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Remarks on complete intersection

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Let (R,m) be a Noetherian local ring. The monomial conjecture asserts that for any integer $n \geqq 0$ and for any system of parameters $a_1,\ldots,a_d$ of R we have

$$(a_1a_2\ldots a_d)^n \notin (a_1^{n+1},\ldots,a_d^{n+1}) .$$

The monomial conjecture holds if R contains a field or $\dim R \leqq 2$. The purpose of this note is to show that the monomial conjecture is equivalent to the following property (P) of Gorenstein local rings.

(P) An ideal of height 0 is (0) if it is contained in a parameter ideal.

We begin with a reformulation of the monomial conjecture.
Let $a_1,\ldots,a_d$ be a system of parameters of a Noetherian local ring (R,m) and let $\underline{a}^n = (a_1^n,\ldots,a_d^n)$. We define an R-homomorphism

$$f_n : R/\underline{a}_n \to R/\underline{a}^{n+1}$$

by $f_n(1) = a_1a_2\ldots a_d \bmod \underline{a}^{n+1}$. Then the direct limit of the direct system $\{ R/\underline{a}^n ; n = 1,2,\ldots \}$ is the local cohomology module $H_m{}^d(R)$. Let

$$\Phi_n(\underline{a}) : R/\underline{a}_n \to H_m{}^d(R)$$

be the canonical homomorphism. Then $\Phi_1(\underline{a}) \neq 0$ if and only if

$$(a_1a_2\ldots a_d)^n \notin (a_1^{n+1},\ldots,a_d^{n+1})$$

for all $n \geqq 0$.

Let I be an ideal of R with $ht(I) = 0$. Since $(0 : I)$ is an R/I-module we have an isomorphism

$$(0 : I) \simeq (0 : I) \otimes_R R/I .$$

Hence we get an R-homomorphism

$$(0 : I) \otimes_R R/I \rightarrow R .$$

Tensoring this with the direct system $\{R/\underline{a}^n ; n = 1,2,....\}$, we get a commutative diagram

$$\begin{array}{ccc} (0 : I) \otimes_R R/(I , \underline{a}^n) & \rightarrow & R/\underline{a}^n \\ 1 \otimes \Psi_n(\underline{a}) \downarrow & & \downarrow \Phi_n(\underline{a}) \\ (0 : I) \otimes_R H_m{}^d(R/I) & \underset{\theta}{\rightarrow} & H_m{}^d(R) \end{array} ,$$

where $\Psi_n(\underline{a}) : R/(I , \underline{a}^n) \rightarrow H_m{}^d(R/I)$ is the canonical homomorphism. θ induces a homomorphism

$$\theta^* : H_m{}^d(R/I) \rightarrow Hom_R((0 : I) , H_m{}^d(R)) .$$

Lemma 1. If R is Gorenstein then θ^* is an isomorphism.

Proof. Since R is Gorenstein, $H_m{}^d(R)$ is isomorphic to the injective envelope of the residue field R/m . The lemma follows from the local duality.

In order to characterize unmixed ideals in a Gorenstein local ring, we have:

Lemma 2. Let (R, m) be a Gorenstein local ring and I an ideal of height 0. Then I is unmixed if and only if $(0 : (0 : I)) = I$.

Proof. If $(0 : (0 : I)) = I$ it is clear that I is unmixed. Conversely, suppose that I is unmixed. For any $p \in \mathrm{Ass}(R/I)$, R_p is a 0-dimensional Gorenstein local ring and we have $(0 : (0 : I))R_p = IR_p$. Since I is unmixed, we have $(0 : (0 : I)) = I$.

Now we are ready to prove:

Theorem 3. The following statements are equivalent:

(1) The monomial conjecture holds.

(2) Every Gorestein local ring has the property (P).

Proof. (1) ⇒ (2): Let (R, m) be a Gorenstein local ring and I an unmixed ideal of height 0. Suppose that $I \neq (0)$ and let $J = (0 : I)$. Since I is unmixed we have $I = (0 : J)$, by lemma 2. Let $\underline{a} = a_1, \ldots, a_d$ be a system of parameters of R. We have a commutative diagram

$$\begin{array}{ccc} I \otimes_R R/(J, \underline{a}) & \rightarrow & R/\underline{a} \\ \downarrow & & \downarrow \\ I \otimes_R H_m^d(R/J) & \rightarrow & H_m^d(R) . \end{array}$$

By lemma 1, we have an isomorphism

$$H_m^d(R/J) \rightarrow \mathrm{Hom}_R(I, H_m^d(R)) .$$

The image α of the identity of $R/(J, \underline{a})$ in $H_m^d(R/J)$ is not trivial by the monomial conjecture. α induces a non-trivial homomorphism

$$\alpha^* : I \rightarrow H_m{}^d(R) .$$

From the above commutative diagram, we see that I is not contained in $\underline{a}$.

(2) ⇒ (1) : Suppose that the monomial conjecture is not true. Then, there is a Noetherian local ring A and a system of parameters of $a_1,\dots,a_d$ of A such that

$$(a_1a_2\dots a_d)^n \in (a_1{}^{n+1},\dots,a_d{}^{n+1})$$

for some $n \geqq 0$. We may assume that A is a complete local domain. There is a Gorenstein local ring R and an ideal I of R such that $A = R/I$. We can assume that $\dim A = \dim R$. Since the monomial conjecture does not hold for A , we have $I \neq (0)$. Let $J = (0 : I)$. If the monomial conjecture does not holds for R/I there is a system of parameters $x_1,\dots,x_d$ of R such that $J \subset \underline{x}$,by lemma 1. By assumption (2), we get $J = (0)$,but this is impossible .

Corollary 4. Let R be a Gorenstein local ring containing a field and let I be an unmixed ideal of R . Suppose that there is a sytem of parameters $a_1,\dots,a_d$ of R such that $(a_1,\dots,a_h) \subset I \subset (a_1,\dots,a_d)$,with $ht(I) = h$. Then $I = (a_1,\dots,a_h)$.

Example (J.R. Strooker and J. Stückrad). Let k be a field and $R = k[[X,Y,U,V]]/(XY - UV,V^2,YV) = k[[x,y,u,v]]$. Then R is a Cohen-Macaulay local ring but not Gorenstein. (y^2) is an unmixed ideal of R and contained in a parameter ideal $(x + y,u)$.

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References

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