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Period maps and their extensions

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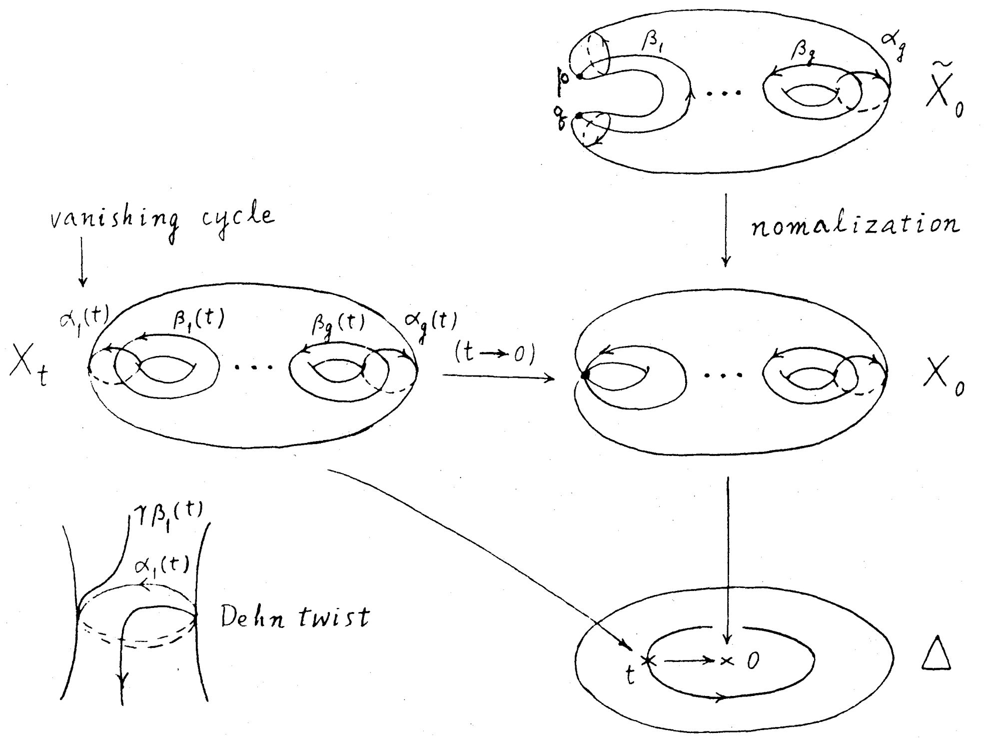
§0. Introduction

In this survey, we shall explain the works of Griffiths, Deligne, Schmid, Clemens, Mumford, Cattani, Kaplan, Steenbrink, Zucker et.al., concerning period maps and their extensions.

As an introduction to our topic, we review here the outline of a proof of the generic Torelli theorem for curves via degeneration.

Let $\{X_t\}_{t\in\Delta}$ be a degeneration of compact complex smooth curves X_t of genus g over a punctured disk $t \in \Delta^*$ to a curve X_0 with one node over the origin $0 \in \Delta$ as in the following figure:



Let $\alpha_1(t)$, ..., $\alpha_g(t)$, $\beta_1(t)$, ..., $\beta_g(t)$ be a flat symplectic basis of $H_1(X_t, \mathbf{Z})$, $t \in \Delta^*$, where $\alpha_1(t)$ is a vanishing cycle, and let $\omega_1(t), ..., \omega_g(t) \in H^0(X_t, \Omega^1_{X_t})$ be the basis of differential forms of the first kind, normalized by $\int_{\alpha_i(t)} \omega_j(t) = \delta_{ij}$. We denote by $\varphi(t) = (\varphi_{ij}(t)) := \left(\int_{\beta_i(t)} \omega_j(t)\right)$ the period matrix. Then, by the Dehn twist, we have the Picard-Lefschetz formula for the action of the local monodromy γ:

$$\begin{aligned} &\gamma\beta_1(t) = \beta_1(t) + \alpha_1(t), \quad \text{the other } \alpha_i(t),\ \beta_i(t) \text{ are invariant.} \\ &\gamma\varphi_{11}(t) = \varphi_{11}(t) + 1 \quad \text{hence} \quad \varphi_{11}(t) = (\log t)/2\pi\sqrt{-1} + s(t), \\ &\text{where } s(t) \text{ as well as the other } \varphi_{ij}(t) \text{ are single-valued.} \end{aligned} \tag{0.1}$$

By the hyperbolicity of the Siegel upperhalf space, $\beta_i(t)$ for $i \neq 1$, $s(t)$, $\varphi_{ij}(t)$ for $(i,j) \neq (1,1)$ extend over the puncture. This is the most primitive form of the more general fact: the 'Nilpotent Orbit Theorem' of Schmid in §2.

Let $\widetilde{X}_0$ be the normalzation of X_0. Let p, q and β_i $(i \neq 1)$ be the inverse images on $\widetilde{X}_0$ of the double point of X_0 and the 1-cycles $\beta_i(0)$ on X_0, respectively. Moreover, let $\widetilde{\omega}_1$ and ω_j $(j \neq 1)$ be the induced differential forms of the third kind and the first kind on $\widetilde{X}_0$, respectively. Then, as $t \to 0$, we have from (0.1) that

$$\varphi(t) \bmod \gamma = \left(\begin{array}{c|c} t\exp 2\pi\sqrt{-1}s(t) & \cdots \\ \hline \vdots & \cdots \end{array}\right) \to \left(\begin{array}{c|ccc} 0 & \int_q^p \omega_2 & \cdots & \int_q^p \omega_g \\ \hline \int_{\beta_2} \widetilde{\omega}_1 & & & \\ \vdots & & \varphi^1(0) & \\ \int_{\beta_g} \widetilde{\omega}_1 & & & \end{array}\right) \tag{0.2}$$

From this, we can observe the following:

(i) There are the following correspondences:

modulus of $\widetilde{X}_0$

$\leftrightarrow$ (2,2)-block $\varphi^1(0)$ of the last matrix in (0.2)

$\leftrightarrow$ 'graded piece gr^W of the limiting mixed Hodge structure'

$\to$ point of the Satake-Baily-Borel compactification

It follows that we can recover $\widetilde{X}_0$ from $\varphi^1(0)$ by the induction hypothesis.

The differential of $\varphi^1(t)$ by t at $t = 0$ is dual to the multiplication map $R_1 \otimes R_1 \to R_2$ of the canonical ring $R = \bigoplus R_n$ of $\widetilde{X}_0$ which is surjective if $\widetilde{X}_0$ is non-hyperelliptic (Noether's theorem). This means that the infinitesimal Torelli theorem holds in the direction of the moduli space of the normalizations $\widetilde{X}_0$.

