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Perfect Isometries for Blocks with Abelian Defect Groups and Klein Four Inertial Quotients

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1. Alperin's weight conjecture for the case of abelian defect groups

Let p be a prime number, k an algebraically closed field of characteristic p , O a complete discrete valuation ring with residue field k and quotient field K of characteristic zero, G a finite group , b a p-block of G (i.e. a primitive idempotent of Z(kG)), P a defect group of b , e a root of b in $C_G(P)$ and E the inertial quotient $N_G(P,e)/PC_G(P)$. We assume that K is large enough.

Alperin's weight conjecture states that the number l(b) of isomorphism classes of simple kGb-modules can be calculated by the function of its local structure. When P is abelian, this is equivalent to the following one. ([1])

Conjecture 1. If P is abelian, then l(b) is the number of isomorphism classes of simple $kN_G(P,e)e$-modules.

This is known to be true if $|E| \leq 3$ by the results of Brauer (cf.[3],Proposition (6G)) and Usami [13] (except the case $|E| = 3$ and $p = 2$). Here we introduce the result which proves it in the case E is a Klein four group (and in the case $|E|=3$ and $p=2$).

2. Reformed conjecture

First we want to reform Conjecture 1 in terms of a suitable k^*-central extension of E . Setting $\bar{N}_G(P,e) = N_G(P,e)/P$, $\bar{C}_G(P) = C_G(P)/P$ and denoting by $\bar{e}$ the image of e in $k\bar{C}_G(P)$, it is well known from Brauer that $k\bar{C}_G(P)\bar{e}$ is a simple k-algebra (i.e. a full matrix algebra over k) and , in particular, we have $Z(k\bar{C}_G(P)\bar{e}) \cong k$; hence, by Skolem-Noether's theorem , we have an exact sequence

$$1 \longrightarrow k^* \longrightarrow (k\bar{C}_G(P)\bar{e})^* \xrightarrow{\pi} \mathrm{Aut}(k\bar{C}_G(P)\bar{e}) \longrightarrow 1$$

so that $(k\bar{C}_G(P)\bar{e})^*$ can be seen as a k^*-central extension. Since $N_G(P,e)$ acts on $k\bar{C}_G(P)\bar{e}$, we have a group homomorphism ρ: $\bar{N}_G(P,e) \longrightarrow \mathrm{Aut}(k\bar{C}_G(P)\bar{e})$ and then $\hat{\bar{N}}_G(P,e)$ is the k^*-central extension of $\bar{N}_G(P,e)$ induced by $(k\bar{C}_G(P)\bar{e})^*$: that is to say, $\hat{\bar{N}}_G(P,e)$ is the subgroup of

$$(\bar{a}\ ,\ \bar{n}) \in (k\bar{C}_G(P)\bar{e})^* \times \bar{N}_G(P,e)$$

such that $\pi(\bar{a}) = \rho(\bar{n})$ and we get a commutative and exact diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & k^* & \longrightarrow & (k\bar{C}_G(P)\bar{e})^* & \xrightarrow{\pi} & \mathrm{Aut}(k\bar{C}_G(P)\bar{e}) & \longrightarrow & 1 \\
 & & \uparrow \mathrm{id} & & \uparrow \hat{\rho} & & \uparrow \rho & & \\
1 & \longrightarrow & k^* & \longrightarrow & \hat{\bar{N}}_G(P,e) & \longrightarrow & \bar{N}_G(P,e) & \longrightarrow & 1
\end{array}$$

