have optimal solutions.*

In the next section, we shall begin with proving this theorem which was roughly stated in [4]. We shall study whether this theorem is valid or not on an infinite network. Related to the f-flows, we shall consider an extremum problem which is analogous to the extremal width of a and b (cf. [5]).

2. Max-flow min-cut theorem on a finite network

In this section, we always assume that G is a finite network, i.e., X and Y are finite sets. To apply the duality theory in [2], we shall formulate Problem (1.1) as a usual linear programming problem on paired spaces.

Let us take

$$\mathcal{X} = \mathcal{Y} = L(Y) \times R,\ \mathcal{Z} = \mathcal{W} = L(X) \times L(Y) \times L(Y),$$
$$\mathcal{P} = L(Y) \times R,\ \mathcal{Q} = \{0\} \times L^+(Y) \times L^+(Y),$$
$$T\mathbf{x} = T(w,t) = (\textstyle\sum_{y\in Y} K(\cdot,y)w(y) - tf, w, -w),$$
$$\mathbf{y}_0 = (0,-1),\ \mathbf{z}_0 = (0,-C,-C).$$

Define bilinear functionals:

$$(\mathbf{x},\mathbf{y})_1 = ((w,t),(w',t'))_1 = \sum_{y\in Y} w(y)w'(y) + tt'$$

for $\mathbf{x} = (w,t), \mathbf{y} = (w',t') \in L(Y) \times R$;

$$(\mathbf{z},\mathbf{w})_2 = ((u,v,w),(u',v',w'))_2 = < u,u' > + \sum_{y\in Y} v(y)v'(y) + \sum_{y\in Y} w(y)w'(y)$$

for $\mathbf{z} = (u,v,w), \mathbf{w} = (u',v',w') \in L(X) \times L(Y) \times L(Y)$. Then $\mathcal{X}$ and $\mathcal{Y}$ (resp. $\mathcal{Z}$ and $\mathcal{W}$) are paired linear spaces with respect to $(\cdot,\cdot)_1$(resp.$(\cdot,\cdot)_2$). We see that the quintuple $\{T,P,Q,\mathbf{y}_0,\mathbf{z}_0\}$ is a linear program and

$$-M(\mathbf{F}(f);C) = \inf\{(\mathbf{x},\mathbf{y}_0)_1; \mathbf{x} \in P, T\mathbf{x} - \mathbf{z}_0 \in Q\}.$$

Denote by T^* the adjoint of T. Then

$$T^*(u,w_1,w_2) = (\sum_{x\in X} K(x,\cdot)u(x) + w_1 - w_2, - < u,f >).$$

The dual problem is to find the value

$$\tilde{M}^* = \sup\{(\mathbf{z}_0,\mathbf{w})_2; \mathbf{w} \in \mathcal{Q}^+, \mathbf{y}_0 - T^*\mathbf{w} \in \mathcal{P}^+\},$$

where $\mathcal{P}^+$ and $\mathcal{Q}^+$ are dual cones of $\mathcal{P}$ and $\mathcal{Q}$ respectively and given by

$$\mathcal{P}^+ = \{0\} \times \{0\}, \mathcal{Q}^+ = L(X) \times L^+(Y) \times L^+(Y).$$

Rewriting the right hand side of $\tilde{M}^*$, we see that $-\tilde{M}^*$ is equal to the value of the following extremum problem: Minimize the objective function

$$\sum_{y\in Y} C(y)[w_1(y) + w_2(y)]$$

subject to $w_1, w_2 \in L^+(Y), < u, f >= 1$ and

$$\sum_{x \in X} K(x,y)u(x) + w_1(y) - w_2(y) = 0 \quad \text{on} \quad Y.$$

Therefore we have

$$-\tilde{M}^* = V := \inf\{\|u\|_C; u \in L(X), < u, f >= 1\}.$$

Since $\mathcal{X}$ and $\mathcal{Z}$ are finite dimensional and $\mathcal{P}$ and $\mathcal{Q}$ are polyhedral cones, there is no duality gap (cf. [2]), i.e., $M(\mathbf{F}(f); C) = \tilde{M}^*$. It follows that $M(\mathbf{F}(f); C) = V$. By an easy calculation, we obtain

(2.1) $\quad V = \min\{\|u\|_C / \mid < u, f > \mid; u \in L(X), < u, f > \neq 0\},$

and hence

(2.2) $\quad V = \min\{\|u\|_C / \mid < u, f > \mid; u \in \mathbf{V}(f)\},$

where $\mathbf{V}(f) = \{u \in L(X); 0 \leq u(x) \leq 1 \text{ on } X, < u, f > \neq 0\}$.