(ii) There are the following correspondences:

position of $p, q \in \widetilde{X}_0$

$\leftrightarrow$ (1,2)-block of the last matrix in (0.2)

$\leftrightarrow$ (2,1)-block of the last matrix in (0.2), by reciprocity

$\leftrightarrow$ 'extension data of the limiting mixed Hodge structure'

$\rightarrow$ point on a fiber from the smooth toroidal compactification

to the Satake-Baily-Borel compactification

The Abel-Jacobi map $\widetilde{X}_0 \times \widetilde{X}_0 -$ (diagonal) $\rightarrow J(\widetilde{X}_0)$ is given by the (1,2)-block of the last matrix in (0.2), which is injective, and hence we can recover the position of $p, q \in \widetilde{X}_0$.

The partial differential of this block by p as well as by q is the canonical map $\widetilde{X}_0 \rightarrow \mathbf{P}^{g-1}$ which is injective if $\widetilde{X}_0$ is non-hyperelliptic. This means that the infinitesimal Torelli theorem holds in the direction of the moduli space of the positions of $p, q \in \widetilde{X}_0$.

(iii) Since $t = 0$ is a local equation of the boundary component of the compactification of the moduli space of curves, we get from the (1,1)-block of the middle matrix in (0.2) the infinitesimal Torelli theorem in the direction normal to the moduli space of the singular curves X_0.

Thus the induction on the genus g proceeds and we get the generic Torelli theorem for curves.

§1. Linear-algebraic generalizations of Hodge Structure

We recall here the notions of (Cohomological) (Filtered) (Mixed) Hodge Complex introduced by Deligne and generalized by El Zein, which yield more flexible frame-works whose axioms are easier to check. We first list up series of definitions and then explain their relationship and the fundamental results.

(1.1) Definition ([D2], [E]). *A* **Hodge Structure** *(HS for short) of weight w defined over $\mathbf{Q}$ is $(H_{\mathbf{Q}}, (H_{\mathbf{C}}, F))$ such that*

(0) *$H_{\mathbf{Q}}$ is a $\mathbf{Q}$-module of finite type and F is a decreasing filtration on $H_{\mathbf{C}} := \mathbf{C} \otimes H_{\mathbf{Q}}$.*

(i) *F and $\bar{F}$ are w-opposed, i.e., $\mathrm{gr}_F^p \mathrm{gr}_{\bar{F}}^q = 0$ unless $p + q = w$.*

A **Mixed HS** *(MHS for short) defined over $\mathbf{Q}$ is $((H_{\mathbf{Q}}, W), (H_{\mathbf{C}}, F)) =: (H, W)$ such that*

(0) *H is as above (HS.0) and W is an increasing filtration on $H_{\mathbf{Q}}$.*

(i) *gr_k^W is an HS of weight k.*

A **Filtered MHS** *(FMHS for short) defined over $\mathbf{Q}$ is $((H_{\mathbf{Q}}, W, G), (H_{\mathbf{C}}, F)) =: (H, G)$ such that*

(0) *H is as above (MHS.0) and G is an increasing filtration on $H_{\mathbf{Q}}$.*

(i) *gr_i^G is an MHS.*

A **Hodge Complex** *(HC for short) of weight k defined over $\mathbf{Q}$ is $(K_{\mathbf{Q}}, (K_{\mathbf{C}}, F), \alpha)$ such that*

(0) *$K_{\mathbf{Q}} \in D^+(\mathbf{Q})$, $(K_{\mathbf{C}}, F) \in D^+F(\mathbf{C})$, $\alpha : \mathbf{C} \otimes K_{\mathbf{Q}} \simeq K_{\mathbf{C}}$ in $D^+(\mathbf{C})$, i.e., quasi-isomorphism.*

(i) *$E_1 = E_\infty$ for $(K_{\mathbf{C}}, F)$.*

(ii) *$H^n(K)$ is an HS of weight $k + n$ for all n.*

A **Mixed HC** *(MHC for short) defined over $\mathbf{Q}$ is $((K_{\mathbf{Q}}, W), (K_{\mathbf{C}}, W, F), \alpha)$*

$=: (K, W)$ *such that*

(0) $(K_{\mathbf{Q}}, W) \in D^{+}F(\mathbf{Q})$, $(K_{\mathbf{C}}, W, F) \in D^{+}F_2(\mathbf{C})$, $\alpha : \mathbf{C} \otimes (K_{\mathbf{Q}}, W) \simeq (K_{\mathbf{C}}, W)$ *in* $D^{+}F(\mathbf{C})$, *i.e., filtered quasi-isomorphism.* $H^n(K_{\mathbf{Q}})$ *is a* $\mathbf{Q}$*-module of finite type for all* n.

(i) $\mathrm{gr}_k^W(K)$ *is an HC of weight* k *for all* k.

A **Filtered MHC** *(FMHC for short) defined over* $\mathbf{Q}$ *is* $((K_{\mathbf{Q}}, W, G), (K_{\mathbf{C}}, W, G, F), \alpha) =: (K, G)$ *such that*

(0) *For* $A = \mathbf{Q}$ *and* $\mathbf{C}$, *there exists* $(K_A, G, L) \in D^{+}F_2(A)$ *which satisfies* $\alpha : \mathbf{C} \otimes (K_{\mathbf{Q}}, G, L) \simeq (K_{\mathbf{C}}, G, L)$ *in* $D^{+}F_2(\mathbf{C})$, *i.e., bifiltered quasi-isomorphism, and* $(K_A, W) \simeq (K_A, G * L)$ *in* $D^{+}F(A)$. $(K_{\mathbf{C}}, W, G, F) \in D^{+}F_3(\mathbf{C})$. $H^n(K_{\mathbf{Q}})$ *is a* $\mathbf{Q}$*-module of finite type for all* n.

(i) $\mathrm{gr}_i^G(K)$ *is an MHC for all* i.

(ii) $E_2 = E_\infty$ *for* $(K_{\mathbf{Q}}, G)$.

A **Cohomological HC** *(CHC for short) of weight* k *defined over* $\mathbf{Q}$ *on a topological space* X *is* $(K_{\mathbf{Q}}, (K_{\mathbf{C}}, F), \alpha)$ *such that*

(0) $K_{\mathbf{Q}} \in D^{+}(\mathbf{Q}_X)$, $(K_{\mathbf{C}}, F) \in D^{+}F(\mathbf{C}_X)$, $\alpha : \mathbf{C} \otimes K_{\mathbf{Q}} \simeq K_{\mathbf{C}}$ *in* $D^{+}(\mathbf{C}_X)$.

(i) $R\Gamma(K)$ *is an HC of weight* k.

A **Cohomological MHC** *(CMHC for short) defined over* $\mathbf{Q}$ *on a topological space* X *is* $((K_{\mathbf{Q}}, W), (K_{\mathbf{C}}, W, F), \alpha) =: (K, W)$ *such that*

(0) $(K_{\mathbf{Q}}, W) \in D^{+}F(\mathbf{Q}_X)$, $(K_{\mathbf{C}}, W, F) \in D^{+}F_2(\mathbf{C}_X)$, $\alpha : \mathbf{C} \otimes (K_{\mathbf{Q}}, W) \simeq (K_{\mathbf{C}}, W)$ *in* $D^{+}F(\mathbf{C}_X)$. $H^n(X, K_{\mathbf{Q}})$ *is a* $\mathbf{Q}$*-module of finite type for all* n.