Now , the twisted algebra $k_*\hat{\bar{N}}_G(P,e)$ is the quotient of the full group algebra by the ideal generated by the elements $\lambda(\bar{a}\ ,\ \bar{n}\ ) - (\lambda\bar{a}\ ,\ \bar{n}\ )$ where λ runs over k^* and $(\bar{a}\ ,\ \bar{n}\ )$ over $\hat{\bar{N}}_G(P,e)$. (We can define $O_*\hat{\bar{N}}_G(P,e)$, since there is a unique section $k^* \longrightarrow O^*$ of the canonical homomorphism $O^* \longrightarrow k^*$.) Moreover, we have an injective group homomorphism

$$\bar{C}_G(P) \longrightarrow \hat{\bar{N}}_G(P,e)$$

mapping $\bar{z} \in \bar{C}_G(P)$ on $(\bar{z}\bar{e}\ ,\ \bar{z}) \in \hat{\bar{N}}_G(P,e)$ and its image is a normal subgroup of $\hat{\bar{N}}_G(P,e)$ intersecting trivially the image of k^* , so the corresponding quotient is a k^*-central extension of E . We denote by $\hat{E}$ the opposite one ; that is to say , denoting by $\hat{\bar{N}}_G(P,e)^\circ$ the set $\hat{\bar{N}}_G(P,e)$ endowed with opposite product, we have the exact sequence

$$1 \longrightarrow \bar{C}_G(P) \longrightarrow \hat{\bar{N}}_G(P,e)^\circ \xrightarrow{\hat{\sigma}} \hat{E} \longrightarrow 1$$

where $\bar{z} \in \bar{C}_G(P)$ maps on $(\bar{z}\bar{e}\ ,\ \bar{z})^{-1}$. The following more or less known lemma explains the role of $\hat{E}$ (see also [9], Proposition 14.6 in [11] , Proposition 2.1 in [10] and Lemma 2.5 in [12]).

Lemma 1. With the notation above , there is an algebra isomorphism

$$k\bar{N}_G(P,e)\bar{e} \cong k\bar{C}_G(P)\bar{e} \otimes_k k_*\hat{E}$$

mapping $\bar{n}\bar{e}$ on $\hat{\rho}(\hat{\bar{n}}) \otimes \hat{\sigma}(\hat{\bar{n}})^{-1}$, where $\bar{n} \in \bar{N}_G(P,e)$ and $\hat{\bar{n}}$ is an element of $\hat{\bar{N}}_G(P,e)$ lifting $\bar{n}$.

Let $\hat{L}$ be the semidirect product of $\hat{E}$ and P . Since the number of isomorphism classes of simple $k_*\hat{L}$-modules is equal to the number of isomorphism classes of simple $k_*\hat{E}$-modules, we can reform Conjecture 1 by Lemma 1 as follos.

Conjecture 2. If P is abelian , then l(b) is the number of isomorphism classes of simple $k_*\hat{L}$-modules.

Hence we must study the relation between $OG\hat{b}$ and $O_*\hat{L}$,where $\hat{b}$ denotes the unique primitive idempotent lifting b to Z(OG) . We denote respectively by $L_K(\hat{L})$ and $L_K(G,b)$ the Grothendieck groups of the categories of $K_*\hat{L}$-modules and ordinary K-representations of G in b . We expect that there exists a special kind of bijective isometry between $L_K(\hat{L})$ and $L_K(G,b)$.

3. Preliminaries and the main theorem

Following [2] and [6] , we consider Brauer morphism Br_Q for a p-subgroup Q of G and (b,G)-Brauer pairs. Note that (P,e) is a maximal (b,G)-Brauer pair and for a p-subgroup Q of P, $(Q,e^{C_G(Q)})$ is a (b,G)-Brauer pair contained in (P,e). One of the typical properties of blocks with abelian defect groups is the following one.

Lemma 2.(Proposition 4.21 in [2]) Assume that P is abelian. If (Q,f) is a (b,G)-Brauer pair such that $(Q,f) \subset (P,e)$ and x an element of G such that $(Q,f)^x \subset (P,e)$, then there are $z \in C_G(Q)$ and $n \in N_G(P,e)$ such that x = zn . In particular, if U is a set of representatives for the orbits of E in P , then $\{(u, e^{C_G(u)})\}_{u \in U}$ is a set of representatives for the conjugacy classes of (b,G)-Brauer elements.

