数理解析研究所講究録
第 803 巻 1992 年 65-106

AN INTRODUCTION TO MORIHIKO SAITO'S THEORY OF MIXED HODGE MODULES

VERSION 0.5

YUJI SHIMIZU

Mathematical Institute
Tohoku University

The following text includes the manuscript for my talk at RIMS in the workshop "Algebraic Geometry and Hodge Theory". Due to laziness (and business) of the author, the publication of this volume has been delayed. I apologize for it to the other contributors.

The full text (Version 1.0) is to contain §§4,5 as well. A version close to 1.0 will probably appear soon in a kind of lecture notes consisting of all the survey papers in Part I.

AN INTRODUCTION TO MORIHIKO SAITO'S THEORY OF MIXED HODGE MODULES

YUJI SHIMIZU

Mathematical Institute
Tohoku University

§ Introduction

0.1 We would like to introduce you to the theory of mixed Hodge modules which is inaugurated and built up by Morihiko Saito [S1,2]. This is a generalization of Deligne's mixed Hodge theory and gives us a tool in parallel with the theory of constructible sheaves (or complexes) in the classical topology on complex analytic spaces or in the étale topology on algebraic varieties over a field.

This exposition is intended to be slightly technical and contains precise definitions, main results and indications on their proofs. For a person who needs only the main results, there is a short neat exposition by Saito himself [S3].

We made efforts for this exposition to be comprehensible and omitted proofs of the results, for which we indicated their sources. This exposition is not self-contained being a part of a series of lectures prepared to learn mixed Hodge modules, which was given during the workshop "Algebraic Geometry and Hodge theory" at RIMS, December 1991.

Among others we assume the following prerequisites :

(1) on D-modules, perverse sheaves and the Riemann-Hilbert correspondence, cf.[H],[Bo],[BBD],
(2) on variations of (mixed) Hodge structures (or period maps) and their extensions, cf.[U],[Sc],[CK],[SZ],
(3) on vanishing-cycle functors for D-modules, cf.[MuS],[Sa].

This exposition does not include applications of mixed Hodge modules. As to Kollár's conjecure and the Kodaira vanishing, Masahiko Saito's exposition [MaS] in this volume. As to the representation theory, cf.[T],[L]. As to the Hodge(-type) conjectures, cf.[S13].

We briefly explain the contents, cf.the contents below.

In §1, we give the filtered version of materials (direct images, vanishing-cycle functors, etc.), treated in [H],[MuS]. These are necessary to give the definition of

Hodge modules and their polarization, which is the theme of §2. We also state two basic results (2.4.2, 2.4.4) relating polarizable Hodge modules and polarizable variations of Hodge structure. The former result relies on the stability theorem in §4 and the latter one relies on the combinatorial description of D-modules (or perverse sheaves) in the normal crssing situation, cf.5.3.

In §3 we give the definition of mixed Hodge modules. The theory of gluing mixed Hodge modules is stated as an application of Beilinson's functor ξ_g. The definition of standard operations is given in 3.4 and the treatment of derived category is briefly mentioned in 3.5. To understand these topics, a certain knowledge of sheaf theory is indispensable.

In order to lower the burden, we pretended in the main text to work on smooth spaces. This attitude is guaranteed by the consideration in §Appendix.

We recollected in §§Complements the definition of and comments on key technical notions : strictness (C.1.1,C.1.2), compatible filtrations (C.1.3), relative monodromy filtrations (C.2).

CONTENTS

§Introduction

§1 Filtered D-modules

§1.1 Induced filtered D-modules; §1.2 Filtered differential complexes and operations; §1.3 Vanishing-cycle functors

§2 Hodge modules

§2.1 Filtered D-modules with k-structure and strict support decomposition; §2.2 Definition of Hodge modules; §2.3 Polarization for Hodge modules; §2.4 Variations of Hodge structure and Hodge modules

§3 Mixed Hodge modules

§3.1 Specialization and prolongation; §3.2 Mixed Hodge modules; §3.3 More about the extendability; §3.4 Standard operations; §3.5 Derived category of mixed Hodge modules

§4 Stability and decomposition theorem

§5 Admissible variations and mixed Hodge modules

§Appendix – Hodge modules on singular spaces

§Complements 1 – Strictness and compatible filtrations –

§Complements 2 – Relative monodromy filtrations –

§References

0.2 Motivations for mixed Hodge modules

There are several motivations to introduce the notion of mixed Hodge modules. The first is the Hodge theory with degenerating coefficients (= VHS, i.e., variations of Hodge structure) over a curve by Zucker [Z1], which includes the case of cohomology of open varieties. Its relativization or localization was an issue. It is also related to the third motivation.

The second is the problem of finding a notion of "good" variations of mixed Hodge structure (= VMHS) in the sense of Deligne [Weil II, 1.8.14-15]. This is tightly related with the above issue and is answered by the notion of admissible VMHS by Steenbrink-Zucker [SZ] and Kashiwara [K1], cf.[S2,3.27].

The third is the problem of showing existence of pure Hodge structure on intersection cohomologies of singular algebraic varieties over the field of complex numbers $\mathbb{C}$. Brylinski formulated a conjecture on this problem using Kashiwara-Kawai's canonical good filtration of regular holonomic D-modules, which turned out to be intractable.

For this problem, there was a model to follow, namely, the theory of mixed perverse sheaves by Beilinson-Bernstein-Deligne-Gabber [BBD] over algebraic varieties defined over finite fields. It gave us the so-called decomposition theorem which holds also for VHS of geometric origin over varieties over $\mathbb{C}$. Thus finding a proof of it without methods in characteristic p is the fourth motivation.

Below these motivations, there immersed, as an analogy or a phylosophical support, yoga of weights of Grothendieck, which was a strong impetus to Deligne's mixed Hodge theory or to the weight formalism.

0.3 We enumerate more techinical motivations for the notion of mixed Hodge modules.

First, consideration of limit (or limiting) mixed Hodge structure [Sc,St] led to the study of nearby-cycle functors for one-parameter families. Its generalization to the case of higher dimensional base constitutes a step in the proof of stability [S1,5.3.1].

Second, the theory of Brieskorn lattices [S9] required a fine study of (microlocal version of) Gauss-Manin systems or projective direct images and led to the nomination of "V-filtration".

Third, the interpretation of VHS in terms of D-modules led to use systematically filtered D-modules and the strictness C.1.2 of their de Rham complexes led to the notion of filtered differential complexes.

Fourth, the key to the inductive definition of mixed Hodge modules and their polarizations was Gabber's theorem which states that the (relative) monodromy filtration equals the weight filtration in the ℓ-adic situation.