(i) $\mathrm{gr}_k^W(K)$ *is a CHC of weight* k *for all* k.

A **Cohomological FMHC** *(CFMHC for short) defined over* $\mathbf{Q}$ *on a topological space* X *is* $((K_{\mathbf{Q}}, W, G), (K_{\mathbf{C}}, W, G, F), \alpha) =: (K, G)$ *such that*

(0) *For* $A = \mathbf{Q}$ *and* $\mathbf{C}$, *there exists* $(K_A, G, L) \in D^{+}F_2(A_X)$ *which satisfies* $\alpha : \mathbf{C} \otimes (K_{\mathbf{Q}}, G, L) \simeq (K_{\mathbf{C}}, G, L)$ *in* $D^{+}F_2(\mathbf{C}_X)$ *and* $(K_A, W) \simeq (K_A, G * L)$ *in*

$D^+F(A_X)$. $(K_{\mathbf{C}}, W, G, F) \in D^+F_3(\mathbf{C}_X)$. $H^n(X, K_{\mathbf{Q}})$ *is a* $\mathbf{Q}$*-module of finite type for all* n.

(i) $\mathrm{gr}_i^G(K)$ *is an CMHC for all* i.

(ii) $E_2 = E_\infty$ *for hypercohomology of* $(K_{\mathbf{Q}}, G)$.

In the above definition of (C)FMHC, $G * L$ means the *convolution* of two filtrations, which is defined by $(G * L)_k := \sum_{i+j=k} G_i \cap L_j$.

The second remark is that, for three filtrations F, G, H on K, in $\mathrm{gr}_H\,\mathrm{gr}_G\,\mathrm{gr}_F K$ gr_H and gr_G commute but gr_G and gr_F do not commute. Hence, besides $\mathrm{gr}_H\,\mathrm{gr}_G\,\mathrm{gr}_F K$ and $\mathrm{gr}_H\,\mathrm{gr}_G\,\mathrm{gr}_F K'$ being quasi-isomorphic, we need the following additional axiom for *trifiltered quasi-isomorphism* (i.e., isomorphism in D^+F_3):

$\mathrm{gr}_H\,\mathrm{gr}_G K$ and $\mathrm{gr}_H\,\mathrm{gr}_G K'$ are quasi-isomorphic.

Then, for biregular filtrations, if (K, F, G, H) and (K', F, G, H) are trifiltered quasi-isomorphic, K and K' applied by any 0, 1 or 2 of gr_F, gr_G and gr_H are quasi-isomorphic.

(1.2) The following diagram indicates the relationship among the nine notions defined in (1.1).

$$\begin{array}{ccccc}
HS & \overset{\mathrm{gr}^W}{\Longleftarrow} & MHS & \overset{\mathrm{gr}^G}{\Longleftarrow} & FMHS \\
{\scriptstyle H}\Uparrow{\scriptstyle (a)} & & \uparrow & & \uparrow \\
HC & \Longleftarrow & MHC & \underset{(b)}{\Longleftarrow} & FMHC \\
{\scriptstyle R\Gamma}\Uparrow & & \uparrow & & \uparrow \\
CHC & \Longleftarrow & CMHC & \underset{(b)}{\Longleftarrow} & CFMHC
\end{array}$$

where $\Leftarrow$ and $\Uparrow$ stand for Definitions in (1.1) whereas $\uparrow$ stands for Theorem in (1.3) below. The conditions $E_1(F) = E_\infty(F)$ and $E_2(G) = E_\infty(G)$ are assumed at (a) and (b), respectively.

(1.3) Fundamental Results. **(i)** [D2, III] *For a MHC (K, W), we have*

(a) *$E_1(K,W)$ is a complex in (HS), $E_2(K,W) = E_\infty(K,W)$, $E_1(K,F) = E_\infty(K,F)$, $E_2(\mathrm{gr}_F K, W) = E_\infty(\mathrm{gr}_F K, W)$.*

(b) *All the filtrations on $E_r(K,W)$ induced from F in various ways coinside for any r (actually for $r=2$).*

(c) *$((H^n(K_{\mathbf{Q}}), W[n]), (H^n(K_{\mathbf{C}}), F))$ is a MHS for all n.*

(ii) [E, I] *For a FMHC (K, W), we have, moreover, that*

(b′) *All the filtrations on $E_r(K,G)$ induced from W (resp. F) in various ways coinside for any r (actually for $r=2$).*

(c′) *$((H^n(K_{\mathbf{Q}}), W[n], G[n]), (H^n(K_{\mathbf{C}}), F))$ is a FMHS for all n.*

Remark. Our definition of (C)FMHC is slightly more restrictive (but more natural) than the one in [E]. In fact, it can be easily verified that

(i) $(G * L)R\Gamma = R\Gamma(G * L)$ (by using canonical resolution of Godement).

(ii) $(K, W) \underset{FQIS}{\simeq} (K, G*L)$ implies $\mathrm{gr}_k^W G_i K \underset{QIS}{\simeq} \oplus_{i' \leq i} \mathrm{gr}_k^W \mathrm{gr}_{i'}^G K$ for all i, k.

Hence, by [E, I] or [SZ, (6.8)], we see that $\mathrm{Dec}\, W$ and gr^G commute.

§2. Limit of Variation of Hodge Structure

In this section, we shall review two main theorems of Schmid [S], i.e., the Nilpotent Orbit and SL_2 Orbit Theorems, and then explain their applications. For simplicity, we shall restrict ourselves to one variable case, i.e., Variation of Hodge Structure (VHS for short) over one dimensional parameter space, through out this article except explicitely stated to the contrary.

As a good brief introduction to the theory of Griffiths on VHS [G1], we recommend the readers an exposition by Deligne in Bourbaki Seminar [D.1]. We use the notion there such as 'polarized Hodge structure defined over $\mathbf{Z}$' freely (cf. (1.1)).

(2.1) Let $F_r = \{F_r^p\}$ be a reference Hodge filtration of weight w on $H_{\mathbf{C}} = \mathbf{C} \otimes H_{\mathbf{Z}}$ polarized by a quadratic form S. We denote $f^p := \dim_{\mathbf{C}} F_r^p$ and set $f = (f^0, ..., f^p)$. Then the classifying space D and its 'compact dual' $\check{D}$ are defined by

$$\begin{aligned}
\check{D} &:= \{F \in \mathrm{Flag}(H_{\mathbf{C}}, f) \mid S(F^p, F^q) = 0 \text{ for } p + q = w + 1\}, \\
D &:= \{F \in \check{D} \mid S(C_r x, \bar{x}) > 0 \text{ for } 0 \neq x \in H_{\mathbf{C}}\} \\
&= \{\text{HS on } H_{\mathbf{C}} \text{ of weight } w \text{ with type } f \text{ polarized by } S\}.
\end{aligned}$$

Here C_r is the Weil operator which is defined by $C_r x := (-1)^{p-q} x$ for $x \in H_r^{pq} := F_r^p \cap \bar{F}_r^q$. The orthogonal groups $G_{\mathbf{C}} := O(S, \mathbf{C})$, $G_{\mathbf{R}} := O(S, \mathbf{R})$ act transitively on $\check{D}$, D with isotropy subgroups B, V, respectively. It is important that V is compact but not necessarily maximal. Let Γ is a subgroup of $G_{\mathbf{Z}} := O(S, \mathbf{Z})$. It is known that Γ acts on D properly discontinuously.