It is not difficult to handle $O_*\hat{L}$, since there are a finite subgroup L' of $\hat{L}$ and a p-block b' of L' such that the inclusion L' $\subset$ L induces a bijective isometry $L_K(\hat{L}) = L_K(L',b')$ and an algebra isomorphism $O_*\hat{L} \cong OL'\hat{b}'$ (see Remark 5 in section 1 in [8], Lemma 5.5 and Proposition 5.15 in [11]). Furthermore P is also a defect group of b' and E is also the inertial quotient of b', since P is the normal Sylow p-subgroup of L' and $(P,Br_P(b'))$ is the unique maximal (b',L')-Brauer pair. We remark that

(3.1) $(Q, Br_Q(b'))$ is the unique (b',L')-Brauer pair for a fixed p-subgroup Q of P .

From now on we introduce some general notation and results without any hypothesis until we state Theorem 1 (i.e. E is arbitrary and we do not assume that P is abelian for the moment). We denote by $CF_K(G)$ and $CF_O(G)$ the sets of K- and O-valued central functions over G, so that $CF_O(G) \subset CF_K(G)$, and we identify $CF_K(G)$ with $K \otimes_O CF_O(G)$ and with the set of central K-linear forms over KG (or OG). We denote respectively by $L_K(G)$ and $L_k(G)$ the Grothendieck groups of the categories of KG- and kG-modules (of finite dimension) and we identify $L_K(G)$ with its image in $CF_O(G)$. We also identify any element of $L_k(G)$ with its Brauer character; that is to say, denoting respectively by $BCF_K(G)$ and $BCF_O(G)$ (BCF for "Brauer central function") the sets of K- and O-valued G-central functions over the set $G_{p'}$ of elements of G of order prime to p , we also identify $L_k(G)$ with its image in $BCF_O(G)$ and $BCF_K(G)$ with $K \otimes_O BCF_O(G)$. Recall that the inclusion $L_k(G) \subset BCF_O(G)$ induces an isomorphism

(3.2) $$O \otimes_Z L_k(G) \cong BCF_O(G).$$

Following Brauer, we denote by

$$d_G : CF_K(G) \longrightarrow BCF_K(G)$$

the restriction map, which fulfills

(3.3) $$d_G(L_K(G)) = L_k(G) .$$

Moreover we denote by $CF_K^{\circ}(G)$ the kernel of d_G and set $L_K^{\circ}(G) = CF_K^{\circ}(G) \cap L_K(G)$. It is clear that d_G induces a bijection between the orthogonal subspace of $CF_K^{\circ}(G)$ and $BCF_K(G)$, and then the inverse map determines a section of d_G

$$e_G : BCF_K(G) \longrightarrow CF_K(G)$$

and induces an scalar product on $BCF_K(G)$; thus, d_G and e_G become adjoint maps.

More generally, following Broué [4], for any p-element u of G we consider the "twisted" restriction

$$d_G^u : CF_K(G) \longrightarrow BCF_K(C_G(u))$$

mapping $\chi \in CF_K(G)$ on the $C_G(u)$-central function over $C_G(u)_{p'}$ which maps $s \in C_G(u)_{p'}$ on $\chi(us)$, and denote by

$$e_G^u : BCF_K(C_G(u)) \longrightarrow CF_K(G)$$

the adjoint K-linear map , which is a section of d_G^u .

It is well-known that any idempotent of Z(kG) determines a selfadjoint projector over $CF_K(G)$ which stabilizes $CF_O(G)$ and $L_K(G)$, and commutes with $e_G \circ d_G$, so that it determines a self-adjoint projector over $BCF_K(G)$ stabilizing $BCF_O(G)$ and $L_k(G)$. In particular, for any element χ of $CF_K(G)$ or $BCF_K(G)$, we denote by $b.\chi$ the image of χ by the projector determined by b and set

$$b.CF_K(G) = CF_K(G,b) \text{ and } b.BCF_K(G) = BCF_K(G,b) .$$

Moreover, for any p-element u of G , we have (cf. [4] Appendixes)

(3.4) $$d_G^u(b.\chi) = Br_u(b).d_G^u(\chi) \text{ and } e_G^u(Br_u(b).\varphi) = b.e_G^u(\varphi)$$

for any $\chi \in CF_K(G)$ and any $\varphi \in BCF_K(C_G(u))$ (where $Br_u = Br_{\langle u \rangle}$).