Our next step is to show that $\mathbf{V}(f)$ can be replaced by $\mathbf{U}(f)$ in (2.2). To do this, we need a discrete analogue to the coarea formula.

LEMMA 2.1. *Let $u \in L^+(X)$ and $u(X) = \{\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_n\}$ with $\alpha_0 = 0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n$ and put $A_k = \{x \in X; u(x) \geq \alpha_k\}$. Then*

$$\sum_{x \in X} u(x)f(x) = \sum_{k=1}^{n} (\alpha_k - \alpha_{k-1}) \sum_{x \in A_k} f(x).$$

PROOF. Put $\beta_k = \sum_{x \in A_k} f(x)$ for $0 \leq k \leq n$ and let $A_{n+1} = \emptyset$ and $\beta_{n+1} = 0$. By the relation

$$B_k := A_k - A_{k+1} = \{x \in X; u(x) = \alpha_k\},$$

we see that

$$\begin{aligned} \sum_{x \in X} u(x)f(x) &= \sum_{k=1}^{n+1} \sum_{x \in B_{k-1}} u(x)f(x) \\ &= \sum_{k=1}^{n+1} \alpha_{k-1}(\beta_{k-1} - \beta_k). \end{aligned}$$

Changing the order of summation, we obtain the desired relation.

LEMMA 2.2. *Let $u, \{\alpha_k\}$ and A_k be the same as above and put $Q_k = A_k \ominus (X - A_k)$ for $k = 1, \cdots, n$. Then*

$$\sum_{y \in Y} C(y) \mid \sum_{x \in X} K(x,y)u(x) \mid = \sum_{k=1}^{n} (\alpha_k - \alpha_{k-1}) \sum_{y \in Q_k} C(y).$$

PROOF. Note that $B_j \cap B_k = \emptyset$ if $j \neq k$ and

$$| \sum_{x \in X} K(x,y)u(x) | = \alpha_k - \alpha_j$$

if $y \in B_j \ominus B_k$ and $j < k$. Note that if the endpoints of y belong to B_j, i.e., $\{x \in X; K(x,y) \neq 0\} \subset B_j$, then

$$| \sum_{x \in X} K(x,y)u(x) | = 0.$$

Put

$$\mu_{jk} = \sum_{y \in B_j \ominus B_k} C(y) \quad \nu_j = \sum_{y \in Q_j} C(y)$$

with $\nu_{n+1} = 0$. Then it is easily seen that

$$\sum_{k=0}^{j} \mu_{kj} = \sum_{y \in A_j \ominus (X - A_j)} C(y) = \sum_{y \in Q_j} C(y) = \nu_j$$

and similarly

$$\sum_{k=j+1}^{n} \mu_{jk} = \sum_{y \in Q_{j+1}} C(y) = \nu_{j+1}.$$

By the above observation, we have

$$\begin{aligned} \sum_{y \in Y} C(y) | \sum_{x \in X} K(x,y)u(x) | &= \sum_{j=0}^{n} \sum_{k=j+1}^{n} \mu_{jk}(\alpha_k - \alpha_j) \\ &= \sum_{j=1}^{n} \alpha_j \sum_{k=0}^{j} \mu_{kj} - \sum_{j=0}^{n} \alpha_j \sum_{k=j+1}^{n} \mu_{jk} \\ &= \sum_{j=1}^{n} \alpha_j \nu_j - \sum_{j=0}^{n} \alpha_j \nu_{j+1}. \end{aligned}$$

Now we shall prove a fundamental lemma.