A *period map (= polarized VHS)*

$$\Phi : \Delta^* \to D/\Gamma$$

is defined as a holomorphic map with *horizontal* local liftings, where 'horizontal' means that the Gauss-Manin connection ∇ and the Hodge filtration F, which are the pull-backs by local liftings of Φ, satisfy $\nabla F^p \subset \Omega^1_{\Delta^*} \otimes F^{p-1}$ for all p.

Now we pose

Naive Question. *Does Φ extend over the puncture? Where is $\Phi(t)$ going, as $t \to 0$?*

This is our leading question in the present article.

Let $\tau : U \to \Delta^*$, $t = \tau(z) := \exp 2\pi\sqrt{-1}(z)$, be the universal covering and consider a lifting $\widetilde{\Phi}$ of Φ:

$$\widetilde{\Phi} : U \to D.$$

Take $\gamma \in \Gamma$ such that $\widetilde{\Phi}(z+1) = \gamma\widetilde{\Phi}(z)$, called the *local monodromy*. It is known by Borel that all eigenvalues of γ are roots of unity. Let m be the smallest positive integer such that γ^m is unipotent, and denote

$$N := \frac{1}{m}\log\gamma^m = \frac{1}{m}\sum_{k\geq 1}(-1)^{k+1}\frac{1}{k}(\gamma^m - 1)^k,$$
$$\widetilde{\Psi}(z) := \exp(-mzN)\widetilde{\Phi}(mz) : U \to \check{D},$$

then $\widetilde{\Psi}(z+1) = \widetilde{\Psi}(z)$ and hence it drops to

$$\Psi(t) := \widetilde{\Psi}(\tau^{-1}(t)) : \Delta^* \to \check{D}$$

Nilpotent Orbit Theorem [Sc]. *Ψ extends to $\Psi : \Delta \to \check{D}$, and $F_\infty :=$ $\Psi(0)$ defines a nilpotent orbit $\widetilde{\Phi}_\infty(z) := \exp(zN)F_\infty$, i.e., $\widetilde{\Phi}_\infty : \{\mathrm{Im}\, z > \alpha\} \to D$ is a period map for $\alpha \gg 0$, and $\widetilde{\Phi}_\infty$ approximates the original $\widetilde{\Phi}$. More precisely,*

$$d(\widetilde{\Phi}_\infty(z), \widetilde{\Phi}(z)) < (\mathrm{Im}\, z)^\beta / \exp(2\pi \mathrm{Im}\, z/m) \quad \textit{for some} \quad \beta \geq 0 \quad (\mathrm{Im}\, z \gg 0),$$

where $d(\ ,\ )$ is a $G_{\mathbf{R}}$-invariant metric on D.

Remark. In the situation of §0, according to (0.1), the local monodromy γ is represented by the matrix

$$\gamma = \begin{pmatrix} 1_g & N_1 \\ 0 & 1_g \end{pmatrix}, \quad N_1 := \begin{pmatrix} 1 & \\ & \mathrm{O} \end{pmatrix}$$

with respect to the basis of $H^1(X_t, \mathbf{Z})$ dual to $(\beta_1(t), \ldots, \beta_g(t), \alpha_1(t), \ldots, \alpha_g(t))$, and its action on the period matrix $\varphi(t)$ is

$$\begin{pmatrix} 1_g & N_1 \\ 0 & 1_g \end{pmatrix}\begin{pmatrix} \varphi(t) \\ 1_g \end{pmatrix} = \begin{pmatrix} \varphi(t) + N_1 \\ 1_g \end{pmatrix}, \quad \text{hence} \quad \gamma\varphi(t) = \varphi(t) + N_1.$$

We denote by $\psi(t)$ the $g \times g$ -matrix whose entries are defined by

$$\psi_{ij}(t) := \begin{cases} s(t) & \text{for } (i,j) = (1,1), \\ \varphi_{ij}(t) & \text{otherwise.} \end{cases}$$

Then, by an affine coordinates induced by the Plücker coordinates of Grass$(H_{\mathbf{C}}, g)$, we have the following correspondences between the period maps in this subsection and the period matrices in §0 and their modifications:

$$\Phi(t) = \varphi(t) \bmod \gamma, \ \Psi(t) = \psi(t), \ F_\infty = \Psi(0) = \psi(0), \ \widetilde{\Phi}_\infty(z) = \psi(0) + zN_1.$$

(2.2) We take the standard generators of the Lie algebras $\mathfrak{sl}(2, \mathbf{R})$ and $\mathfrak{su}(1,1)$ which are related by the Cayley transformation $\operatorname{ad} c_1$, where

$$c_1 := \exp \frac{\pi i}{4} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix},$$

as follows:

$$\begin{array}{lllll} \mathfrak{sl}(2,\mathbf{R}) \ni & y := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & n_+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & n_- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\ \operatorname{ad} c_1 \downarrow & \quad \updownarrow & \quad \updownarrow & \quad \updownarrow \\ \mathfrak{su}(1,1) \ni & z := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, & x_+ := \frac{1}{2}\begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}, & x_- := \frac{1}{2}\begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}. \end{array}$$

Note that the generators $\{y, n_+, n_-\}$ and $\{z, x_+, x_-\}$ are closely related to the 'N-filtration' (= 'monodromy weight filtration') and the Hodge filtration, respectively.

Definition. *A representation $\rho : SL(2, \mathbf{R}) \to G_{\mathbf{R}}$ is horizontal at a base point $r \in D$ if $\rho_*(x_+) \in \mathfrak{g}_{\mathbf{C}}^{-1,1} := \{X \in \mathfrak{g}_{\mathbf{C}} \,|\, XH_r^{p,q} \subset H_r^{p-1,q+1}$ for all p, $q\}$, where $\mathfrak{g}_{\mathbf{C}} := \operatorname{Lie} G_{\mathbf{C}}$.*

Remark. A horizontal SL_2-representation ρ induces an equivariant horizontal map $\widetilde{\rho}$:

$$\begin{array}{ccc} \mathrm{SL}(2,\mathbf{C}) & \xrightarrow{\rho} & G_{\mathbf{C}} \\ \downarrow & & \downarrow \\ \mathbf{P}^1 & \xrightarrow{\tilde{\rho}} & \check{D} \\ \Psi & & \Psi \\ i & \longrightarrow & r \end{array}$$

This is a generalization into the present context of the notion of '(H_1)-homomorphism' in the case of symmetric domains of Hermitian type (cf. e.g. [Sa]).