Consequently, for any $\chi \in CF_K(G,b)$ and any (b,G)-Brauer element (u,g) we consider the central function

$$\chi^{(u,g)} = e_G^u(g.d_G^u(\chi))$$

which still belongs to $CF_K(G,b)$. Notice that we have

$$\chi^{(u,g)}(u) = \chi(u\hat{g}),$$

where $\hat{g}$ is the unique primitive idempotent of $Z(OC_G(u))$ lifting g. We remark that

$$(3.5) \qquad \chi = \sum_{(u,g)} \chi^{(u,g)}$$

and for any $\chi, \chi' \in CF_K(G,b)$ we get

$$(3.6) \qquad (\chi, \chi')_G = \sum_{(u,g)} (\chi^{(u,g)}, \chi'^{(u,g)})_G$$

where (u,g) runs over a set of representatives for the conjugacy classes of (b,G)-Brauer elements.

Following [6], a central function λ over P is called (G,e)-stable if, for any (b,G)-Brauer element (u,g) such that $(\langle u\rangle, g) \subset (P,e)$ and any $x \in G$ such that $(\langle u^x\rangle, g^x) \subset (P,e)$, we have $\lambda(u^x) = \lambda(u)$. In that case, for any $\chi \in CF_K(G,b)$, we consider the new central function

$$\lambda * \chi = \sum_{(u,g)} \lambda(u) \chi^{(u,g)}$$

where (u,g) runs over a set of representatives such that $(\langle u\rangle, g) \subset (P,e)$ for the conjugacy classes of (b,G)-Brauer elements, which still belongs to $CF_K(G,b)$ and does not depend on the choice of the set of representatives. We remark that

$$g.d_G^u(\lambda * \chi) = \lambda(u)(g.d_G^u(\chi)) .$$

Then, by the main result in [6], if λ and χ are generalized characters, so is $\lambda * \chi$. Notice that, by Lemma 2, if P is abelian, a central function over P is (G,e)-stable if and only if it is E-stable. We denote by $CF_O(P)^E$ the O-module of E-stable O-valued central functions over P .

We are ready to state our main theorem (Theorem 1.5 in [12]).

Theorem 1. With the notation above, assume that P is abelian and E is a Klein four group. Then there is a bijective isometry

$$\Delta : CF_O(\hat{L}) \longrightarrow CF_O(G,b)$$

such that

$$\Delta(L_K(\hat{L})) = L_K(G,b)$$

and

(3.7) $$\Delta(\lambda * \eta) = \lambda * \Delta(\eta)$$

for any $\lambda \in CF_O(P)^E$ and any $\eta \in CF_O(\hat{L})$.

(3.7) implies that Δ fulfills Definition 4.3 in [5] (i.e. (3.7) guarantees the existence of a local system in Broué's terms) and therefore, by Lemma 4.5 in [5], Δ is a perfect isometry in Broué's terms. (We discuss perfect isometries in section 5.) By Proposition 1.3 and Theorem 1.5 in [5] we have a following corollary.

Corollary 1. If P is abelian and E is a Klein four group , then the following hold with the notation of Theorem 1.

(i) Δ is a perfect isometry from $L_K(\hat{L})$ onto $L_K(G,b)$.

(ii) Δ induces a bijective isometry from $L_k(\hat{L})$ onto $L_k(G,b)$ and hence Alperin's weight conjecture (Conjecture 2) holds in this case.

(iii) The algebra isomorphism Δ^* from $Z(K_*\hat{L})$ onto $Z(KG\hat{b})$ determined by the isometry Δ maps $Z(O_*\hat{L})$ onto $Z(OG\hat{b})$.

