数理解析研究所講究録
第 801 巻 1992 年 163-173



# Symbolic Powers, Rees Algebras and Applications

西村 純一（京大理）

## Introduction.

局所環 $A$ の完備化 $\hat{A}$ の任意の $\hat{\mathfrak{p}} \in \mathrm{Ass}(\hat{A})$ 、或いは任意の $\hat{\mathfrak{p}} \in \mathrm{Min}(\hat{A})$ に対し、$\dim \hat{A}/\hat{\mathfrak{p}} = \dim A$ であるとき、$A$ は清純 (unmixed) 、或いは擬清純 (quasi-unmixed) という。よく知られているように、局所整域 $A$ が強鎖状 (universally catenary) である事と、$A$ が擬清純である事とは同値である。従って、$A$ が擬清純であれば、$A$ の任意の素イデアル $P$ に対し、$A/P$ も擬清純である。が、$A$ が清純であれば、$A$ の任意の素イデアル $P$ に対し、$A/P$ も清純であるかどうか、は長い間未解決であった。
しかし、Brodmann–Rotthaus [2] と Ogoma [11] は、ほぼ同時独立に、反例を構成した。我々は、ここで彼らの例を統一的に再構成する。
第 1 節では、先ず Hochster [5] に従い、ネーター環の素イデアルの $n$ 乗と記号的 $n$ 乗とがすべての $n$ で一致するための判定条件を述べる。その応用として、剰余環が Cohen–Macaulay ではないが、$n$ 乗と記号的 $n$ 乗とがすべての $n$ で一致する素イデアルの例を考察する。この例は、最終節で利用される。
第 2 、3 節では、Rotthaus [12] 、Ogoma [10] 、Heitmann [4] 、Brodmann–Rotthaus [1] 等に負う、与えられた完備局所環が完備化である局所整域の基本構成法、及び、特異な素元を持つ局所整域の構成法を、簡単に復習する。
最終第 4 節は、前節までの応用として、Brodmann–Rotthaus 、Ogoma の例の出来るだけ分かり易い再構成を与える。

## 1 Hochster's Theorem and Example.

Throughout this paper, all rings are commutative with identity. We use fully the notation and terminology of EGA [3], Matsumura [7] and Nagata [8]. The set of natural numbers and that of non-negative integers are denoted respectively by $\mathbf{N}$ and $\mathbf{N}_0$.

Throughout this section, $R$ denotes a Noetherian domain, $P = (p_1, \ldots, p_k)$ a prime ideal of $R$ and $\boldsymbol{p}$ the $k$-tuple $p_1, \ldots, p_k$. As mentioned in Introduction, we first recall Hochster's criteria for the equality of ordinal and symbolic powers of a prime ideal $P$ of a Noetherian domain $R$, namely, criteria for the equality $P^n = P^{(n)} = P^n R_P \cap R$ for any $n$. Then, for the later use, we review Hochster's third example of a height 2 prime ideal $P$ of a polynomial ring $R$ in 4 variables over a field $K$ such that $R/P$ is not Cohen–Macaulay but $P^n = P^{(n)}$ for any $n$.

To state Hochster's criteria, we fix notation. If $P = (p_1, \ldots, p_k)$ is a prime ideal of a Noetherian domain $R$, taking $k+1$ algebraically independent indeterminates $t_1, \ldots, t_k, q$ over $R$, we set $S = R[t_1, \ldots, t_k]$. We define an increasing sequence of ideals of $S$ recursively as follows:

$$J_0(\boldsymbol{p}) = (0),\ J_{n+1}(\boldsymbol{p}) = \{\textstyle\sum_{i=1}^k s_i t_i \mid s_i \in S,\ \sum_{i=1}^k s_i p_i \in J_n(\boldsymbol{p})\} + J_n(\boldsymbol{p}),$$
$$J(\boldsymbol{p}) \ = \textstyle\bigcup_n J_n(\boldsymbol{p}).$$