Finally, Beilinson's work on the derived category of perverse sheaves [B1] served as a model for deriving standard functors for mixed Hodge modules (6 operations of Grothendieck and ψ, ϕ).

0.4 Some cautions and indications to the reader are in order.

A complex space always means a separated and reduced complex analytic space. An algebraic variety (over $\mathbb{C}$) means a separated reduced scheme of finite type over $\mathbb{C}$.

We consider only right D-modules in this exposition. We sometimes omit the adjetive "right". The reader who likes left ones may consult the summary in [S5].

The reader is recommended to look quickly §Complements 1 at the end of this exposition. §1 may be omitted and refered when necessary except the preparation on nearby and vanishing cycle functors. §Complements 2 may be refered to when the reader doesn't know the relative monodromy filtration.

We abbreviate the adjective "*quasi-unipotent and regular*" (cf.1.3.3) as "q-r" throughout this exposition.

§1 Filtered D-modules

We give preparations on filtered versions of standard notions in the theory of D-modules, especially vanishing-cycle functors.

§1.1 Induced filtered D-modules

1.1.1 The sheaf of rings of (linear) differential operators with holomorphic coefficients D_X on a complex manifold X (of complex dimension D_X) is naturally filtered by the order of operators : $F_pD_X = \{P \in D_X | ordP \leq p\}$.

A right filtered D-modules or (D, F)-module (M, F) is, by definition, a right D-module endowed with an increasing filtration $\{F_pM\}$ which is exhaustive (i.e. $\cup_p FpM = M$), (locally) discrete (i.e. $F_pM = 0$ $(p \gg 0)$ and compatible with (D, F) (i.e. $F_pM \cdot F_qD \subset F_{p+q}M(\forall\, p, q))$. Right filtered D-modules form an exact category $MF(D_X)$. In fact, $MF(D_X)$ is a full subcategory of the category of graded modules over the graded ring (Rees ring) $\oplus_p F_pD$ (which are locally discrete).

1.1.2 In order to treat operations on filtered D-modules, we introduce the notion of induced D-modules. To a filtered $\mathcal{O}_X$-module (L, F) (which is locally discrete), we associate a filtered D-module $(M, F) =: (L, F) \otimes_{\mathcal{O}} (D_X, F)$:

$$
\begin{aligned}
M &= L \otimes_{\mathcal{O}} D_X \\
F_pM &= \sum_i F_{p-i}L \otimes F_iD_X
\end{aligned}
$$

A filtered D-modules obtained in this way is called an induced (filtered) D-module.

There is a canonical resolution of $(\mathcal{O}, F)$ by induced modules :

$$
\begin{aligned}
&(D_X \otimes_{\mathcal{O}} \wedge^{-\cdot}T_X, F) \to (\mathcal{O}, F) \\
&F_p((D_X \otimes_{\mathcal{O}} \wedge^{-i}T_X) = F_{p+i} \otimes \wedge^{-i}T_X
\end{aligned}
$$

It is locally isomorphic to the Koszul complex $(D_X; \partial_i(1 \geq i \geq d_X))[d_X]$.

Using this, we obtain the following :

Lemma 1.1.3. *For a filtered D-module (M, F), we have a filtered resolution by induced modules :*

$$
(M, F) \otimes (D_X \otimes_{\mathcal{O}} \wedge^{-\cdot}T_X, F) \to (M, F)
$$

We can prove this lemma by taking Gr^F of the morphism.

We recall the derived categories of filtered (induced) D-modules very briefly. For their definitions, see [S1, §2].

Proposition 1.1.4. *We have the following commutative diagram of functors : ($\star = b, +, -, \infty$)*

$$\begin{array}{ccccc} D^\star F_i^f(D_X) & \hookrightarrow & D^\star F_i(D_X) & \stackrel{\sim}{\rightarrow} & D^\star F(D_X) \\ \wr\downarrow & & \wr\downarrow & & \wr\downarrow \\ DF_i^f(D_X)^\star & \hookrightarrow & DF_i(D_X)^\star & \stackrel{\sim}{\rightarrow} & DF(D_X)^\star. \end{array}$$

Here $\stackrel{\sim}{\rightarrow}$ means equivalence of categories and $\hookrightarrow$ means fully faithful.

Comments on the notations : $D^\star F()$ denotes the derived category obtained starting from the category of complexes with boundedness condition $\star$ while $DF()^\star$ denotes the subcategory of $DF()$ consisting the objects with boundedness condition $\star$ on their cohomologies. The superscript f in the first row means the condition :

$$G_i(M^\cdot, F) := ((F_i M^\cdot)D_X, F) = (M^\cdot, F) \quad i \gg 0 \text{ (loc.)}$$

while the superscript f in the second row means the condition :

$$Gr^F Gr_i^G(M^\cdot, F) \text{ is acyclic} \quad i \gg 0 \text{ (loc.)}$$

The meaning of the following theorem is clear.

Proposition 1.1.5. *We have the equivalence of categories :*

$$D^b_{coh} F_i^f(D_X) \stackrel{\sim}{\rightarrow} D^b_{coh} F(D_X),$$

where the left hand side consists of objects $(M^\cdot, F)$ satisfying the condition :

$$\mathcal{H}^j Gr_i^F Gr_i^G M^\cdot \textit{ is coherent over } \mathcal{O}_X \ \forall i, j$$

The proof is reduced to the following :

Lemma. *Let (M, F) be an object of $MF(D_X)$ such that $Gr^F M$ is coherent over $Gr^F D_X$. Then there exist a filtered quasi-isomorphism $(L^\cdot, F) \to (M, F)$ such that*

$$L^j = 0 \qquad \textit{for} \quad j \notin [-2d_X, 0]$$
$$(L^j, F) = \oplus_p L_p^j \otimes_{\mathcal{O}_X} (D_X, F[p])$$

The proof is carried out as follows. We can find a resolution $(L^\cdot, F)$ satisfying the second condition in the lemma. Then we put

$$(K, F) = Ker((L^{-2d_X+1}, F) \to (L^{-2d_X+2}, F)).$$

To check that it is induced, free of finite type, one notes that $Gr^F K$ is locally a direct factor of $Gr^F L^{-2d_X+1}$ since

$$Ext^i_{GrD}(Gr^F K, N) \cong Ext^{i+2d_X}_{GrD}(Gr^F M, N) = 0.$$

Then one can conclude by Nakayama's lemma.