(iv) Δ preserves the height of irreducible ordinary characters. In particular , all the irredicible ordinary characters of G in b have height zero and Alperin-McKay conjecture holds in this case.

(v) Δ preserves the elementary divisors of the Cartan matrices.

4. Local systems for blocks with abelian defect groups

Before we introduce Theorem 2 in this section , we assume only that P is abelian (and E is arbitrary).

By Lemma 2 E controles the fusion of (b,G)-Brauer pairs (resp. (b',L')-Brauer pairs). Then in the summation of (3.5) we have only to make u run over U.

Applying inductive method, we hope to construct a bijective isometry Δ_Q from $CF_O(C_{\hat{L}}(Q))$ onto $CF_O(C_G(Q),f)$ for each p-subgroup Q of P where $f = e^{C_G(Q)}$. (That is to say , first we construct it for Q=P and we hope to construct it for Q=1 finally.) We note that (P,e) is also a maximal $(f,C_G(Q))$-Brauer pair and (u,g) is a $(f,C_G(Q))$-Brauer element contained in it for any element u of P, where $g = e^{C_G(Q\langle u\rangle)}$. By Lemma 2 $C_E(Q)$ controles the fusion of $(f,C_G(Q))$-Brauer pairs , and by (3.5) for any $\chi \in CF_K(C_G(Q),f)$

$$(4.1) \qquad \chi = \sum_{u \in U_Q} e^u_{C_G(Q)} \, (g.d^u_{C_G(Q)}(\chi))$$

where U_Q is a set of representatives for the orbits of $C_E(Q)$ in P. Similarly we note that $(P, Br_P(b'))$ is the maximal $(Br_Q(b'), C_{L'}(Q))$-Brauer pair and $(u, Br_{Q\langle u\rangle}(b'))$ is a $(Br_Q(b'), C_{L'}(Q))$-Brauer element contained in it for any element u of P by (3.1). By Lemma 2 and (3.5) for any $\eta \in CF_K(C_{\hat{L}}(Q))$

(4.2) $$\eta = \sum_{u \in U_Q} e^u_{C_{\hat{L}}(Q)}(d^u_{C_{\hat{L}}(Q)}(\eta)),$$

since we can omit $Br_{Q\langle u\rangle}(b')$ by (3.1) and (3.4).

Let X be an E-stable non-empty set of subgroups of P and assume that X contains any subgroup of P containing an element of X. We call any map Γ (G,b)-local system over X, if Γ is defined over X and sends $Q \in X$ to a bijective isometry

$$\Gamma_Q : BCF_K(C_{\hat{L}}(Q)) \cong BCF_K(C_G(Q), f)$$

where $f = e^{C_G(Q)}$, and fulfills the following two conditions:

(4.3) For any $Q \in X$, any $\eta \in BCF_K(C_{\hat{L}}(Q))$ and any $s \in E$, we have

$$\Gamma_Q(\eta)^s = \Gamma_{Q^s}(\eta^s).$$

(4.4) For any $Q \in X$ and any $\eta \in L_K(C_{\hat{L}}(Q))$, the sum

$$\sum_{u \in U_Q} e^u_{C_G(Q)}(\Gamma_{Q\langle u\rangle}(d^u_{C_{\hat{L}}(Q)}(\eta)))$$

is a generalized character of $C_G(Q)$.