**Theorem 1.1.** (Hochster [5]) *The following conditions on a prime ideal $P = (p_1, \ldots, p_k)$ of a Noetherian domain $R$ are equivalent:*

(1.1.1) *$P^n = P^{(n)}$ for every positive integer $n$, and the associated graded ring of $R_P$ is a domain.*

(1.1.2) *$PS + J(\boldsymbol{p})$ is prime.*

(1.1.3) *For some integer $n > 0$, $PS + J_n(\boldsymbol{p})$ is a prime of height $k$. In this case, $PS + J_n(\boldsymbol{p}) = PS + J(\boldsymbol{p})$.*

(1.1.4) *There is a height $k$ prime $Q$ of $S$ such that $Q \subset PS + J(\boldsymbol{p})$. In this case, $Q = PS + J(\boldsymbol{p})$.*

(1.1.5) *$q$ is a prime element in the subring $R[q, p_1/q, \ldots, p_k/q]$ of $R[q, 1/q]$.*

As applications of the theorem above, Hochster observed three examples of prime ideals whose ordinal and symbolic powers are equal. The first example is a prime ideal generated by an $R$-sequence, the second one is the prime ideal generated by the $k$ by $k$ minors of a $k$ by $k+1$ matrix of indeterminates over a field. Though they are interesting as well, we shall only look close at the third one for our later purpose.

Let $X, Y$ be indeterminates over a field $K$. Set $A = K[X, XY, Y^2, Y^3]$, which is not Cohen–Macaulay. Let $x, z_1, z_2, z_3$ be indeterminates over $K$ and set $R = K[x, z_1, z_2, z_3]$. Let $\phi\colon R \to A$ be the $K$-homomorphism which maps $x, z_1, z_2, z_3$ to $X, XY, Y^2, Y^3$, respectively. Let $P = \operatorname{Ker}\phi$. Then

**Example 1.2** (cf.[5, p.61]). $P = (z_2^3 - z_3^2, z_2x^2 - z_1^2, z_2z_1 - xz_3, z_2^2x - z_3z_1) = (p_1, p_2, p_3, p_4)$ is a height 2 prime ideal of $R$ where $R/P$ is not Cohen–Macaulay but $P^n = P^{(n)}$ for every positive integer $n$.

Indeed, $J_1(\boldsymbol{p})$ contains $a = xt_1 - z_3t_3 - z_2t_4, b = z_1t_1 - z_2^2t_3 - z_3t_4, c = z_2t_2 + z_1t_3 - xt_4$ and $d = z_3t_2 + z_2xt_3 - z_1t_4$. Then, $e = t_1t_2 + z_2t_3^2 - t_4^2 \in J_2(\boldsymbol{p})$. Hence, $Q = (p_1, p_2, p_3, p_4, a, b, c, d, e)S \subset PS + J_2(\boldsymbol{p}) \subset PS + J(\boldsymbol{p})$. Hochster shows that $Q$ is a height 4 prime ideal of $S = R[t_1, t_2, t_3, t_4]$.

We end the first section by notifying that the five relations $a, b, c, d, e$ above appear again in a very crucial step of our reconstruction of Brodmann–Rotthaus peculiar unmixed local domain given below.

## 2 Construction of Peculiar Local Domains.

**2.0 Notation.** Let $K_0$ be a countable field, for example, $\mathbf{Q}$ the field of rational numbers, $\mathbf{F}_q$ the finite field with $q$ elements, or $\overline{\mathbf{F}}_p$ the algebraic closure of the prime field of characteristic $p > 0$, etc..., and let $K$ be a purely transcendental extension field of countable degree over $K_0$, that is, $K = K_0(a_{ik})$ with transcendental basis $\{a_{ik} \mid i = 1, \ldots, n;\ k = 1, 2, \ldots\}$. Further, for any $k \in \mathbf{N}$, let

(2.0.1) $$K_k = K_{k-1}(a_{1k}, \ldots, a_{nk}) = K_{k-1}(a_{ik}) \text{ and } K = \bigcup_k K_k\,.$$