§1.2 Filtered differential complexes and operations

1.2.1 In order to treat the direct image, one introduces the notion of filtered differential complexes.

Put

$$DR(M) := M \otimes_{\mathcal{D}_X} \mathcal{O}$$

for a right D-module. If M is an induced module $L \otimes_{\mathcal{O}_X} D_X$, then we have

$$DR(M) = L.$$

Consider the following functor :

$$DR : M_i(D_X) \to M(\mathbb{C}_X).$$

It is faithful, i.e.,

$$Hom_D(M_1, M_2) \to Hom_{\mathbb{C}}(DR(M_1), DR(M_2))$$

is injective for $M_i = L_i \otimes_{\mathcal{O}_X} D_X$. We denote its image by $Hom_{Diff}(L_1, L_2)$. Put

$$F_pHom_{Diff}(L_1, L_2) :=Hom_{\mathcal{O}}(L_1, L_2 \otimes_{\mathcal{O}} F_pD)$$
$$\subset Hom_{\mathcal{O}}(L_1, L_2 \otimes D) =: Hom_{Diff}(L_1, L_2).$$

An element in this subspace is called a *differential morphism* of order $\leq p$.

Next put for filtered $\mathcal{O}$-modules (L_i, F)

$$Hom_{Diff}((L_1, F), (L_2, F)) := \{\psi \in Hom_{Diff}(L_1, L_2); F_pL_1 \to L_1 \xrightarrow{\psi} L_2 \to L_2/F_{p-q-1}(\forall p, q)\}.$$

An element ψ is called a *filtered diferential morphism.*

Let us denote by $MF(\mathcal{O}_X, Diff)$ the category of filtered $\mathcal{O}_X$-modules (L, F) with $F_pL = 0$ for $p << 0$ as objects and filtered differential morphisms as morphisms. Denote by $MF^f(\mathcal{O}_X, Diff)$ the full subcategory of those objects with finite filtration.

1.2.2 In order to relate filtered differential complexes with filtered D-modules, consider the following functors. First put

$$DR^{-1}(L, F) := (L, F) \otimes (D_X, F),$$

which gives rise to the equivalence :

$$DR^{-1} : MF^{(f)}(\mathcal{O}_X, Diff) \to MF_i^{(f)}(D_X).$$

Next consider

$$\widetilde{DR}(M, F) := (M, F) \otimes_D (D_X \otimes_{\mathcal{O}} \Lambda^{-\cdot}T_X, F)$$

whose $-i$-th term is equal to $(M, F)\otimes_{\mathcal{O}}(\Lambda^iT_X, F)$ under the convention $Gr_p^F \Lambda^iT_X = 0\,(p \neq i)$. This complex is locally isomorphic to the (shifted) Koszul complex $K(M; \partial_i(1 \geq i \geq d_X))[d_X]$.

Lemma. *$\widetilde{DR}(M,F)$ is a complex in $MF(\mathcal{O}_X, Diff)$ and the following holds :*

$$DR^{-1}(\widetilde{DR}(M,F)) \cong (M,F) \otimes (D_X \otimes_{\mathcal{O}} \Lambda^{-\cdot} T_X, F)$$

which is quasi-isomorphic to (M,F) by Lemma 1.1.3.

Then we have the following statement, cf.Proposition 1.1.4 :

Proposition 1.2.3. *DR^{-1} gives rise to the following equivalences of categories :*

$$\begin{aligned} D^\star F(\mathcal{O}_X, Diff) &\cong D^\star F(D_X), \quad D^\star F^f(\mathcal{O}_X, Diff) \cong D^\star F_i^f(D_X) \\ DF(\mathcal{O}_X, Diff)^\star &\cong DF(D_X)^\star, \quad DF^f(\mathcal{O}_X, Diff)^\star \cong DF_i^f(D_X)^\star \\ D^b_{coh}F(\mathcal{O}_X, Diff) &\cong D^b_{coh}F(D_X) \end{aligned}$$

$\widetilde{DR}$ is quasi-inverse to DR^{-1} by the above Lemma.

1.2.4 "direct image"

Let $f : X \to Y$ be a morphism. We define the direct image f_* for filtered D-modules to be the following composite :

$$DF(D_X)_f \overset{DR^{-1}\cdot\widetilde{DR}}{\longrightarrow} DF_i(D_X) \overset{DR_{X/Y}}{\longrightarrow} DF_i(f^{-1}D_Y) \overset{f_*}{\to} DF_i(D_Y).$$

Here $DF(D_X)_f$ denotes the category of filtered D_X-modules whose support is porper over Y. $DR_{X/Y}$ is the relative de Rham functor :

$$DR_{X/Y}(M,F) := (M,F) \otimes_D (D_{X\to Y}, F)$$

where we put $(D_{X\to Y}, F) = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}(D_Y, F)$ so that $DR_{X/Y}(M,F) \cong (L,F) \otimes_{f^{-1}\mathcal{O}_Y} (f^{-1}D_Y, F)$ if $(M,F) = (L,F) \otimes_{\mathcal{O}} (D_X, F)$. The right most f_* denotes the usual direct image :

$$f_*(M,F) = (\cup_p f_* F_p M, F), \ F_p(f_* M) = f_* F_p M.$$

Using the equivalence in Proposition 1.2.3, we can also define the direct image for filtered diffrential complexes :

$$f_* : D^\star F^{(f)}(\mathcal{O}_X, Diff) \to D^\star F^{(f)}(\mathcal{O}_Y, Diff).$$

1.2.5 "dual"

For $(M,F) \in MF(D_X)$, we want to consider its dual $\mathbb{D}(M,F)$. First put

$$\begin{aligned} FHom_D((M,F),(M',F)) &:= (\cup_p F_p Hom_D((M,F),(M',F)), F) \\ F_p Hom_D((M,F),(M',F)) &:= F_p = \{\phi \in Hom_D(M,M'); \phi(F_i M) \subset F_{p+i}M' \ (\forall i)\}. \end{aligned}$$

Put
$$\mathbb{D}(M,F) := FHom_D((M,F), K^{\cdot}_X \otimes_{\mathcal{O}} (D_X, F)),$$
where $K^{\cdot}_X$ is a resolution of $\omega_X[d_X]$ by $\mathcal{O}_X$-injective D_X-modules such that $K^i_X = 0$, $i < -d_X$. (We can take one such of finite length.)

Then $K^{\cdot}_X \otimes_{\mathcal{O}} (D_X, F)$ has two right D_X-module structures defined as

$$(m \otimes P)r(g) = m \otimes Pg, \quad (m \otimes P)r(\partial_i) = m \otimes P\partial_i$$
$$(m \otimes P)t(g) = mg \otimes P, \quad (m \otimes P)t(\partial_i) = m\partial_i \otimes P - m \otimes \partial_i P$$

for $g \in \mathcal{O}_X, P \in D_X, m \in K^{\cdot}_X$. We use t to consider Hom_D and r to equip a right D-module structure on $FHom_D$.