LEMMA 2.3. *The relation $V = M^*(\mathbf{U}(f); C)$holds and there exists $\varphi \in \mathbf{U}(f)$ such that $M^*(\mathbf{U}(f); C) = \|\varphi\|_C / \, |< \varphi, f >|$.*

PROOF. Let us put $M^* = M^*(\mathbf{U}(f); C)$. Clearly, $V \leq M^*$. Suppose that $V < M^*$, i.e., there exists $\varepsilon > 0$ such that $M^* \geq V + \varepsilon$. Then

(2.3) $\qquad \|\varphi\|_C \geq (V + \varepsilon) \, |< \varphi, f >|$

holds for all $\varphi \in \mathbf{U}(f)$. Since (2.3) holds trivially for $\varphi \in \mathbf{U}(X) - \mathbf{U}(f)$, (2.3) holds for all $\varphi \in \mathbf{U}(X)$. For any proper subset A of X, we have $\varphi_A \in \mathbf{U}(X)$ and by (2.3)

$$\sum_{y \in A\ominus(X-A)} C(y) \geq (V+\varepsilon)\, |< \varphi_A, f >|\,.$$

Let $u \in \mathbf{V}(f)$ and $u(X) = \{\alpha_0, \alpha_1, \cdots, \alpha_n\}$ with $\alpha_0 = 0 < \alpha_1 < \cdots < \alpha_n \leq 1$ and put $A_k = \{x \in X; u(x) \geq \alpha_k\}$. Multiplying both sides of the above inequality (with $A = A_k$)) by $\alpha_k - \alpha_{k-1}$ and summing both sides over k, we have by Lemmas 2.1 and 2.2

$$\begin{aligned} \|u\|_C &= \sum_{k=1}^{n}(\alpha_k - \alpha_{k-1}) \sum_{y \in A_k\ominus(X-A_k)} C(y) \\ &\geq \sum_{k=1}^{n}(\alpha_k - \alpha_{k-1})(V+\varepsilon)\, |< \varphi_{A_k}, f >| \\ &\geq (V+\varepsilon)\, |\sum_{k=1}^{n}(\alpha_k - \alpha_{k-1}) \sum_{x \in A_k} f(x)\,| \\ &= (V+\varepsilon)\, |< u, f >|\,. \end{aligned}$$

Namely we have $V + \varepsilon \leq \|u\|_C / \, |< u, f >|$ for all $u \in \mathbf{V}(f)$, and hence $V + \varepsilon \leq V$. This is a contradiction. Thus $V = M^*$. Since $\mathbf{U}(f)$ contains only a finite number of elements, there exists $\varphi \in \mathbf{U}(f)$ such that $M^* = \|\varphi\|_C / \, |< \varphi, f >|$.

Summing up (2.2), (2.3) and Lemma 2.3, we complete the proof of Theorem 1.1.

3. Max-flow min-cut theorems on an infinite network

In order to study a max-flow problem on an infinite network, we consider the subset $\mathbf{F}_0(f) = \mathbf{F}(f) \cap L_0(Y)$ of the set of f-flows. In this section, we always assume the following condition:

(3.1) $\quad f \in L_0(X), f \neq 0$ and $< f, 1 >= 0$.

Let $\{G_n\}(G_n =< X_n, Y_n >)$ be an exhaustion of G, i.e., each G_n is a finite subnetwork of G and $\{G_n\}$ approximates G increasingly. For simplicity, we assume that $Sf \subset X_1$. Define $C_n \in L^+(Y)$ by $C_n(y) = C(y)$ for $y \in Y_n$ and $C_n(y) = 0$ for $y \in Y - Y_n$ and consider the following extremum problems:

(3.2) $\quad$ Find $M_n = M(\mathbf{F}(f); C_n)$;

(3.3) $\quad$ Find $M_n^* = M^*(\mathbf{U}(f); C_n)$.

We shall be concerned with the limits of $\{M_n\}$ and $\{M_n^*\}$.

LEMMA 3.1. $\lim_{n\to\infty} M(\mathbf{F}(f); C_n) = M(\mathbf{F}_0(f); C)$.

PROOF. If w is a feasible solution of Problem (3.2), then $w \in L_0(Y)$ by the condition $| w(y) | \leq C_n(y)$ on Y, and hence $M_n \leq M_{n+1} \leq M(\mathbf{F}_0(f); C)$. For any $\varepsilon > 0$, there exists $w \in \mathbf{F}_0(f)$ such that

$$M(\mathbf{F}_0(f); C) - \varepsilon < I(w), \quad | w(y) | \leq C(y) \text{ on } Y.$$

There exists n_0 such that $Sw \subset Y_n$ for all $n \geq n_0$. Then w is a feasible solution of Problem (3.2) for $n \geq n_0$, and hence $M(\mathbf{F}_0(f); C) - \varepsilon < I(w) \leq M_n$ for all $n \geq n_0$.