We examine these conditions to give more explicit expression. For any $Q \in X$ and any $\eta \in CF_K(C_{\hat{L}}(Q))$, the sum

(4.5) $$\Delta_Q(\eta) = \sum_{u \in U_Q} e^u_{C_G(Q)}(\Gamma_{Q\langle u\rangle}(d^u_{C_{\hat{L}}(Q)}(\eta)))$$

is certainly an element of $CF_K(C_G(Q), f)$ (cf. (3.4), (4.1) and (4.2)), and we have, setting $g = e^{C_G(Q\langle u\rangle)}$, (4.6) $$\Delta_Q(\eta)^{(u,g)} = e^u_{C_G(Q)}(\Gamma_{Q\langle u\rangle}(d^u_{C_{\hat{L}}(Q)}(\eta)))$$

and therefore, for any $\eta' \in CF_K(C_{\hat{L}}(Q))$ we get (cf.(3.6))

$$\begin{aligned}(\Delta_Q(\eta), \Delta_Q(\eta'))_{C_G(Q)} \\ = \sum_{u\in U_Q} (d^u_{C_{\hat{L}}(Q)}(\eta), d^u_{C_{\hat{L}}(Q)}(\eta'))_{C_{\hat{L}}(Q\langle u\rangle)} \\ = (\eta, \eta')_{C_{\hat{L}}(Q)}\end{aligned}$$

(recall that $e^u_{C_G(Q)}$ and $e^u_{C_{\hat{L}}(Q)}$ are isometries !). Hence for any $Q \in X$ we get a bijective isometry

(4.7) $$\Delta_Q = \sum_{u\in U_Q} e^u_{C_G(Q)} \circ \Gamma_{Q\langle u\rangle} \circ d^u_{C_{\hat{L}}(Q)}$$

from $CF_K(C_{\hat{L}}(Q))$ onto $CF_K(C_G(Q),f)$ and condition (4.3) insures that Δ_Q does not depend on the choice of U_Q whereas condition (4.4) demands that $L_K(C_G(Q),f)$ contains $\Delta_Q(L_K(C_{\hat{L}}(Q))$ which actually implies the equality

(4.8) $$\Delta_Q(L_K(C_{\hat{L}}(Q)) = L_K(C_G(Q),f)$$

since both members have orthonormal basis of the same cardinal (cf.(4.7)). Moreover, notice that $d_{C_G(Q)} \circ \Delta_Q = \Gamma_Q \circ d_{C_{\hat{L}}(Q)}$ (cf. (4.5)) and therefore we get (cf.(3.3) and (4.8))

$$\Gamma_Q(L_k(C_{\hat{L}}(Q))) = L_k(C_G(Q),f)$$

which then implies (cf.(3.2))

(4.9) $$\Gamma_Q(BCF_0(C_{\hat{L}}(Q))) = BCF_0(C_G(Q),f).$$

Consequently , since (4.9) is true for any $R \in X$ and the maps $d^u_{C_{\hat{L}}(R)}$ and $e^u_{C_G(R)}$ send 0-valued functions to 0-valued functions, we have for any $Q \in X$

(4.10) $$\Delta_Q(CF_0(C_{\hat{L}}(Q))) = CF_0(C_G(Q),f) .$$

An immediate consequence of the definition of Δ_Q, which does not depend on conditions (4.3) and (4.4), is that for any $Q \in X$, any $\lambda \in CF_K(P)^{C_E(Q)}$ and any $\eta \in CF_K(C_{\hat{L}}(Q))$ we have

$$(4.11) \qquad \Delta_Q(\lambda * \eta) = \lambda * \Delta_Q(\eta).$$

These (4.8),(4.10) and (4.11) show that for any $Q \in X$, Δ_Q (for $C_{\hat{L}}(Q)$ and $(C_G(Q),f)$) fulfills the similar conditions to Δ (for $\hat{L}$ and (G,b)) in Theorem 1. (Hence, if $1 \in X$, then $\Delta = \Delta_1$ is a required one in Theorem 1.) Since $C_{\hat{L}}(P) \cong k^* \times P$, we can easily prove that there are exactly two (G,b)-local systems defined over $\{P\}$ (cf.(4.11)). (Notice that ,up to sign, there is just one possibility for the isometry Γ_P .)