Let $u$, $z_1, \ldots,$ $z_n$ be $n+1$ indeterminates over $K$, and set

(2.0.2) $$S_0 = K_0[u, z_1, \ldots, z_n] \text{ and } \mathfrak{N}_0 = (u, z_1, \ldots, z_n)S_0,$$

(2.0.3) $$S_k = S_{k-1}[a_{1k}, \ldots, a_{nk}] \text{ and } \mathfrak{N}_k = (u, z_1, \ldots, z_n)S_k.$$

Then, $S_k = K_0[a_{ih}, u, z_1, \ldots, z_n]$ with $i = 1, \ldots, n$ and $1 \leq h \leq k$. Further, we set

$$S = \bigcup_k S_k = K_0[a_{ik}, u, z_1, \ldots, z_n] \text{ and } \mathfrak{N} = (u, z_1, \ldots, z_n)S, \tag{2.0.4}$$

where $i = 1, \ldots, n$; $k = 1, 2, \ldots$. Moreover, let

$$R_0 = (S_0)_{\mathfrak{N}_0} = K_0[u, z_1, \ldots, z_n]_{(u,z_1,\ldots,z_n)} \text{ with } \mathfrak{n}_0 = (u, z)R_0, \tag{2.0.5}$$

$$R = S_{\mathfrak{N}} = K[u, z_1, \ldots, z_n]_{(u,z_1,\ldots,z_n)} \text{ with } \mathfrak{n} = (u, z_1, \ldots, z_n)R. \tag{2.0.6}$$

Then $R$ is a countable regular local ring. With notation as above, let

$$\mathcal{P} = \left\{ p \in S \;\middle|\; \begin{array}{l} \text{for each height one } \mathfrak{p} \in \operatorname{Spec}(R), \text{ there is} \\ \text{at least one element } p \text{ such that } p \in \mathfrak{p} \end{array} \right\}. \tag{2.0.7}$$

Then, $\mathcal{P}$ is a *countable* set and we may assume that $u \in \mathcal{P}$ and that $\mathcal{P}$ contains an infinite number of elements of $S_0$.

**2.1 Numbering.** Now we come to an important lemma due to Heitmann [4], which guarantees a *good* enumeration on $\mathcal{P}$.
With notation and assumptions above, we fix a surjective mapping $\rho\colon \mathbf{N} \to \mathcal{P}$, called a *numbering* on $\mathcal{P}$, which, if we set $\rho(i) = p_i$, satisfies the following:

$$p_1 = u, \text{ and } p_i \in S_{i-2} \text{ for any } i \geq 2. \tag{2.1.1}$$

We define:

$$q_k = p_1 \cdots p_k, \tag{2.1.2}$$

$$z_{i0} = z_i \text{ and } z_{ik} = z_i + a_{i1}q_1 + \cdots + a_{ik}q_k^k. \tag{2.1.3}$$

Then $P_k = (z_{1k}, \ldots, z_{nk})R$ is a prime ideal of height $n$ for any $k \geq 0$.

**Lemma 2.2 (Heitmann's Lemma** [4]**).** *Notation being as above, let $\rho$ be a numbering on $\mathcal{P}$ which satisfies* (2.1.1). *Then*

(2.2.1) $\quad p_h \notin P_k$ *whenever* $k \geq h - 1$,

(2.2.2) $\quad (z_{1k}, \ldots, z_{mk})S_k$ *is a prime ideal, generated by an $S_k$-regular sequence $z_{1k}, \ldots, z_{mk}$ for any $m$ $(1 \leq m \leq n)$.*

**2.3 Relations.** Taking polynomials in $n$ variables over $K_0$ without constant term: $F_1(Z), \dots, F_r(Z) \in K_0[Z_1, \dots, Z_n]$, we set

$$\alpha_{jk} = \frac{1}{q_k^k} F_j(z_{1k}, \dots, z_{nk}) \in L = Q(R),\ j = 1, \dots, r. \tag{2.3.1}$$