We see easily the following:

REMARK 3.2. The value of Problem (3.2) is equal to the value of the following max-flow problem on G_n :

(3.4) Maximize t subject to $w \in L(Y_n), | w(y) | \leq C_n(y)$ on Y and

$$\sum_{y \in Y_n} K(x, y) w(y) = t f(x) \text{ on } X_n.$$

Related to Problem (3.3), consider the following min-cut problem on G_n :

(3.5) Find $M^*(\mathbf{U}(f; X_n); C_n) = \inf\{\sum_{y \in Y_n} C_n(y) \mid \sum_{x \in X_n} K(x, y)\varphi(x) \mid; \varphi \in \mathbf{U}(f; X_n)\}$,

where $\mathbf{U}(f; X_n)$ is the set of all $\varphi \in L(X_n)$ such that $\varphi(X_n) = \{0, 1\}$ and $\sum_{x \in X_n} \varphi(x) f(x) \neq 0$.

LEMMA 3.3. *$M_n^* = M^*(\mathbf{U}(f; X_n); C_n)$ holds and there exists $\varphi \in \mathbf{U}(f)$ such that $M_n^* = \|\varphi\|_{C_n} / |< \varphi, f >|$.*

PROOF. The equality follows from our construction. Problem (3.5) has an optimal solution $\varphi' \in \mathbf{U}(f; X_n)$ by Theorem 1.1 and the extension φ of φ' to $X - X_n$ by 0 belongs to $\mathbf{U}(f)$ and satisfies our requirement.

LEMMA 3.4. *$\lim_{n \to \infty} M_n^* = M^*(\mathbf{U}(f); C)$ and there exists $\varphi \in \mathbf{U}(f)$ such that $M^*(\mathbf{U}(f); C) = \|\varphi\|_C / |< \varphi, f >|$.*

PROOF. By definition, $M_n^* \leq M_{n+1}^* \leq M^*(\mathbf{U}(f); C)$ is clear. There exists $\varphi_n \in \mathbf{U}(f)$ such that $M_n^* = \|\varphi_n\|_{C_n} / |< \varphi_n, f >|$. Since $f \in L_0(X)$, it should be noted that the set $\{|< \varphi, f >|; \varphi \in \mathbf{U}(f)\}$ contains only a finite number of real numbers which are apart from 0, so that there exists $\alpha > 0$ such that

(3.6) $|< \varphi, f >| \geq \alpha > 0$ for all $\varphi \in \mathbf{U}(f)$.

Since $\varphi_n(X) = \{0, 1\}$, we may assume that $\{\varphi_n\}$ converges pointwise to $\tilde{\varphi} \in L(X)$ by choosing subsequences if necessary. We see by (3.6) that $\tilde{\varphi} \in \mathbf{U}(f)$. Since $f \in L_0(X)$,

$<\varphi_n, f> \to <\tilde{\varphi}, f>$ as $n \to \infty$. It follows that

$$\begin{aligned}\liminf_{n\to\infty} M_n^* &\geq \sum_{y\in Y} \liminf_{n\to\infty} C_n(y) \mid \sum_{x\in X} K(x,y)\varphi_n(x) \mid / \mid <\varphi_n, f> \mid \\ &\geq \sum_{y\in Y} C(y) \mid \sum_{x\in X} K(x,y)\tilde{\varphi}(x) \mid / \mid <\tilde{\varphi}, f> \mid \\ &\geq M^*(\mathbf{U}(f); C).\end{aligned}$$

This completes the proof.

By Theorem 1.1 and Lemmas 3.1, 3.3 and 3.4 and Remark 3.2, we obtain the following:

THEOREM 3.5. $M(\mathbf{F}_0(f); C) = M^*(\mathbf{U}(f); C)$ *holds and there exists an optimal solution of the min-cut problem.*

In the special case where $f = \varphi_{\{b\}} - \varphi_{\{a\}}$, this theorem was proved in [5].