We want to extend X and Γ step by step. Assume that X does not contain all the subgroups of P and let Q be a subgroup of P which is maximal such that $Q \notin X$. We will discuss now a necessary and sufficient condition to extend Γ to a (G,b)-local system Γ' over the union X' of X and the E-orbit of Q. Since any subgroup R of P properly containing Q belongs to X, for any $u \in P - Q$ we still have the map (as in (4.6))

$$e^u_{C_G(Q)} \circ \Gamma_{Q\langle u\rangle} \circ d^u_{C_{\hat{L}}(Q)} \; : CF_K(C_{\hat{L}}(Q)) \longrightarrow CF_K(C_G(Q),f)$$

where $f = e^{C_G(Q)}$. We consider the sum

$$\Delta^\circ_Q = \sum_{u \in U_Q - Q} e^u_{C_G(Q)} \circ \Gamma_{Q\langle u\rangle} \circ d^u_{C_{\hat{L}}(Q)}$$

where , as above , U_Q is a set of representatives for the orbits of $C_E(Q)$ in P and by condition (4.3) again , Δ°_Q does not depend on the choice of U_Q .

Denote by $\bar{f}$ the image of f in $k\bar{C}_G(Q)$, where we set $\bar{C}_G(Q) =$

$C_G(Q)/Q$. We also set $\bar{C}_{\hat{L}}(Q) = C_{\hat{L}}(Q)/Q$. In [12] we proved following propositions (Proposition 3.7 and Proposition 3.11).

Proposition 1. With the notation and the hypothesis above , Δ_Q° induces a bijective isometry

$$\bar{\Delta}_Q^{\circ} : CF_K^{\circ}(\bar{C}_{\hat{L}}(Q)) \cong CF_K^{\circ}(\bar{C}_G(Q) , \bar{f})$$

such that $\bar{\Delta}_Q^{\circ}(L_K(\bar{C}_{\hat{L}}(Q)) \subset L_K^{\circ}(\bar{C}_G(Q) , \bar{f})$.

Proposition 2. With the notation and the hypothesis above, the (G,b)-local system Γ over X can be extended to a (G,b)-local system Γ' over X' if and only if the bijective isometry $\bar{\Delta}_Q^{\circ}$ can be extended to a $N_E(Q)$-stable bijective isometry

$$\bar{\Delta}_Q : CF_K(\bar{C}_{\hat{L}}(Q)) \cong CF_K(\bar{C}_G(Q) , \bar{f})$$

such that $\bar{\Delta}_Q(L_K(\bar{C}_{\hat{L}}(Q)) = L_K(\bar{C}_G(Q) , \bar{f})$.

Now we try to extend $\bar{\Delta}_Q^{\circ}$ to a $N_E(Q)$-stable bijective isometry $\bar{\Delta}_Q$. When E is a Klein four group , we obtain the following slightly stronger theorem (Theorem 4.2 in [12]) than Theorem 1. Unfortunately we do not succeed when E is cyclic of order 4 .

Theorem 2. If P is abelian and E is a Klein four group , then there is a (G,b)-local system over the set of all the subgroups of P.

5. Perfect isometry

In this section we introduce some Broué's terms. Let (H,f) (resp. (H',f')) be a pair of a finite group H and its block f (resp. a finite group H' and its block f').

Definition 1 (Definition 1.4 and Proposition 4.1 in [5]). A bijective isometry I from $L_K(H,f)$ onto $L_K(H',f')$ is called a perfect isometry if it induces a bijection from $CF_O(H,f)$ onto $CF_O(H',f')$ and a bijection from $BCF_K(H,f)$ onto $BCF_K(H',f')$. (We can extend I K-linearly.)

Such special kind of bijective isometry has various properties as follows.

Proposition 3 (Proposition 1.3 and Theorem 1.5 in [5]). If I is a perfect isometry from $L_K(H,f)$ onto $L_K(H',f')$, then the following hold.

(i) I induces a bijective isometry from $L_k(H,f)$ onto $L_k(H',f')$ and then $l(f) = l(f')$.

(ii) I induces a bijective isometry from the Z-module generated by the characters of projective $OH\hat{f}$-modules onto the Z-module generated by the characters of projective $OH'\hat{f}'$-modules.