Then, by definition

$$\begin{aligned} \alpha_{j(k+1)} &= \frac{1}{q_{k+1}^{k+1}} F_j(z_{1(k+1)}, \dots, z_{n(k+1)}) \\ &= \frac{1}{q_{k+1}^{k+1}} F_j(z_{1k} + a_{1(k+1)} q_{k+1}^{k+1}, \dots, z_{nk} + a_{n(k+1)} q_{k+1}^{k+1}). \end{aligned}$$

Thus

$$\alpha_{jk} = \frac{q_{k+1}^{k+1}}{q_k^k} \alpha_{j(k+1)} + \frac{q_{k+1}^{k+1}}{q_k^k} r_{jk} \text{ with } r_{jk} \in R. \tag{2.3.2}$$

Let $B = \bigcup_k R[\alpha_{jk}] \subset L$ with $j = 1, \dots, r$ and $k = 1, 2, \dots$. Then

**Lemma 2.4.** *With notation as above, let $M = (u, z_1, \dots, z_n)B$. Then $M$ is a maximal ideal of $B$.*

Notation being as above, for $i = 1, \dots, n$ and $j = 1, \dots, r$, we set

$$\begin{aligned} \zeta_i &= z_i + a_{i1} q_1 + \dots + a_{ik} q_k^k + \dots = z_i + \textstyle\sum_{k=1}^{\infty} a_{ik} q_k^k, \\ f_j &= F_j(\zeta_1, \dots, \zeta_n) \in \hat{R} = K[[u, z_1, \dots, z_n]] = K[[u, \zeta_1, \dots, \zeta_n]]. \end{aligned}$$

Let $A = B_M \subset L$ be a quasi-local domain with maximal ideal $\mathfrak{m} = MA$. Then

**Theorem 2.5.** $(A, \mathfrak{m})$ *is a* Noetherian *local domain which satisfies the following conditions:*

$$\tilde{\imath}\colon K[[u, \zeta_1, \dots, \zeta_n]]/(F_1(\zeta), \dots, F_r(\zeta)) \cong \hat{R}/(f_1, \dots, f_r) \xrightarrow{\sim} \hat{A}, \tag{2.5.1}$$

$$\hat{\mathfrak{p}} = (\tilde{\imath}(\zeta_1), \dots, \tilde{\imath}(\zeta_n))\hat{A} \text{ is a prime ideal of } \hat{A} \text{ and } \hat{\mathfrak{p}} \cap A = (0), \tag{2.5.2}$$

(2.5.3) $A/\mathfrak{p}$ *is essentially of finite type over $K$ for any non-zero prime* $\mathfrak{p} \in \operatorname{Spec}(A)$.

## 3 Local Domains with Odd Non-zero Primes.

To begin with, we make a minor change of the previous notation.

**3.0 Notation.** Let $K_0$, $K$ and $K_k$ be as in (2.0.1). Take $n+2$ indeterminates $u$, $x$, $z_1, \dots, z_n$ over $K$, and set

$$(3.0.1) \qquad S_0 = K_0[u, x, z_1, \dots, z_n] \text{ and } \mathfrak{N}_0 = (u, x, z_1, \dots, z_n)S_0,$$

$$(3.0.2) \qquad S_k = S_{k-1}[a_{1k}, \dots, a_{nk}] \text{ and } \mathfrak{N}_k = (u, x, z_1, \dots, z_n)S_k.$$

Then, $S_k = K_0[a_{ih}, u, x, z_1, \dots, z_n]$ with $i = 1, \dots, n$ and $1 \le h \le k$. Further, we set

$$(3.0.3) \qquad S = \bigcup_k S_k = K_0[a_{ik}, u, x, z_1, \dots, z_n] \text{ and } \mathfrak{N} = (u, x, z_1, \dots, z_n)S,$$

where $i = 1, \dots, n$; $k = 1, 2, \dots$. Moreover, let

$$(3.0.4) \qquad R_0 = (S_0)_{\mathfrak{N}_0} = K_0[u, x, z_1, \dots, z_n]_{(u,x,z)} \text{ with } \mathfrak{n}_0 = (u, x, z)R_0,$$

$$(3.0.5) \qquad R = S_{\mathfrak{N}} = K[u, x, z_1, \dots, z_n]_{(u,x,z)} \text{ with } \mathfrak{n} = (u, x, z_1, \dots, z_n)R.$$

Then $R$ is a countable regular local ring.