4. Extremal width of a network

Denote by $\mathbf{Q}(f)$ the set of all cuts generated by $\varphi \in \mathbf{U}(f)$, i.e.,

$$\mathbf{Q}(f) = \{S\varphi \ominus (X - S\varphi); \varphi \in \mathbf{U}(f)\}$$

and consider the following extremum problem of minimizing

$$H(W) := \sum_{y\in Y} r(y)W(y)^2$$

subject to $W \in L^+(Y)$ and

$$\sum_{y\in Q} W(y)/ \mid <\varphi, f> \mid \geq 1 \text{ for all } Q = S\varphi \ominus (X - S\varphi) \in \mathbf{Q}(f).$$

Let $\mu(\mathbf{Q}(f))^{-1}$ be the value of this problem. In the case where $f = \varphi_{\{b\}} - \varphi_{\{a\}}$, this value is called the extremal width between $\{a\}$ and $\{b\}$ of N in [5].

Denote by $E^*(\mathbf{Q}(f))$ the set of all feasible solutions of this problem, i.e..

$$E^*(\mathbf{Q}(f)) = \{W \in L^+(Y); M^*(\mathbf{U}(f); W) \geq 1\}.$$

Then we have

$$\mu^*(\mathbf{Q}(f))^{-1} = \inf\{H(W); W \in E^*(\mathbf{Q}(f))\}.$$

We shall consider the extremum problem of finding the following value related to f-flows:

$$d^*(\mathbf{F}_0(f)) = \inf\{H(w); w \in \mathbf{F}_0(f), I(w) = 1\}.$$

We shall prove

THEOREM 4.1. *Assume Condition (3.1). Then* $d^*(\mathbf{F}_0(f)) = \mu^*(\mathbf{Q}(f))^{-1}$.

PROOF. Let $w \in \mathbf{F}_0(f), I(w) = 1$ and put $W(y) = \mid w(y) \mid$. For any $\varphi \in \mathbf{U}(f)$,

$$\begin{aligned} |< \varphi, f >| &= \mid \sum_{y \in Y} w(y) \sum_{x \in X} K(x,y)\varphi(x) \mid \\ &\leq \sum_{y \in Y} W(y) \mid \sum_{x \in X} K(x,y)\varphi(x) \mid, \end{aligned}$$

so that $W \in E^*(\mathbf{Q}(f))$. Thus $\mu^*(\mathbf{Q}(f))^{-1} \leq H(W) = H(w)$, and hence $\mu^*(\mathbf{Q}(f))^{-1} \leq d^*(\mathbf{F}_0(f))$. On the other hand, let $W \in L^+(Y)$ satisfy $M^*(\mathbf{U}(f); W) \geq 1$. Then by Theorem 3.5,

$$M(\mathbf{F}_0(f); W) = M^*(\mathbf{U}(f); W) \geq 1.$$

For any positive number $t < 1$, there exists $w \in \mathbf{F}_0(f)$ such that $\mid w(y) \mid \leq W(y)$ and $I(w) > t$. Clearly $w' := w/I(w) \in \mathbf{F}_0(f)$ and $I(w') = 1$, so that

$$d^*(\mathbf{F}_0(f)) \leq H(w/I(w)) < H(W)/t^2.$$

Letting $t \to 1$, we have $d^*(\mathbf{F}_0(f)) \leq H(W)$, and hence $d^*(\mathbf{Q}(f)) \leq \mu^*(\mathbf{Q}(f))^{-1}$. This completes the proof.

Related to the above flow problems, let us consider the following extremum problem of minimizing the Dirichlet sum:

(4.1) Find $d(f) = \inf\{D(u); u \in L(X) \text{ and } < u, f >= 1\}$,

where $D(u) := H(du)$ and

$$du(y) = -r(y)^{-1} \sum_{x \in X} K(x,y)u(x).$$

We have the following reciprocal relation:

THEOREM 4.2. *Assume Condition (3.1). Then* $d(f)d^*(\mathbf{F}_0(f)) = 1$.