When $(M^{\cdot}, F)$ is induced from $(L, F) \in MF^f(\mathcal{O}_X, Diff)$, then we have

$$\mathbb{D}(M,F)^i = Hom_D((L,F),(K^i_X, F)) \otimes_{\mathcal{O}} (D_X, F)$$

where F on K^i_X is the trivial filtration.

Lemma 1.2.6. *$\mathbb{D}$ induces the following contravariant functor :*

$$\mathbb{D} : D^- F^f_i(D_X) \to D^+ F^f_i(D_X),$$

which preserves the subcategory $D^b_{coh} F^f_i(D_X)$ as well as $D^b_{coh} F(D_X)$.

1.2.7 We can define "dual" for a filtered differential complex $(L^{\cdot}, F) \in C^- F^f(\mathcal{O}_X, Diff)$:
$$\mathbb{D}(L^{\cdot}, F) := Hom_D((L^{\cdot}, F), (K^{\cdot}_X, F))$$
Then the functor $\mathbb{D}$ preserves the category $D^b_{coh} F^f(\mathcal{O}_X, Diff)$ and commutes with DR^{-1}.

Denote by $Forget$ the forgetful functor :

$$DF(\mathcal{O}_X, Diff) \to D(\mathbb{C}_X).$$

It is easy to see that $Forget(\widetilde{DR}(K^{\cdot}_X, F)) \simeq a^!_X \mathbb{C} \simeq \mathbb{C}_X(d_X)[2d_X]$. The natural morphism

$$Forget(Hom_{Diff}((L^{\cdot}, F), \widetilde{DR}(K^{\cdot}_X, F))) \to Hom_{\mathbb{C}}(Forget(L^{\cdot}, F), a^!_X \mathbb{C})$$

induces the following functorial morphism :

$$Forget(\mathbb{D}(L^{\cdot}, F)) \to \mathbb{D}(Forget(L^{\cdot}, F)).$$

Proposition. *The above functorial morphism is isomorphic when restricted to* $D^b_{coh}F^f(\mathcal{O}_X, Diff)$.

§1.3 Vanishing-cycle functors

1.3.1 Let X_0 be a closed smooth subvariety of X of codimension 1 defined by a local equation t and set

$$X_0 \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} X^* = X \setminus X_0.$$

Then we dispose the filtration $V.D_X$ recalled in Mutsumi Saito's expository lecture [MuS].

Consider the category $MV(D_X)$ of (right) coherent D-modules equipped with a V-filtration. It is an abelian category and its morphisms are strictly compatible with the filtration V.

First recall the following elementary

Proposition 1.3.2 ([S1,3.1.8]). *Let M be an object of $MV(D_X)$. Then the following hold.*

(1) *The smallest subobject M' of M such that $M'|_{X^*} = M|_{X^*}$ is given by $(V_\alpha M)D_X$ for any $\alpha < 0$.*
(2) $M/M' \simeq i_*Coker(Gr^V_{-1}M \xrightarrow{\partial_t} Gr^V_0 M)$.
(3) $H^0_{[X_0]}(M) = i_*Ker(Gr^V_0 M \xrightarrow{t} Gr^V_{-1}M)$.

We now proceed to consider the filtered version of the V-filtration.

Definition 1.3.3

Let (M, F) be a coherent filtered D-module.

1) (M, F) is said to be *quasi-unipotent and regular* (abbreviated as "q-r") along X_0 if

(1) M admits a V-filtration along X_0.
(2) $(F_pV_\alpha M)t = F_pV_{\alpha-1}M$ for $\alpha < 0$.
(3) $(F_pGr^V_\alpha M)\partial_t = F_{p+1}Gr^V_{\alpha+1}M$ for $\alpha > -1$.
(4) $Gr^F Gr^W_i Gr^V_\alpha M$ is coherent over $Gr^F D_{X_0}$.

Here W is the monodromy fltration of $t\partial_t - \alpha$ on $Gr^V_\alpha M$. (Note that $Gr^V_0 D_X \simeq D_{X_0}[t\partial_t]$.) (When (1), (2), (3) are satisfied, we say that (M, F) admits a V-filtration along X_0.)

2) Let f be a (local) holomorphic function on X and $i_f : X \to X \times \mathbb{C}$ its graph map. (M, F) is said to be q-r along f if $i_{f_*}(M, F)$ is q-r along $X \times \{0\}$.

Recall that $i_{f_*}(M, F) \simeq (M \otimes_{\mathbb{C}} \mathbb{C}[\partial_t], F)$, where F is given by the convolution of F on M and the degree with respect to ∂_t.

The following lemma is fundamental for the later consideration.

Proposition 1.3.4. (i) (2) $\Longleftrightarrow F_pV_{<0}M = V_{<0}M \cap j_*j^{-1}F_pM \quad (\forall p)$.
(i') (2) *& $t : Gr_0^V M \to Gr_{-1}^V M$; strict* $\Longleftrightarrow F_pV_0M = V_0M \cap j_*j^{-1}F_pM \quad (\forall p)$.
(ii) *If* $(Gr_{-1}^V M)\partial_t = Gr_0^V M$, *i.e.,* $M = (V_{<0}M)D_X$, *then* (3) *for* $\alpha \geq -1$
$\Longleftrightarrow F_pM = \sum_{i\geq 0}(F_{p-i}V_{<0}M)\partial_t^i \quad (\forall p)$.
(iii) *If* $SuppM \subset X_0$, *then* (3) $\Longleftrightarrow F_p = \sum_{i\geq 0} F_{p-i}M_0 \otimes \partial_t^i \quad (\forall p)$.
Here M_0 *is given by* $M_0 = Gr_0^V M = V_0M = Ker(t : M \to M)$ *so that* $M = M_0 \otimes_{\mathbf{C}} \mathbb{C}[\partial_t]$.

Remark 1.3.5 Assume that (M, F) is q-r along X_0.
1) If $(Gr_{-1}^V M)\partial_t = Gr_0^V M$, the following holds :

$$F_pM = \sum_{i\geq 0}(V_{<0}M \cap j_*j^{-1}F_{p-i}M)\partial_t^i \quad (\forall p).$$

If $t : (Gr_0^V M, F) \to (Gr_{-1}^V M, F)$ is strict and injective, the following holds :

$$F_pM = \sum_{i\geq 0}(V_0M \cap j_*j^{-1}F_{p-i}M)\partial_t^i \quad (\forall p).$$

2) $V_{<0}$ is determined by $M|_{X^*}$. Moreover, if M is given by a VHS (L, F) on $X|_{X^*}$ $(L = L_{\mathcal{O}})$,

$$M|_{X^*} = \Omega_{X^*}^{d_X} \otimes L \quad F_pM|_{X^*} = \Omega_{X^*}^{d_X} \otimes F_{p+d_X}L.$$

As to the V-filtration, we have the following relation :

$$V_{<0}M = \Omega_{X^*}^{d_X} \otimes \widetilde{L}^{>-1}$$
$$V_0M = \Omega_{X^*}^{d_X} \otimes \widetilde{L}^{\geq -1}.$$

Here $\widetilde{L}^{\geq -1}$ (resp. $\widetilde{L}^{>-1}$) denotes Deligne's canonical extension of L with eigenvalues of the residue operator $res\nabla$ contained in $(-1, 0]$ (resp. $[-1, 0)$).