**3.1 Numbering.** Putting $\mathrm{Spec}(R)^* = \mathrm{Spec}(R) \setminus \{xR\}$, let

$$(3.1.1) \qquad \mathcal{P}^* = \left\{ p \in S \;\middle|\; \begin{array}{l} \text{for each height one } \mathfrak{p} \in \mathrm{Spec}(R)^*, \text{ there is} \\ \text{at least one element } p \text{ such that } p \in \mathfrak{p} \text{ and} \\ p \notin xR \end{array} \right\}.$$

Then, $\mathcal{P}^*$ is a *countable* set and we may assume that $u \in \mathcal{P}^*$. Denoting by $\overline{s}$ the image of $s \in S$ in $\overline{S} = S/xS$ (or in $\overline{R} = R/xR$), we may further assume that $\overline{\mathcal{P}} = \{\overline{p} \in \overline{S} \mid p \in \mathcal{P}^*\}$ satisfies the same condition as in (2.0.7), namely,

$$(3.1.2) \qquad \overline{\mathcal{P}} = \left\{ \overline{p} \in \overline{S} \;\middle|\; \begin{array}{l} \text{for each height one } \overline{\mathfrak{p}} \in \mathrm{Spec}(\overline{R}), \text{ there is} \\ \text{at least one element } \overline{p} \text{ such that } \overline{p} \in \overline{\mathfrak{p}} \end{array} \right\}.$$

Next, we fix a surjective mapping $\rho^* \colon \mathbf{N} \to \mathcal{P}^*$ with $\rho^*(i) = p_i$, a *numbering* on $\mathcal{P}^*$, which satisfies the following:

$$(3.1.3) \qquad p_1 = u, \text{ and } p_i \in S_{i-2} \text{ for any } i \ge 2.$$

Moreover, we remark that, if $\rho^*$ is the numbering above, then the induced mapping $\overline{\rho}: \mathbf{N} \to \overline{\mathcal{P}}$, which maps $i$ to $\overline{p}_i$, is also an enumeration on $\overline{\mathcal{P}}$ satisfying:

(3.1.4) $\quad \overline{p}_1 = \overline{u}$, and $\overline{p}_i \in \overline{S}_{i-2} = S_{i-2}/xS_{i-2}$ for any $i \geq 2$.

Hence, $\overline{\rho}$ becomes a numbering on $\overline{\mathcal{P}}$. As in Section 2, we define:

(3.1.5) $\quad z_{i0} = z_i$ and $z_{ik} = z_i + a_{i1}q_1 + \cdots + a_{ik}q_k^k$ with $q_k = p_1 \cdots p_k$.

Then $Q_k = (x, z_{1k}, \ldots, z_{nk})R$ is a prime ideal of height $n+1$ for any $k \geq 0$. And the same proof as in Lemma 2.2 shows:

**Lemma 3.2.** *With notation as above, if $\rho^*$ is a numbering on $\mathcal{P}^*$ which satisfies* (3.1.3)*, then*

(3.2.1) $\quad p_h \notin Q_k$ *whenever* $k \geq h-1$,

(3.2.2) $\quad (x, z_{1k}, \ldots, z_{mk})S_k$ *is a prime ideal, generated by an* $S_k$*–regular sequence* $x$, $z_{1k}$, $\ldots$, $z_{mk}$ *for any* $m\,(1 \leq m \leq n)$.