(iii) The bijection between primitive idempotents of $ZKH\hat{f}$ and $ZKH'\hat{f}'$ defined by I induces an algebra isomorphism between $ZOH\hat{f}$ and $ZOH'\hat{f}'$.

(iv) I preserves the height of irreducible ordinary characters and the elementary divisors of the Cartan matrices.

Let (P,f_P) be a maximal (f,H)-Brauer pair and for any p-subgroup Q of P , let (Q,f_Q) be a (f,H)-Brauer pair contained in it.

Definition 2. Let $\mathbb{B}r_f(H)$ be the category whose objects are (f,H)-Brauer pairs and whose morphisms from (Q,f_Q) to (R,f_R) are the homomorphisms from Q to R induced by the inner automorphisms of G which send (Q,f_Q) onto a pair contained in (R,f_R). This is called the Brauer category of H for f.

Hypothesis for pairs (H,f) and (H',f') (Hypothesis 4.2 in [5]). We suppose that P is a defect group of f and f'. We also suppose that the inclusions of P in H and H' induce a equivalence of Brauer categories $\mathbb{B}r_f(H)$ and $\mathbb{B}r_{F'}(H')$.

Definition 3(Definition 4.3 in [5]). With the above Hypothesis, a linear map I from $CF_K(H,f)$ to $CF_K(H',f')$ is called compatible with the fusion, if for every cyclic subgroup $\langle u\rangle$ of P, there exists a linear map $I_{p'}^{\langle u\rangle}$ from $BCF_K(C_H(u),f_{\langle u\rangle})$ onto $BCF_K(C_{H'}(u), f'_{\langle u\rangle})$ such that

$$(f'_{\langle u\rangle}\cdot d^u_{H'})\circ I = I_{p'}^{\langle u\rangle}\circ(f_{\langle u\rangle}\cdot d^u_H).$$

Here the family $\left\{I_{p'}^{\langle u\rangle} \mid \langle u\rangle \subseteqq P\right\}$ is called the local system of I.

Definition 4 (Definition 4.6 and "good definition" in its

Remark in [5]). We say that the pair (H,f) and (H',f') are the same type (in "good definition") , if the following conditions are satisfied.

(i) The Brauer categories $\mathbb{Br}_f(H)$ and $\mathbb{Br}_{f'}(H')$ are equivalent.

(ii) There exists a family of perfect isometries

$$\left\{ I^Q \ : \ L_K(C_H(Q), f_Q) \longrightarrow L_K(C_{H'}(Q), f'_Q) \ \middle| \ Q \subseteqq P \right\}$$

such that if for any Q we denote by

$$I^Q_{p'} \ : \ BCF_K(C_H(Q), f_Q) \longrightarrow BCF_K(C_{H'}(Q), f'_Q)$$

the map induced by I^Q , then I^Q is compatible with the fusion and its local system is

$$\left\{ I^{Q\langle u\rangle}_{p'} \ \middle| \ \langle u\rangle \subseteqq C_P(Q) \right\} .$$

Broué conjectured that if b has an abelian defect group P , and (P,e) is a maximal (b,G)-Brauer pair , then (G,b) and $(N_G(P,e),e)$ are the same block type (Conjecture 6.1 in [5]). By Lemma 2 $\mathbb{Br}_b(G)$ and $\mathbb{Br}_e(N_G(P,e))$ are equivalent. Notice that by (4.5) for any p-subgroup Q of P and any $u \in U_Q$ we have

$$g \cdot d^u_{C_G(Q)} \circ \Delta_Q = \Gamma_{Q\langle u\rangle} \circ d^u_{C_{\hat{L}}(Q)}$$

and in particular, Γ_Q is the restriction of Δ_Q to $BCF_K(C_{\hat{L}}(Q))$. Then by (4.8),(4.10) and Theorem 2,this conjecture holds when E is a Klein four group (and it also holds when $|E| \lneqq 3$).

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