Recall that the condition "$(Gr_{-1}^V M)\partial_t = Gr_0^V M$" is equivalent to the one "M is a minimal extension of $M|_{X^*}$". This, combined together with the ones (2) and (3) for $\alpha \geq -1$, implies that F_pM is determined by $(M, F)|_{X^*}$.

Now we state a criterion for q-r-ness in the case $SuppM \subset g^{-1}(0)$.

Proposition 1.3.6 ([S1,3.2.6]). *Let* i_g *be the graph morphism for a function* g, $i_0 : X = X \times \{0\} \to X$ *the natural inclusion. Suppose that* $SuppM \subset g^{-1}(0)$ *for a coherent filtered D-module* (M, F).

The following conditions are equivalent :

(1) $(F_pM)g \subset F_{p-1}M$ *for any* p.
(2) (M, F) *is q-r along* g.
(3) *One has a (canonical) isomorphism :* $i_{g*}(M, F) \simeq i_{0*}(M, F)$.

The following compatibility of ψ, ϕ and DR is indispensable for showing the fact that the axiom of polarization is satisfied by the Hodge module coming from a polarized VHS.

Proposition 1.3.7 (cf.[MuS]). *Assume that (M, F) is q-r along X_0. Put $K = DR_X(M)$ and let $T = T_s T_u$ be the Jordan decomposition of the local monodromy T along X_0. Then the isomorphism*

$$DR_{X_0}(Gr_\alpha^V M) \cong \begin{cases} \psi_{e(\alpha)} DR_X M[-1] & -1 \leq \alpha < 0 \\ \phi_1 DR_X M[-1] & \alpha = 0 \end{cases}$$

$$\begin{array}{ccc} t\partial_t - \alpha & \leftrightarrow & N \\ \partial_t & \leftrightarrow & can \\ t & \leftrightarrow & Var \end{array}$$

is given by A_α, B_α (briefly described below). $e(\alpha) = exp(2\pi i\alpha)$ and $\psi_\lambda(K)$ denotes $Ker(T_s - \lambda : \psi(K) \to \psi(K))$.

We only recall the construction of the above isomorphism very briefly.

Let $\widetilde{S}^* \to S^* = S \setminus \{0\}$ be the universal covering of the punctured disc S^*, $\tilde{j} : \widetilde{S}^* \to S$ the natural projection. Consider the following diagram :

$$\begin{array}{ccccc} \widetilde{X}^* = X \times_S \widetilde{S}^* & \to & X^* & & \\ \tilde{j} & \searrow & \downarrow & & \\ & & X & \xleftarrow{i} & X_0 \end{array}$$

Here $X \to S$ is given by the equation t.

Recall the definition of nearby-cycle and vanishing-cycle functors :

$$\psi L := i^{-1} R\tilde{j}_* \tilde{j}^{-1} L$$
$$\phi L := Cone(i^{-1}L[-1] \to i^{-1} R\tilde{j}_* \tilde{j}^{-1} L[-1])[1]$$

and also the moderate nearby-cycle functor, cf.[MuS] :

$$\psi_\alpha^{al} N := i^{-1} V_\alpha N \otimes_{\mathbb{C}\{t\}} \widetilde{\mathcal{O}}^\alpha [t^{-1}]$$
$$\widetilde{\mathcal{O}}^\alpha := t^\alpha \mathbb{C}\{t\}[logt] \subset (\tilde{j}_* \mathcal{O}_{\widetilde{S}^*})_0$$

(L is an arbitrary complex of sheaves and N will be $DR_{X/S}M$ in the following.)

The isomorphism in the proposition is the composition of three morphisms. The first one is

$$DR_{X_0} Gr_\alpha^V M \cong Gr_\alpha^V DR_{X/S} M$$

where $DR_{X/S}$ is defined by

$$DR_{X/S}V_\alpha M = V_\alpha M \otimes^L_{D_{X/S}} \mathcal{O}_X$$
$$D_{X/S} = \{P \in D_X; [P,t] = 0\}.$$

The second one is $A_\alpha : V_0\widetilde{DR_S}\psi^{al}_\alpha N \cong Gr^V_\alpha N[1]$ given by the following diagram :

$$\begin{array}{ccccc} V_{-1}\psi^{al}_\alpha N & = & i^{-1}V_\alpha N \otimes_{\mathbf{C}\{t\}} \widetilde{\mathcal{O}}^{\alpha+1} & \overset{(*)}{\to} & Gr^V_\alpha N \\ \partial_t \downarrow & & \downarrow & & \downarrow \\ V_0\psi^{al}_\alpha N & = & i^{-1}V_\alpha N \otimes_{\mathbf{C}\{t\}} \widetilde{\mathcal{O}}^{\alpha} & \to & 0 \end{array}$$

The cone of the first (and the second) column is $V_0\widetilde{DR_S}\psi^{al}_\alpha N$. The morphism (*) is given by

$$\sum u_j \otimes t^{\alpha+1}(logt)^j/j! \mapsto u_0.$$

The third morphism is

$$B_\alpha : V_0\widetilde{DR_S}\psi^{al}_\alpha N \hookrightarrow \widetilde{DR_S}\psi^{al}_\alpha N \to ImB_\alpha$$
$$\subset \widetilde{DR_S}\psi N = \widetilde{DR_S}\psi DR_{X/S}M \simeq \psi DR_X M$$

which is induced from the natural morphism $\psi^{al}_\alpha M \to \psi M$, and the image ImB_α is identified with $\psi_{e(\alpha)}DR_X M$ $(\alpha \neq 0)$. The case for $\alpha = 0$ is treated similarly.

§2 Hodge modules

In this §, we give the definition of (polarizable) Hodge modules after some preparations in 2.1.

§2.1 Filtered D-modules with k-structure and strict support decomposition

2.1.1 Let k be a subfield of $\mathbb{R}$ and X a complex manifold as in §1. We use mostly $k = \mathbb{Q}$ or $\mathbb{R}$.