The following result was first obtained by Deligne:

T×SL₂-Decomposition Theorem [Sc]. *A (polarized) Hodge structure over* $\mathbf{R}$ *with horizontal* SL_2*-action decomposes into an (orthogonal) direct sum of irreducible factors. The irreducible ones are classified, up to isomorphism, as* $H(k_1) \otimes S(k_2)$ *or* $E(p,q) \otimes S(k_2)$ *for* $(k_2 \geq 0)$ (for the notation, see [Sc]).

The idea of proof is as follows: Let $T := R_{\mathbf{C}/\mathbf{R}}(\mathbf{G}_m)$ be the restriction of defining field of one-dimensional algebraic torus $\mathbf{G}_m$, i.e., 'Weil's restriction of scalars'. Then a real Hodge structure is equivalent to a real T-module (cf. [D.1]). By the horizontality, T and SL_2 commute and we get a real $T\times$ SL_2-module, which is complete reducible because $T\times$ SL_2 is reductive.

SL_2-Orbit Theorem can be roughly stated as

SL₂-Orbit Theorem [Sc]. *Any nilpotent orbit* (N, F) *can be asymptotically approximated by the orbit of a horizontal* SL_2*-representation* ρ*, i.e.,* $\exp(zN)F = g(-iz)\widetilde{\rho}(z)$*, where* $g:$ *(a neighborhood of* $\infty \in \mathbf{P}^1$*)* $\to G_{\mathbf{C}}$ *such that* $\rho_* n_+ = N, \ldots$

For more precise statement, see [Sc].

(2.3) The following are corollaries of Nilpotent+SL_2-Orbit Theorems:

Norm Estimate of N-filtration [Sc]. *In the situation of (2.1), there exists unique filtration $L = L(N)$, called N-filtration, which satisfies the condition (i) below. L is also characterized by (ii).*

(i) $NL_j \subset L_{j-2}$, $N^j : \mathrm{gr}_j^L \xrightarrow{\sim} \mathrm{gr}_{-j}^L$ *for all* j.

(ii) $L_j = \{x \in H_{\mathbf{Q}} \mid S(C_t x, \bar{x}) = O((\log \frac{1}{|t|})^j)$ *as* $|t| \to 0\}$, *where C_t is the Weil operator defined by $F_t = \Phi(t)$.*

This estimate (ii) is important and, in fact, SL_2-Orbit Theorem was prepared so as to prove it.

Nilpotent Orbit and Limiting MHS [Sc]. *For $(N, F) \in \mathfrak{g}_{\mathbf{Q}} \times \check{D}$ with $NF^p \subset F^{p-1}$, the following are equivalent:*

(i) *(N, F) is a nilpotent orbit, i.e.,* $\exp(zN)F \in D$ *for* $\mathrm{Im}\, z \gg 0$.

(ii) *$(L[w], F)$ is a polarized Mixed Hodge Structure (MHS, for short), on $H_{\mathbf{Q}}$, where polarized means that $P_j(N) := \mathrm{Ker}\{N^{j+1} : \mathrm{gr}_j^L \to \mathrm{gr}_{-j-2}^L\}$, the primitive part, is polarized by $S_{(j)}(x, y) := S(x, N^j y)$ $(x, y \in P_j(N))$ for all $j \geq 0$.*

The outline of a proof is as follows. (i) $\Rightarrow$ (ii): By SL_2-Orbit Theorem, the filtration F and the reference filtration F_r coincide on gr^L, hence we can replace F by F_r. By $\mathrm{T}\times\mathrm{SL}_2$-Decomposition Theorem, the problem can be reduced to an irreducible case, where the assertion is directly verified. The converse can be found in [KK, Prop.1.2.2].

(2.4) Let $\mathcal{X} \to \Delta$ be a semistable degeneration with $\mathcal{X} \subset \mathbf{P}^m \times \Delta$ closed and of relative dimension d. We denote by X_t and X_∞ the fiber $f^{-1}(t)$ over $t \in \Delta$ and the fiber of the induced family via the base extension to the universal covering $U \to \Delta^*$, respectively.

The MHS on X_0 and the limiting MHS on X_∞ are related by

Clemens-Schmid Sequence [Cl]. *The following sequence of (co)homology groups with coefficients in* $\mathbf{Q}$ *is an exact sequence in (MHS):*

$$
\begin{array}{ccccccccc}
H^n(X_0) & \rightarrow & H^n(X_\infty) & \xrightarrow{N} & H^n(X_\infty) & \rightarrow & H_{2d-n}(X_0) & \rightarrow & H^{n+2}(X_0). \\
 & & & & \downarrow & & \downarrow \wr & & \downarrow \wr \\
 & & & & H^{n+1}(\mathcal{X}^*) & \rightarrow & H^{n+2}(\mathcal{X}, \mathcal{X}^*) & \rightarrow & H^{n+2}(\mathcal{X})
\end{array}
$$

The most important part in the above sequence is the exactness at the first $H^n(X_\infty)$, which is proved by looking at gr^L.

Corollary (Monodromy Criteria). **(i)** $p_g(X_t) \geq \sum p_g(X_i)$, *where* $X_0 = \bigcup X_i$ *is the decomposition into irreducible components of the central fiber.*

(ii) $N = 0$ *on* $H^d(X_\infty)$ $\Rightarrow$ *equality holds in (i)* $\Rightarrow$ $N^{d-1} = 0$ *on* $H^d(X_\infty)$.

(iii) $N^n = 0$ *on* $H^n(X_\infty)$ $\Leftrightarrow$ H^n*(dual gragh of* X_0*)* $= 0$ *for* $n = 1, 2$.

(2.5) Now we can answer our Naive Question in (2.1) to some extent.

First we recall the classical case when the classifying space D is a symmetric domain of hermitian type. Note that, in our context, D is of hermitain type if and only if it is a classifying space of the Hodge structures of type $H_{\mathbf{C}} = H^{n,n-1} \oplus H^{n-1,n}$ for odd weight or $H_{\mathbf{C}} = H^{n+1,n-1} \oplus H^{n,n} \oplus H^{n-1,n+1}$ with $\dim H^{n+1,n-1} = 1$ for even weight.

As a consequence of (2.1) and (2.2), we have

Proposition [Sc]. *In the situation of (2.1), if* D *is a hermitian symmetric domain, then*

(i) $(L[w], F_z)$ *is a MHS for* $\mathrm{Im}\, z \gg 0$, *where* $F_z := \widetilde{\Phi}(z) \in D$.

(ii) $F_z \,\mathrm{gr}^{L[w]} \rightarrow F_\infty \,\mathrm{gr}^{L[w]}$ *as* $\mathrm{Im}\, z \rightarrow 0$ *in the Satake topology, where* $F_\infty := \Psi(0) \in \check{D}$.

This is a version of extension of period map in the case of one-parameter by using the Satake compactification. But, in the hermitian case, we have also a version for several parameters or, even more, if we use a toroidal compactification, we can catch up with extension data of the limiting MHS.