PROOF. Let $w \in \mathbf{F}_0(f), I(w) = 1$ and $u \in L(X), < u, f >= 1$. Then

$$\begin{aligned} 1 =< u, f > &= \sum_{y \in Y} w(y) \sum_{x \in X} K(x,y)u(x) \\ &\leq [H(w)]^{1/2}[D(u)]^{1/2}, \end{aligned}$$

so that $1 \leq d(f)d^*(\mathbf{F}_0(f))$. Denote by $\mathbf{F}_2(f)$ the closure of $\mathbf{F}_0(f)$ in the Hilbert space $L_2(Y; r) = \{w \in L(Y); H(w) < \infty\}$ with the inner product

$$H(w, w') = \sum_{y \in Y} r(y)w(y)w'(y).$$

Then we have $d^*(\mathbf{F}_0(f)) = d^*(\mathbf{F}_2(f))$. Let $\{w_n\}$ be a sequence in $\mathbf{F}_0(f)$ such that $I(w_n) = 1$ and $H(w_n) \to d^*(\mathbf{F}_0(f))$ as $n \to \infty$. Since $(w_n + w_m)/2 \in L_0(Y)$ is a f-flow of unit strength , we see by the standard method that $H(w_n - w_m) \to 0$ as $n, m \to \infty$. There exists $\tilde{w} \in L_2(Y; r)$ such that $H(w_n - \tilde{w}) \to 0$ as $n \to \infty$. Clearly $\tilde{w} \in \mathbf{F}_2(f)$ and $I(w_n) \to I(\tilde{w})$ as $n \to \infty$. It follows that $I(\tilde{w}) = 1$ and $d^*(\mathbf{F}_2(f)) = H(\tilde{w})$. For any $w' \in \mathbf{F}_0(0)$(a finite cycle) and for any real number t, we have $\tilde{w} + tw' \in \mathbf{F}_2(f)$, so that $H(\tilde{w}) \leq H(\tilde{w} + tw')$. By the usual variational method, we have $H(\tilde{w}, w') = 0$. We see by the same arguement as in [7] that there exists $\tilde{u} \in$ D(N) such that $d\tilde{u}(y) = \tilde{w}(y)$ on Y. Here $D(N)$ is the set of all $u \in L(X)$ with finite Dirichlet sum. Notice that $H(\tilde{w}, w_m - w_n) = 0$ for all n, m by the above observation, so that $H(\tilde{w}) = H(\tilde{w}, w_n)$. It follows that

$$\begin{aligned} < \tilde{u}, f > &= \sum_{x \in X} \tilde{u}(x) \sum_{y \in Y} K(x, y) w_n(y) \\ &= \sum_{y \in Y} w_n(y) \sum_{x \in X} K(x, y) \tilde{u}(x) \\ &= H(w_n, \tilde{w}) = H(\tilde{w}) = D(\tilde{u}). \end{aligned}$$

Therefore $< f, \tilde{u}/D(\tilde{u}) >= 1$, and

$$d(f) \leq D(\tilde{u}/D(\tilde{u})) = D(\tilde{u})^{-1} = H(\tilde{w})^{-1} = d^*(\mathbf{F}_0(f))^{-1}.$$

Thus $d(f)d^*(\mathbf{F}_0(f)) \leq 1$. This completes the proof.

Theorems 4.1 and 4.2 were proved in [5] in the case where $f = \varphi_{\{b\}} - \varphi_{\{a\}}$.

References

[1] L.R. Ford and D.R. Fulkerson, Flows in networks, Princeton Univ. Press, Princeton, N.J., 1962.

[2] K. S. Kretschmer, Programmes in paired spaces, Canad. J. Math. **13**(1961), 221-238.

[3] R. Nozawa, Max-flow min-cut theorem in an anisotropic network, Osaka J. Math. **27**(1990), 805-842.

[4] G. Strang, Maximal flow through a domain, Math. Programming, **26**(1983), 123-143.

[5] M. Yamasaki, Extremum problems on an infinite network, Hiroshima Math. J. **5**(1975), 223-250.

[6] M. Yamasaki, Max-flow min-cut theorems on an infinite network, RIMS Kokyuroku **680**(1989), 81-88.

[7] M. Yamasaki, Nonlinear Poisson equations on an infinite network, Mem. Fac. Sci. Shimane Univ. **23**(1989), 1-9.