**3.3 Relations.** Taking polynomials in $n+1$ variables over $K_0$ without constant term: $G_1(X,Z), \ldots, G_r(X,Z) \in K_0[X, Z_1, \ldots, Z_n]$, we set

(3.3.0) $\quad F_j(Z) = G_j(0, Z) \in K_0[Z_1, \ldots, Z_n]$,

(3.3.1) $$\beta_{jk} = \frac{1}{q_k^k} G_j(x, z_{1k}, \ldots, z_{nk}) \in L = Q(R),\ j = 1, \ldots, r.$$

Then, by definition

$$\begin{aligned}\beta_{j(k+1)} &= \frac{1}{q_{k+1}^{k+1}} G_j(x, z_{1(k+1)}, \ldots, z_{n(k+1)}) \\ &= \frac{1}{q_{k+1}^{k+1}} G_j(x, z_{1k} + a_{1(k+1)}q_{k+1}^{k+1}, \ldots, z_{nk} + a_{n(k+1)}q_{k+1}^{k+1}).\end{aligned}$$

Thus

(3.3.2) $$\beta_{jk} = \frac{q_{k+1}^{k+1}}{q_k^k}\beta_{j(k+1)} + \frac{q_{k+1}^{k+1}}{q_k^k}s_{jk} \text{ with } s_{jk} \in R.$$

Let $B = \bigcup_k R[\beta_{jk}] \subset L$ with $j = 1, \ldots, r$ and $k = 1, 2, \ldots$. Then, the same reasoning as in Lemma 2.4 gives:

**Lemma 3.4.** *Notation being as above, let $M = (u, x, z_1, \ldots, z_n)B$. Then $M$ is a maximal ideal of $B$ and $B/M \cong R/\mathfrak{n} \cong K$.*

With notation as above, for $i = 1, \ldots, n$ and $j = 1, \ldots, r$, we set

$$\begin{aligned} \zeta_i &= z_i + a_{i1}q_1 + \ldots + a_{ik}q_k^k + \ldots = z_i + \textstyle\sum_{k=1}^{\infty} a_{ik}q_k^k, \\ g_j &= G_j(x, \zeta_1, \ldots, \zeta_n) \in \hat{R} = K[[u, x, z_1, \ldots, z_n]] = K[[u, x, \zeta_1, \ldots, \zeta_n]], \\ f_j &= F_j(\zeta_1, \ldots, \zeta_n) \in \hat{R}/x\hat{R} = K[[u, z_1, \ldots, z_n]] = K[[u, \zeta_1, \ldots, \zeta_n]]. \end{aligned}$$

Let $A = B_M \subset L$ be a quasi-local domain with its maximal ideal $\mathfrak{m} = MA$. On the other hand, let $\tilde{\phi}$, $\phi$ be ring homomorphisms:

$$\begin{aligned} &\tilde{\phi}: K_0[U, X, Z][T_1, \ldots, T_r] \to K_0[U, X, Z][\frac{G_1}{U}, \ldots, \frac{G_r}{U}] \text{ with } T_j \mapsto G_j/U, \\ &\phi: K_0[U, Z][T_1, \ldots, T_r] \to K_0[U, Z][\frac{F_1}{U}, \ldots, \frac{F_r}{U}] \text{ with } T_j \mapsto F_j/U. \end{aligned}$$

Under the circumstances, we get:

**Theorem 3.5.** *Regarding $K_0[U, Z]$ as $K_0[U, X, Z]/XK_0[U, X, Z]$, suppose that*

(3.5.0) $\quad \operatorname{Ker} \phi = K_0[U, Z] \otimes_{K_0[U,X,Z]} \operatorname{Ker} \tilde{\phi}$.