Extension to S.B.B. Compactification [Bo]. *In the case of several parameters, every holomorphic map* $\Phi : \Delta^{*k} \times \Delta^{n-k} \to D/\Gamma$ *extends to a holomorphic map* $\Phi^S : \Delta^n \to (D/\Gamma)^S$ *into the Satake-Baily-Borel compactification.*

In the situation of the above theorem, let B be the boundary component containing $\widetilde{\Phi}^S(\underline{i\infty}, \underline{0})$. Then $\operatorname{Im}\widetilde{\Phi}^S \subset \bigcup_{\bar{B}' \supset B} B'$.

Extension to Toroidal Compactification [AMRT]. *In the above situation of several parameters, we see that every local monodromy* $\gamma_i \in \overline{C}(B) \cap U(B)_{\mathbf{Z}}$, *hence* $\widetilde{\Phi}$ *induces* $\Phi' : \Delta^{*k} \times \Delta^{n-k} \to D/U(B)_{\mathbf{Z}}$, *and the following are equivalent:*

(i) *There exists a cone* $\sigma_\alpha \subset C(B)$ *such that* $\gamma_i \in \sigma_\alpha$ *for all* i.

(ii) Φ' *extends holomorphically to* ${\Phi'}^M : \Delta^n \to (D/U(B)_{\mathbf{Z}})_{\{\sigma_\alpha\}}$.

(iii) Φ *extends holomorphically to* $\Phi^M : \Delta^n \to (D/\Gamma)^M$ *into a toroidal compactification.*

(For the notaion such as $C(B), U(B)_{\mathbf{Z}}, \sigma_\alpha$ etc., see [AMRT] where they use F in stead of B for a boundary component.)

In the case of non-hermitian type, there is a speculation by Griffiths [G2]. Cattani and Kaplan obtained the following result from (2.1) and (2.2):

Extension in Case of Weight 2 [CK]. *In the case of Hodge structures of weight 2, there is a partial compactification* $(D/\Gamma)^C$ *of Satake type, which is obtained by adding up all limit points of extensions of period maps over a punctured disk, i.e.,*

(i) *Any period map* $\Phi : \Delta^* \to D/\Gamma$ *extends continuously to* $\Phi^C : \Delta \to (D/\Gamma)^C$.

(ii) *For any point $b \in (D/\Gamma)^C$, there exists a period map $\Phi : \Delta^* \to D/\Gamma$ which extends continuously over the puncture and $\Phi^C(0) = b$.*

They constructed their partial compactification $(D/\Gamma)^C$ in the following way. For any pair $(\rho, 0)$ of a horizontal SL_2-representation and a base point $0 \in D$, set $N := \rho(n_+)$, $L := L(N)$, the N-filtration, and define the associated boundary component $B(\rho)$ and boundary bundle $\mathcal{B}(L)$ by

$$
\begin{aligned}
B(\rho) :=\ & \text{classifying space of} \\
& (S_{(0)}\text{-polarized HS on } P_0 \subset \mathrm{gr}_0^L) \times (S_{(-1)}\text{-polarized HS on } \mathrm{gr}_{-1}^L), \\
\mathcal{B}(L) :=\ & \bigcup \{B(\rho) \,|\, \text{only } L \text{ is fixed}\}, \\
D^C :=\ & \bigcup_{\rho:\,\text{rational}} B(\rho) \underset{(\text{dense})}{\subset} D^* := \bigcup_{L:\,\text{rational}} \mathcal{B}(L).
\end{aligned}
$$

Then the normalizer $N(\mathcal{B}(L))$ becomes a parabolic subgroup of $G_{\mathbf{R}}$ and one can construct a Satake topology on D^*, so that Γ acts on it properly discontinuously.

Remark. As far as the author knows the following problems are still open:

(i) Theory of 'automorphic forms' on $(D/\Gamma)^C$.

(ii) A partial compactification which comprizes the extension data of limiting MHS.

§3. Limit of Variation of Mixed Hodge Structure

In this section, after [SZ] we shall see mixed versions of §2, which are generalizations of the theory of Steenbrink on the limit of VHS in geometric origin [St], and of Hodge theory of Zucker with degenerating coefficients [Z].

(3.1) Analogously to (2.1), we can formulate a *mixed period map* $\Phi : \Delta^* \to D/\Gamma$ over a punctured disk, or equivalently, a *gradedly polarized Variation of Mixed Hodge Structure* (*VMHS* for short) $(\mathbf{V}, G, F, S)$ over Δ^*, where $\mathbf{V}$ is the local system of free $\mathbf{Z}$-modules, G the weight filtration, F the Hodge filtration and $S = \{S_i\}$ a set of polarizations on the $\mathrm{gr}_i^G \mathbf{V}$ (cf. [U1], [SSU]). Let $\Psi : \Delta^* \to D$ be the modification of Φ by monodromy as in (2.1) and let $\widetilde{\mathcal{V}}$ be the canonical extension of $\mathcal{V} = \mathcal{O}_{\Delta^*} \otimes \mathbf{V}$ over the puncture.

In the mixed case, 'Nilpotent Orbit Theorem' does not hold in general. So we need

Definition [SZ]. *A gradedly polarized VMHS* $(\mathbf{V}, G, F, S)$ *over* Δ^* *is admissible if*

(i) *There exists the G-relative N-filtraton M, i.e., an increasing filtration M on $\widetilde{\mathcal{V}}(0)$ which is characterized by $NM_k \subset M_{k-2}$ and $N^k : \mathrm{gr}_{i+k}^M \mathrm{gr}_i^G \xrightarrow{\sim} \mathrm{gr}_{i-k}^M \mathrm{gr}_i^G$.*

(ii) *Ψ extends over the puncture. We set $F_\infty := \Psi(0)$.* (For Ψ, cf (2.1).)

Note that if a gradedly polarized VMHS $(\mathbf{V}, G, F, S)$ over Δ^* is admissible then one can prove that (M, F_∞) induces an MHS on each $G_k \widetilde{\mathcal{V}}(0)$ and the monodromy logarithm N becomes a morphism of MHS of type $(-1, -1)$.

(3.2) Let $(\mathcal{X}, \mathcal{Y}) \to \Delta$ be a semistable degeneration of pairs of relative dimension d, i.e., $\mathcal{X} \to \Delta$ is a semistable degeneration and $\mathcal{Y}$ is a divisor of $\mathcal{X}$ flat over Δ such that $\mathcal{Y} + X_0$ is reduced and with simple normal crossings. We assume that $\mathcal{X} \subset \mathbf{P}^m \times \Delta$ is closed. We denote by X_t, Y_t and X_∞, Y_∞ the fibers $f^{-1}(t)$, $f^{-1}(t) \cap Y$ over $t \in \Delta$ and the fibers of the induced families via the base extension to the universal covering $U \to \Delta^*$, respectively. We also denote $\overset{\circ}{\mathcal{X}} := \mathcal{X} - \mathcal{Y}$, $\overset{\circ}{X}_t := X_t - Y_t$, $\overset{\circ}{X}_\infty := X_\infty - Y_\infty$, $\overset{\circ}{\mathcal{X}}{}^* := \overset{\circ}{\mathcal{X}} - \overset{\circ}{X}_0$, $\overset{\circ}{f} : \overset{\circ}{\mathcal{X}} \to \Delta$. Then it can be seen that $(\mathbf{V}, G, F, S) = (R^n \overset{\circ}{f}_* \mathbf{Q}_{\overset{\circ}{\mathcal{X}}{}^*}, W(\mathcal{Y})[n], F, S)$ is a gradedly polarized VMHS over Δ^* defined over $\mathbf{Q}$ (cf. [U1], [SSU]).