We have the following functors :

$$DR : MF_h(D_X) \to Perv(\mathbb{C}_X)$$
$$\otimes_k \mathbb{C} : Perv(k_X) \to Perv(\mathbb{C}_X).$$

Consider the following category :

$$MF_h(D_X, k) := MF_h(D_X) \times_{Perv(\mathbb{C}_X)} Perv(k_X)$$

whose object is a triple $((M, F), K, \alpha)$ consisting of $(M, F) \in MF_h(D_X)$, $K \in Perv(k_X)$ and an isomorphism $\alpha : DR(M, F) \simeq \mathbb{C} \otimes_k K$ in $Perv(\mathbb{C}_X)$. (We often omit α in the notation.) Such a triple is called a filtered holonomic D-module with a k-structure.

Lemma 2.1.2. *$MF_h(D_X, k)$ is an exact category.*

A sequence is exact iff its images are so in $MF_h(D_X)$ and in $Perv(k_X)$.

$Ker, Coker, Im, Coim$ exist in $MF_h(D_X, k)$.

2.1.3 "twist à la Tate"

For $\mathfrak{M} = (M, F, K) \in MF_h(D_X, k)$ and $n \in \mathbb{Z}$, put

$$\mathfrak{M}(n) := (M(n), F[n], K(n))$$
$$M(n) = M \otimes_{\mathbb{Z}} \mathbb{Z}(n), \ K(n) = K \otimes_{\mathbb{Z}} \mathbb{Z}(n)$$
$$\mathbb{Z}(n) = (2\pi i)^n \mathbb{Z} \subset \mathbb{C}, \ F[n]_k = F_{k-n}.$$

2.1.4 Let us define the direct image functor $\mathcal{H}^i f_*$ for a proper morphism $f : X \to Y$. If $f_*(M, F)$ (cf.1.2.4) is strict (cf.C.1.2), then it makes sense to speak of $\mathcal{H}^i f_*(M, F)$ and we put

$$\mathcal{H}^i f_* \mathfrak{M} := (\mathcal{H}^i f_*(M, F), {}^p\mathcal{H}^i f_* K)$$

which is equipped with the isomorphism

$${}^p\mathcal{H}^i f_* \alpha : DR_Y \mathcal{H}^i f_*(M, F) = {}^p\mathcal{H}^i f_* DR_X(M, F) \simeq {}^p\mathcal{H}^i f_*(K \otimes_k \mathbb{C}).$$

The case we need in a moment is the case when f is a closed immersion. In this case, $f_*(M, F)$ is just a filtered D-module.

2.1.5 "vanishing-cycle functors"

Let g be a (locally defined) holomorphic function on X and denote by $i_g : X \to X \times \mathbb{C}$ its graph. For $\mathfrak{M} = (M, F, K) \in MF_h(D_X, k)$, put

$$(\widetilde{M}, F) = i_{g*}(M, F).$$

We define the functors $\psi_g, \phi_{g,1}$ as follows :

Definition.

$$\psi_g(\mathfrak{M}) := \left(\oplus_{-1\geq\alpha<0}(Gr_\alpha^V\widetilde{M}, F[1]),{}^p\psi_g K\right)$$
$$\phi_{g,1}(\mathfrak{M}) := \left((Gr_0^V\widetilde{M}, F),{}^p\phi_{g,1}K\right)$$

For the comparison isomorphisms (cf.2.1.1) omitted in the above definition, we use the commutation isomorphisms of DR with $\psi_g, \phi_{g,1}$ in §1. We naturally get the following morphisms in $MF_h(D_X, k)$:

$$can : \psi_{g,1} \to \phi_{g,1}, \quad Var : \phi_{g,1} \to \psi_{g,1}(-1)$$
$$N : \psi_g \to \psi_g(-1)$$

Definition. *We say that* $\mathfrak{M}$ *has* strict support Z *if* $SuppM = Z$ *and* $\mathfrak{M}$ *has no subobject nor quotient whose support is smaller than* Z.

Lemma 2.1.6. *Assume that* (M, F) *is q-r along* g.
(1) *The following are equivalent :*
(a) $\mathfrak{M}$ *(or* K *or* (M, F)*) has no subobject (resp. quotient) with support in* $g^{-1}(0)$.
(b) $can : \psi_{g,1} \to \phi_{g,1}$ *is surjective (resp.* $Var : \phi_{g,1} \to \psi_{g,1}(-1)$ *is injective).*
(2) *Moreover, the following are equivalent.*
(a) $\phi_{g,1} = KerVar \oplus Imcan$.
(b) $\mathfrak{M} = (M_1, F, K_1) \oplus (M_2, F, K_2)$ *(in* $MF_h(D_X, k)$ *such that* $SuppM_2 \subset g^{-1}(0)$ *and that* $\mathfrak{M}_1$ *has no suobject nor quotient with support in* $g^{-1}(0)$.

(We call these equivalent properties the *decomposion property* (dec) with respect to g.)

Remark 2.1.7 1) In the above lemma we can forget the filtrations F, for the functor

$$MF_h(D_X, k) \to M_h(D_X) \times_{Perv(\mathbf{C}_X)} Perv(k_X)$$

is faithful.

2) To see the geometric meaning of the property (dec), the following facts are useful.

The largest quotient (resp. subobject) of $\tilde{M} = i_{g*}M$ with support in $X \times \{0\}$ is $\tilde{M}/(V_{<0}\tilde{M})D_{X\times\mathbf{C}}$ (resp. $i_*Ker(t : Gr_0^V \to Gr_{-1}^V\tilde{M})$), where $i : X \times \{0\} \to X \times \mathbb{C}$ is the inclusion. Note that $(V_{<0}\tilde{M})D_{X\times\mathbf{C}}$ is nothing but the minimal extension of $\tilde{M}|_{X\times\mathbf{C}^*}$.

Similarly the largest quotient (resp. subobject) of $\tilde{K} = i_{g*}K$ with support in $X\times\{0\}$ is $\tilde{K}/\tilde{K}_1$ (resp. $\tilde{K}_2$), where $\tilde{K}_1$ (resp. $\tilde{K}_2$) is the perverse sheaf corresponding to $(\tilde{K}|_{X\times\mathbf{C}^*}, (\psi_{g,1}\tilde{K} \rightleftarrows Imcan))$ (resp. $(0, (0 \rightleftarrows KerVar))$) in the theory of gluing of perverse sheaves by Deligne-MacPherson-Verdier, cf.[B2],[V1].

3) The following is a filtered isomorphism thanks to the condition on V-filtration : $(F_pV_\alpha M)t = F_pV_{\alpha-1}M$.

$$Ker(\widetilde{M} \xrightarrow{t} \widetilde{M}) \simeq Ker(Gr_0^V\widetilde{M} \xrightarrow{t} Gr_{-1}^V\widetilde{M})$$

It is not difficult to see the following :

Lemma 2.1.8. *Assume that (M, F) is q-r along g. Then the following are equivalent.*

(a) The decomposition property (dec) holds for any g.