*Then, $(A, \mathfrak{m})$ is a Noetherian local domain with* prime *element $x$ which satisfies the following conditions:*

(3.5.1) $\quad \tilde{\imath}: K[[u, x, \zeta]]/(G_1(x, \zeta), \ldots, G_r(x, \zeta)) \cong \hat{R}/(g_1, \ldots, g_r) \xrightarrow{\sim} \hat{A}$,

(3.5.2) $\quad \tilde{\bar{\imath}}: K[[u, \zeta]]/(F_1(\zeta), \ldots, F_r(\zeta)) \cong (R/xR)^\wedge/(f_1, \ldots, f_r) \xrightarrow{\sim} \hat{A}/x\hat{A}$,

(3.5.3) $\quad \hat{\mathfrak{q}} = (\tilde{\imath}(x), \tilde{\imath}(\zeta_1), \ldots, \tilde{\imath}(\zeta_n))\hat{A}$ *is a prime ideal and* $\hat{\mathfrak{q}} \cap A = xA$,

(3.5.4) $\quad A/\mathfrak{p}$ *is essentially of finite type over $K$ for any non-zero prime* $\mathfrak{p} \in \operatorname{Spec}(A) \setminus \{xA\}$.

## 4 Brodmann - Rotthaus and Ogoma's Examples.

In this section, we start with showing that Brodmann–Rotthaus example (Example 4.1) can be gained as a joint application of Theorem 3.5 and Example 1.2. Next, we shall remark that Ogoma's example (Example 4.2) can be reproduced in the same manner. The crucial point in our reconstruction of these examples is to check the condition (3.5.0). Of course, even though they mentioned implicitly, the same is one of the essential and hard parts of their original work. Namely, Brodmann–Rotthaus use Hochster's relations $a, b, c, d, e$ in Example 1.2 to get (3.5.0). As well, Ogoma wisely calculates the *Kernels* by hand. Nevertheless, we should announce that MACULAY [6] gives us the same result automatically.

**Example 4.1** ([2]). Three dimensional analytically irreducible local domain $A$, hence unmixed, but has $\mathfrak{p} \in \operatorname{Spec}(A)$ such that $A/\mathfrak{p}$ is *not* unmixed.

*Construction.* With notation as in Theorem 3.5, take

$$\begin{aligned} G_1(X,Z) &= Z_2^3 - Z_3^2, & G_2(X,Z) &= Z_2X^2 - Z_1^2, \\ G_3(X,Z) &= Z_2Z_1 - XZ_3, & G_4(X,Z) &= Z_2^2X - Z_3Z_1. \end{aligned}$$

Here, MACAULAY gives us (cf. Example 1.2):

$$\begin{aligned} \operatorname{Ker}\tilde{\phi} = (&UT_1 - G_1,\, UT_2 - G_2,\, UT_3 - G_3,\, UT_4 - G_4, \\ &XT_1 - Z_3T_3 - Z_2T_4,\, Z_1T_1 - Z_2^2T_3 - Z_3T_4,\, Z_2T_2 + Z_1T_3 - XT_4, \\ &Z_3T_2 + Z_2XT_3 - Z_1T_4,\, T_1T_2 + Z_2T_3^2 - T_4^2), \\ \operatorname{Ker}\phi = (&UT_1 - G_1,\, UT_2 - G_2,\, UT_3 - G_3,\, UT_4 - G_4, \\ &- Z_3T_3 - Z_2T_4,\, Z_1T_1 - Z_2^2T_3 - Z_3T_4,\, Z_2T_2 + Z_1T_3, \\ &Z_3T_2 - Z_1T_4,\, T_1T_2 + Z_2T_3^2 - T_4^2). \end{aligned}$$

Consequently, $\operatorname{Ker}\phi = K_0[U,Z] \otimes_{K_0[U,X,Z]} \operatorname{Ker}\tilde{\phi}$. Therefore, by Theorem 3.5, we get a local domain $(A, \mathfrak{m})$ with *prime* element $x$ which satisfies the following conditions:

(4.1.1) $\quad \hat{A} \xrightarrow{\sim} K[[u,x,\zeta]]/(G_1(x,\zeta), \ldots, G_r(x,\zeta)) \cong K[[u,x,xy,y^2,y^3]],$

(4.1.2) $\quad \hat{A}/x\hat{A} \cong K[[u,x,xy,y^2,y^3]]/xK[[u,x,xy,y^2,y^3]].$

**Example 4.2** ([11]). Three dimensional analytically unramified unmixed local domain $A$, which has $\mathfrak{p} \in \operatorname{Spec}(A)$ such that $A/\mathfrak{p}$ is *not* unmixed.