Admissibility of VMHS in Geometric Origin [SZ]. *A gradedly polarized VMHS* $(\mathbf{V}, G, F, S) = (R^n \overset{\circ}{f}_* \mathbf{Q}_{\overset{\circ}{\mathcal{X}}{}^*}, W(\mathcal{Y})[n], F, S)$ *is admissible.*

Outline of Proof. We consider here only the $\mathbf{C}$-structure because of the restriction of the space (for the $\mathbf{Q}$-structure, see [N] as well as [SZ]). It can be seen that $(\widetilde{\mathcal{V}}, G, \nabla) = (R^n f_* \Omega^{\cdot}_f(\log(\mathcal{Y}+X_0)), W(\mathcal{Y})[n]$, connecting homomorphism) gives the canonical extension of $(\mathbf{V}_{\mathbf{C}}, G) = (\mathcal{V}, G, \nabla)$. The limiting FMHS $(H_{\mathbf{C}}, W, G, F) = (H^n(X_\infty), M[n], G[n], F_\infty)$ and the monodromy logarithm N is induced by a CFMHC $(K_{\mathbf{C}}, W, G, F) = (A^{\cdot}_{\mathbf{C}}, M, G, F)$ and its endomorphism ν which are defined by

$$A^{\cdot q}_{\mathbf{C}} := (\Omega^{\cdot}_{\mathcal{X}}(\log(\mathcal{Y}+X_0))/W_q(X_0))[q+1] \quad \text{for } \ p,\, q \geq 0, \quad 0 \ \text{ otherwise},$$

$$d' := (-1)^{q+1}(\text{exterior differential}), \quad d'' := (f^* d\log t/2\pi i)\wedge,$$

$$\nu : A^{pq}_{\mathbf{C}} \to A^{p-1,q+1}_{\mathbf{C}} \quad \text{projection},$$

$$G_i := W(\mathcal{Y})_i, \quad L_j := W(X_0)_{j+2q+1} \text{ on } A^{\cdot q}_{\mathbf{C}}, \quad M := G * L,$$

$$F^p A^{\cdot\cdot}_{\mathbf{C}} := \sum_{p' \geq p} A^{p' \cdot}_{\mathbf{C}},$$

$A^{\cdot}_{\mathbf{C}}$: single complex associated to $A^{\cdot\cdot}_{\mathbf{C}}$, $\quad d := (-1)^q d' + d''$ on $A^{pq}_{\mathbf{C}}$.

Then we see that

$$\mathrm{gr}^M_k A^{\cdot}_{\mathbf{C}} \simeq \sum_{i+j=k} \mathrm{gr}^G_i \mathrm{gr}^L_j A^{\cdot}_{\mathbf{C}} \overset{\mathrm{Res}}{\underset{\sim}{\to}} \sum_{i+j=k} \ \sum_{q \geq \max\{0,-k\}} a_* \Omega^{\cdot}_{\widetilde{\mathcal{Y}}^{(i)} \cap \widetilde{X}_0^{(j+2q+1)}}[-k-2q](-k-q),$$

where $a : \widetilde{\mathcal{Y}}^{(i)} \cap \widetilde{X}_0^{(j+2q+1)} \to X_0$ is the projection from the normalization of the reduced subvariety of X_0 consisting of those points whose multiplicity in $\mathcal{Y}$ is $\geq i$ and multiplicity in X_0 is $\geq j+2q+1$. It follows from the definition that the above isomorphism is a filtered quasi-isomorphism with respect to F, and one can verify the axiom of CFMHC. One can also prove that the filtration M on $A^{\cdot}_{\mathbf{C}}$ induced the G-relative N-filtration on $H^n(X_0, A^{\cdot}_{\mathbf{C}})$ (see [St, (5.9)], also [U2, Appendix to §1]).

Remark. The monodromy logarithm N is related with the residue of the extension of Gauss-Manin connection ∇ in the following way:

$$\begin{array}{ccccc}
H^n(\overset{\circ}{X}_\infty) & & \xrightarrow{\qquad\log\gamma\qquad} & & H^n(\overset{\circ}{X}_\infty) \\
{\scriptstyle\alpha_t}\downarrow\wr & & & & {\scriptstyle\alpha_t}\downarrow\wr \\
\widetilde{\mathcal{V}}(0) & \xrightarrow{\nabla} & d\log t \otimes \widetilde{\mathcal{V}}(0) & \underset{\sim}{\xrightarrow{-2\pi i\mathrm{Res}}} & \widetilde{\mathcal{V}}(0),
\end{array}$$

where α_t depends on the choice of the coordinate t on Δ. This diagram also indicates the relationship between the $\mathbf{Q}$-structure and the $\mathbf{C}$-structure.

(3.3) Let $j : C^* \hookrightarrow C$ be a Zariski open subset of a compact smooth curve and set $\Sigma := C - C^*$.

The following theorem gives the starting point of (Mixed) Hodge Modules of Morihiko Saito.

(Mixed) Hodge Theory with Degenerating Coefficients ([Z], [SZ]). *In the above situation, let $(\mathbf{V}, G, F, S)$ be an admissible gradedly polarized VMHS over C^*. Then the Leray spectral sequence of $\mathbf{V}$ for j becomes a spectral sequence in (MHS), which is functorial in $\mathbf{V}$ and C^*. Moreover, if the weight filtration G is trivial, then $(H^i(C, j_*\mathbf{V}), F)$ is a pure HS.*

Idea of Proof. Since $0 \to \mathbf{V_C} \to \mathcal{V} \xrightarrow{\nabla} \Omega^1_{C^*} \otimes \mathcal{V} \to 0$ is a j_*-acyclic resolution, we have an exact sequence (see Remark in (3.2)):

$$0 \to j_*\mathbf{V_C} \to \widetilde{\mathcal{V}} \xrightarrow{\nabla} \Omega^1_C(\log \Sigma) \otimes \widetilde{\mathcal{V}} \to V/NV \to 0, \quad \text{where} \quad V := \bigoplus_{\sigma \in \Sigma} \widetilde{\mathcal{V}}(\sigma).$$

For any subspace $NV \subset A \subset V$, we denote by $(\mathbf{V}, A)$ the subcomplex of $\widetilde{\mathcal{V}} \xrightarrow{\nabla} \Omega^1_C(\log \Sigma) \otimes \widetilde{\mathcal{V}}$, whose cokernel is A/NV. The weight and the Hodge filtrations are defined by

$$\mathcal{M}_k(\mathbf{V}, V) := (G_k\mathbf{V},\ \oplus_{\sigma\in\Sigma}(N(\sigma)G_k(\sigma) + M_{k-1}(\sigma)G_k(\sigma)))$$
$$F^p(\mathbf{V}, V) := (F^p\widetilde{\mathcal{V}} \xrightarrow{\nabla} \Omega^1_C(\log \Sigma) \otimes F^{p-1}\widetilde{\mathcal{V}}).$$