(b) For any open subset U of X, we have

$$(M, F, K)|_U = \oplus_Z(M_Z, F, K_Z),$$

where Z runs through closed irreducible subsets of X. ("strict support decomposition")

Definition We denote by $MF_h(D_X, k)_{dec}$ the category of filtered D-modules with the condition in the above lemma. Denote also by $MF_h(D_X, k)_Z$ the category of filtered D-modules with strict support Z.

Therefore we have

$$MF_h(D_X, k)_{dec} = \oplus_Z MF_h(D_X, k)_Z$$

(locally finite decomposition).

This is part of the reasons why the decomposition theorem (cf.§5) holds. It should be worthwhile to make the following :

Remark Given $\mathfrak{M} \in MF_h(D_X, k)_Z$ and assume that $g^{-1}(0) \not\supset Z$, and that $can : \psi_{g,1} \to \phi_{g,1}$ is a strict epimorphism. Then we have

$$F_p\widetilde{M} = \sum_i \left(V_{<0}\widetilde{M} \cap j_*j^{-1}F_{p-i}\widetilde{M}\right)\partial_t^i$$

where $j : X \times \mathbb{C}^* \overset{j}{\hookrightarrow} X \times \mathbb{C}$ is the inclusion.

§2.2 Definition of Hodge modules

Definition 2.2.1 $\mathfrak{M} \in MF_h(D_X, k)_Z$ is called a *Hodge module of weight* n if $\mathfrak{M}$ belongs to the category $MH(X, k; n)$ (inductively) defined in the following way :

$MH(X, k; n)$ is the largest full subcategory of $MF_h(D_X, k)_{dec}$ satisfying the following conditions HM1),2) (for variable X, n) :

HM1) If $\mathfrak{M} \in MH(X, k; n)$ has $suppM = \{x\}$, then there exists a k-Hodge structure $(H_{\mathbb{C}}, F, H_k)$ of weight n such that

$$i_{x*}(H_{\mathbb{C}}, F, H_k) \simeq (M, F, K)$$

where $i_x : \{x\} \hookrightarrow X$ is the inclusion.

HM2) Given $\mathfrak{M} \in MH(X, k; n)$, an open subset $U \subset X$, a holomorphic function g defined on U, and a closed subset $Z \subset U$ not contained in $g^{-1}(0)$. If (M_Z, F, K_Z) denotes the component as in Lemma 2.8, then we have

$$Gr_i^W \psi_{g,1}(M_Z, F, K_Z), Gr_i^W \phi_{g,1}(M_Z, F, K_Z) \in MH(U, k; i),$$

where W are the monodromy filtrations shifted by $n-1, n$ respectively.

We put

$$MH(X, k; n)_Z := MF_h(D_X, k)_Z \cap MH(X, k; n).$$

Then we have as for $MF_h(D_X, k)_{dec}$

$$MH(X, k; n) = \oplus_Z MH(X, k; n)_Z.$$

Remark 1) We will realize later that the condition HM2), quasi-unipotence along $\forall g$, and (dec) are strong conditions.
2) The conditions HM1),2) are local ones and stable by taking direct factors in $MF_h(D_X, k)$.

We first collect elementary properties.

Proposition 2.2.2. (1) *For a closed immersion $i : X \hookrightarrow Y$, the following functor is an equivalence of categories :*

$$i_* : MH_Z(X, k; n) \to MH_Z(Y, k; n),$$

especially $MH_{\{x\}}(X, k; n) \approx$ (k-HS of weight n).
(2) *Given a Hodge module $\mathfrak{M} = (M, F, K) \in MH_Z(X, k; n)$. Then it holds that K is an intersection complex $IC_Z(L)$ with coefficient L and $\mathfrak{M}$ is generically a variation of Hodge structure (VHS) of weight $n - d_Z$. Namely, there are a smooth, Zariski open subset $U \subset Z$ such that $K|_U[-d_Z] = L$ is a local system on U, and a VHS $(L \otimes \mathcal{O}_U, F, L)$ of weight $n - d_Z$ such that*

$$\mathfrak{M}|_{X\setminus(Z\setminus U)} \cong i_*(L \otimes \Omega_U^{d_U}, F, L)$$
$$F_p(L \otimes \Omega_U^{d_U} \cong \Omega_U^{d_U} \otimes_{\mathcal{O}} F^{-p-d_Z}(L \otimes \mathcal{O}_U)$$

(3) $Hom(MH_Z(X, k; n), MH_{Z'}(X, k; n')) = 0 \quad$ *if $Z \neq Z'$ or $n > n'$.*

A few comments on 2) in the above Proposition. Considering the holonomicity of M, it is clear that we have $K = IC_Z(L), M = L \otimes \mathcal{O}_U$ for some L. For the local freeness of $Gr_p^F M$, we use the following lemma. (We may assume $X = U$.)

Lemma. *If M is locally free $\mathcal{O}_X$-module of finite type and if $Gr_p^F M \otimes_{\mathcal{O}_X} \mathcal{O}_{X_0}$ is locally free over $\mathcal{O}_{X_0}$, then $Gr_p^F M$ is locally free over $\mathcal{O}_X$.*

The converse to 2) holds, cf.2.4.2.

Definition 2.2.3

We denote by $MF_hW(D_X, k)$ the category of filtered objects in $MF_h(D_X)$ with finite filtration (denoted by W).

Denote by $MHW(X, k)$ the full subcategory of $MF_hW(D_X, k)$ consisting of objects $((M, F, W), (K, W)) \in MF_hW(D_X, k)$ such that $Gr_i^W(M, F, K) \in MH(X, k; i)$ $(\forall i)$.

In §3, we define the category of mixed Hodge modules as a subcategory of $MHW(X, k)$ by imposing some conditions on the filtration W.

Lemma 2.2.4. *The categories $MH_Z(X, k; n), MH(X, k; n), MHW(X, k)$ are abelian categories. All morphisms are strict with respect to F and to (F, W).*

For $\mathfrak{M} = (M, F, K) \in MH_Z(X, k; n)$, we have by definition

$$(\psi_g \mathfrak{M}, W) \in MHW(X, k), \text{ then } \quad Gr_i^W \psi_g \mathfrak{M} \in MH(X, k; i)\ (\forall i),$$

where W is the N-filtration for the monodromy logarithm N centered at $n-1$. Then the above lemma implies, for example, that for a morphism $\mathfrak{M} \to \mathfrak{M}'$ we have

$$(\psi_g \mathfrak{M}, W) \to (\psi_g \mathfrak{M}', W); \text{strict}.$$

If, in addition, $Z \not\subset g^{-1}(0)$, then

$$can : (\psi_{g,1}, W) \to (\phi_{g,1}, W)$$
$$Var : (\phi_{g,1}, W) \to (\psi_{g,1}, W)(-1)$$

are strict in $MF_hW(D_X, k)$, cf.[S1,5.1.17].