*Construction.* Notation being as in Theorem 3.5, take

$$\begin{aligned} G_1(X,Z) &= Z_1Z_3, \quad G_2(X,Z) = Z_1(X+Z_2), \\ G_3(X,Z) &= Z_2Z_3, \quad G_4(X,Z) = Z_2(X+Z_2). \end{aligned}$$

Then, by MACAULAY, we get:

$$\begin{aligned} \operatorname{Ker}\tilde{\phi} = (&UT_1 - G_1,\, UT_2 - G_2,\, UT_3 - G_3,\, UT_4 - G_4, \\ &(X+Z_2)T_1 - Z_3T_2,\, (X+Z_2)T_1 - Z_1T_3, \\ &(X+Z_2)T_2 - Z_1T_4,\, (X+Z_2)T_3 - Z_3T_4,\, T_1T_4 - T_2T_3), \\ \operatorname{Ker}\phi = (&UT_1 - G_1,\, UT_2 - G_2,\, UT_3 - G_3,\, UT_4 - G_4, \\ &Z_2T_1 - Z_3T_2,\, Z_2T_1 - Z_1T_3,\, Z_2T_2 - Z_1T_4, \\ &Z_2T_3 - Z_3T_4,\, T_1T_4 - T_2T_3). \end{aligned}$$

Consequently, $\operatorname{Ker}\phi = K_0[U,Z] \otimes_{K_0[U,X,Z]} \operatorname{Ker}\tilde{\phi}$. Therefore, by Theorem 3.5, we get a local domain $(A, \mathfrak{m})$ with *prime* element $x$ which enjoys the following:

(4.2.1) $\hat{A} \cong K[[u, x, \zeta_1, \zeta_2, \zeta_3]]/(\zeta_1, \zeta_2) \cap (\zeta_3, x + \zeta_2)$,

(4.2.2) $\hat{A}/x\hat{A} \cong K[[u, \zeta_1, \zeta_2, \zeta_3]]/(\zeta_1\zeta_3, \zeta_1\zeta_2, \zeta_2\zeta_3, \zeta_2^2)$.

## References

[1] M. Brodmann – C. Rotthaus, Local domains with bad sets of formal prime divisors, J. Algebra **75**, 386–394(1982).

[2] M. Brodmann – C. Rotthaus, A peculiar unmixed domain, Proc. Amer. Math. Soc. **87**, 596–600(1983).

[3] EGA chapitre IV, IHES Publ. Math. **20** (1964), **24** (1965).

[4] R.C. Heitmann, A non-catenary, normal, local domain, Rocky Mountain J. Math. **12**, 145–148(1982).

[5] M. Hochster, Criteria for equality of ordinary and symbolic powers of primes, Math. Z. **133**, 53–65(1973).

[6] MACAULAY, Macaulay User Manual, May 25, 1989.

[7] H. Matsumura, Commutative Algebra, Benjamin 1970 (second ed. 1980).

[8] M. Nagata, Local Rings, John Wiley 1962 (reprt. ed. Krieger 1975).

[9] J. Nishimura, A few examples of local rings, I, in preparation.

[10] T. Ogoma, Non-catenary pseudo-geometric normal rings, Japan. J. Math. **6**, 147–163(1980).

[11] T. Ogoma, Construction of an unmixed domain $A$ with $A/\mathfrak{p}$ mixed for some prime ideal $\mathfrak{p}$ of $A$ ; making use of Poincaré Series, preprint.

[12] C. Rotthaus, Universell Japanische Ringe mit nicht offenem regulärem Ort, Nagoya Math. J. **74**, 123–135(1979).