$(\mathbf{V}, NV)$ inherites the filtrations as a subcomplex of $(\mathbf{V}, V)$. Then one can observe

$$\mathrm{gr}^{\mathcal{M}}_k(\mathbf{V}, V) \simeq (\mathrm{gr}^G_k\mathbf{V}, (\mathrm{gr}_k N)(\mathrm{gr}^G_k V)) \oplus \mathrm{gr}^M_{k-1}(G_{k-1}/NG_{k-1})[-1](-1),$$
$$\mathrm{gr}^{\mathcal{M}}_k(\mathbf{V}, NV) \simeq (\mathrm{gr}^G_k\mathbf{V}, (\mathrm{gr}_k N)(\mathrm{gr}^G_k V)) \oplus \mathrm{gr}^M_{k-1}((NV \cap G_{k-1})/NG_{k-1})[-1](-1).$$

As we shall see below in the pure case, the first terms in the right hand side of the above formulae together with F become CHC of weight k, and the second terms together with F are also CHC of weight k by virtue of the Admissibility. Hence $((\mathbf{V}, V), \mathcal{M}, F)$ and $((\mathbf{V}, NV), \mathcal{M}, F)$ are CMHC on C.

The Leray spectral sequence of $\mathbf{V}$ for j is nothing but the long exact sequence of hypercohomology groups of the exact sequence

$$0 \to (\mathbf{V}, NV) \to (\mathbf{V}, V) \to Q^{\cdot} \to 0,$$

where $Q^{\cdot}$ is the cokernel with the induced weight and Hodge filtrations.

In the pure case, by Norm Estimate of N-filtration in (2.3), we can take the following L^2-resolution of $j_*\mathbf{V}_{\mathbf{C}}$ with respect to a metric on C which coinsides to the Poincaré metric $\frac{i}{2}\frac{dz\wedge d\bar{z}}{(|z|\log|z|)^2}$ near the punctures Σ:

$$\Omega^{\cdot}(V)_{(2)} : L_0\widetilde{\mathcal{V}} \xrightarrow{\nabla} \Omega^1_C(\log \Sigma) \otimes L_{-2}\widetilde{\mathcal{V}},$$

which carries the Hodge filtration induced from the one on $\widetilde{\mathcal{V}}$. One can show that the inclusion $\Omega^{\cdot}(V)_{(2)} \hookrightarrow (\mathbf{V}, NV)$, together with the filtrations F, is a filtered quasi-isomorphism. Then, by L^2-harmonic theory, one can prove that $H^n(C, j_*\mathbf{V}_{\mathbf{C}}) = H^n(C, \Omega^{\cdot}(V)_{(2)})$ carries a pure HS. One can observe also that

$$\mathrm{gr}^{\mathcal{M}}_k(\mathbf{V}, NV) \overset{\mathrm{Res}}{\underset{\sim}{\to}} \bigoplus_{\sigma\in\Sigma} P_{k-w-1}(\sigma)[-1](-1), \quad \text{where } P_j(\sigma) : \text{primitive part (see (2.3))}.$$

From this one can show that $H^n(C, j_*\mathbf{V})$ is the intersection cohomology.

(3.4) We conclude this survey by adding miscellaneous remarks from [U2].

Remark (Mixed Version of C.S. Sequence). There is a sequence in (MHS), analogous to the one in (2.4) ,arising from a semistable degeneration of pairs $f : (\mathcal{X}, \mathcal{Y}) \to \Delta$. In this case, we can verify the exactness of the part of the 1st and the 2nd cohomology groups under the assumptions: $\mathcal{Y}$ is smooth, Gysin map $H^0(Y_\infty) \to H^2(X_\infty)$ is injective and $H_{2d-1}(X_0) = 0$.

But we do not know the exactness in general.

Remark (**Z**-Structure, Partial Weight Filtration). **Z**-structure and partial weight filtrations are important for the applications to geometry. We see some examples here. There are series of semistable degenerations of pairs $f : (\mathcal{X}, \mathcal{Y}) \to \Delta$ arising from series of degenerations of pairs of a K3 surface with $g+7$ dorinary double points and a smooth curve of genus g ($g \leq 9$). The central fiber (X_0, Y_0) consists of

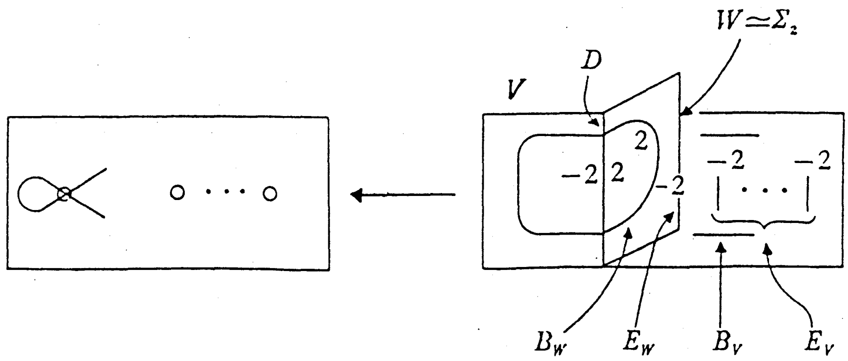
$$X_0 = V + W,\ V: \text{ a K3 surface, } W \simeq \Sigma_2 : \text{ rational ruled surface of degree } -2,$$

$$D := X_0 \cap Y_0 : \ (-2)\text{-curve on } X_0 \text{ and } (2)\text{-section on } \Sigma_2$$

$$Y_0 = Y_0 \cap V + Y_0 \cap W,$$

$$B_V + E_V := Y_0 \cap V = \ (\text{curve of genus } g-1) + (g+6)\ ((-2)\text{-curves}),$$

$$B_W + E_W := Y_0 \cap W = \ ((2)\text{-section}) + ((-2)\text{-section}).$$



In this situation, we can observe the following:

(i) The local monodromy γ on $H^2(\overset{\circ}{X}_\infty, \mathbf{Z})/tor$ splits and $N^2 = 0$.

filtrations G and L. But $(G * L)_0 \subset M_0$ with index 2 on the first term, whereas $(G * L)_0 = M_0$ on the second term (before the shifting [2]).

(iii) By the Mayer-Vietoris sequence, the isomorphism in (ii) is transformed to $(H^2(\overset{\circ}{V}_0, \mathbf{Z})/\mathbf{Z}[\overset{\circ}{D}])/tor \oplus H^2(\overset{\circ}{W}_0, \mathbf{Z}) \xrightarrow{\sim} \mathrm{gr}_0^L(H^2(\overset{\circ}{X}_\infty, \mathbf{Z})/tor)$, where the assertion similar to the last one in (ii) can be also verified.

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