§2.3 Polarization for Hodge modules

We consider the notion of polarization for Hodge modules, which generalizes that for Hodge structure and is indispensable for proving the decomposition theorem.

Lemma 2.3.1.. *For $\mathfrak{M} \in MH_Z(X, k; n)$, $Gr^F M$ is a Cohen-Macaulay $Gr^F D_X$-module. Namely, the filtered object $\mathbb{D}(M, F)$ is strict.*

cf.[S1,5.1.13].

Let $\mathfrak{M}$ be an object of $MH_hW(D_X, k)$ and

$$(M, F, K) = \oplus_Z (M_Z, F, K_Z)$$

its decompositon by strict support. Then a pairing on K

$$S : K \otimes K \to a_X^! k(r)$$

decomposes accordingly : $S = \oplus_Z S_Z$. We call $S_Z : K_Z \otimes K_Z \to a_X^! k(r)$ its Z-component.

S is said to be compatible with F if there exists a morphism $(M, F, K) \to \mathbb{D}(M, F, K)(r)$ such that $K \to \mathbb{D}K(r)$ corresponds to S under

$$Hom(K \otimes K, a_X^! k(r)) \simeq Hom(K, Hom(K, a_X^! k(r))).$$

Definition 2.3.2 An Hodge module $\mathfrak{M} \in MH_Z(X, k; n)$ is said to be polarizable if there is a pairing (called "polarization")

$$S : K \otimes K \to a_X^! k(-n)$$

compatible with F and satisfying the following (inductively defined) conditions :

P1) If $Z = \{x\}$ (thus $(M_Z, F, K_Z) = i_{x*}(H_{\mathbf{C}}, F, H_k)$), then there exists a polarization S' of $(H_{\mathbf{C}}, F, H_k)$ as a VHS (i.e. $S'(u, C\bar{u}) > 0$ $(\forall u \neq 0)$) such that $S = i_{x*}S'$.

P2) If $dim Z > 0, Z \not\supset g^{-1}(0)$, then

$$Gr_i^{W\,p}\psi_g S_Z \cdot (id \otimes N^i) : P_N Gr_{n-1+i}^{W\,p}\psi_g K_Z \otimes P_N Gr_{n-1+i}^{W\,p}\psi_g K_Z \to a_U^! k(1-n-i)$$

is a polarization on $P_N Gr_{n-1+i}^{W\,p}\psi_g(M_Z, F, K_Z)$ for $\forall i \geq 0$.

Here P_N means the primitive part with respect to the monodromy logarithm N. We identified $S_Z : K_Z \otimes K_Z \to a_U^! k(-n)$ with

$$i_{g_*} S_Z : i_{g_*} K_Z \otimes i_{g_*} K_Z \to a_{U \times \mathbf{C}}^! k(-n).$$

Remark 2.3.3 1) The following holds :

$$^p\phi_{g,1} S \cdot (can \otimes id) = {}^p\psi_{g,1} S \cdot (id \otimes Var).$$

2) The condition of polarization implies that $Gr_i^{W\,p}\phi_{g,1} S_Z \cdot (id \otimes N^i)$ is a polarization on $P_N Gr_{n+i}^{W\,p}\phi_g K_Z (\forall i \geq 0)$, since the last perverse sheaf is isomorphic to $P_N Gr_{n+i}^{W\,p}\psi_{g,1} K_Z$ via *can*.

3) If $g^{-1}(0) \supset Z$, then $^p\phi_g S_Z \simeq S_Z$.

Lemma 2.3.4. (1) *Let $i : X \hookrightarrow Y$ be a closed immersion, S a pairing on K for $\mathfrak{M} = (M, F, K) \in MH_Z(X, k; n)$. Then S is a polarization of $\mathfrak{M}$ iff i_*S is that of $i_*\mathfrak{M}$.*

(2) *If S is a polarization of $\mathfrak{M} \in MH_Z(X, k; n)$, then S is $(-1)^n$-symmetric and there exist an open subset $U \subset Z$ such that $K|_U[-d_Z] = L$ is a local system and a polarization $S' : L \otimes L \to k_U(d_Z - n)$ as a VHS such that $S|_{X\backslash(Z\backslash U)} = i_*S''$, where S'' is given by*

$$S'' := (-1)^{d_Z(d_Z-1)/2} S' : L[d_Z] \otimes L[d_Z] \to k_U(d_Z - n)[2d_Z].$$

Denote by $MH_Z(X, k; n)^p$ the full subcategory of $MH_Z(X, k; n)$ consisting of polarizable Hodge modules. Then the above lemma implies the following :

Proposition 2.3.5. *The category $MH_Z(X,k;n)^p$ of polarizable Hodge modules is semi-simple.*

§2.4 Variations of Hodge structure and Hodge modules

We state two fundamental results on the relation betweew VHS and Hodge modules. The proof is not elementary but uses the stability by projective morphism and also the calculation of vanishing cycles in the normal-crossing case.

2.4.1 Consider a VHS $(L_{\mathcal{O}}, F, L)$ of weight $w - d_X$ polarized by

$$S' : L \otimes L \to k(d_X - w).$$

Put

$$\begin{aligned}(M,F) :=&(\Omega_X^{d_X}, F) \otimes_{\mathcal{O}} (L_{\mathcal{O}}, F)\\ K :=&L[d_X]\end{aligned}$$

where $(\Omega_X^{d_X}, F)$ is defined by $Gr_i^F \Omega_X^{d_X} = 0 (i \neq -d_X)$. Then (M,F) is a filtered right D_X-module and $K = DR_X(M)$ is a perverse sheaf. Since $a_X^! k = k(d_X)[2d_X]$, to give a pairing $S : K \otimes K \to a_X^! k(-w)$ amounts to give a pairing

$$\begin{array}{ccc} H^{-2d_X}(K \otimes K) & \xrightarrow{S} & H^{-2d_X}(a_X^! k(-w)) \\ \| & & \| \\ L \otimes L & \xrightarrow{(-1)^{d_X(d_X-1)/2} S'} & k(d_X - w) \end{array}$$

Thus we have defined a pairing on K. We claim that this S is compatible with the filtration F. Namely S is induced by a filtered isomorphism $(M,F) \cong \mathbb{D}(M,F)(-w)$.

This can be seen by observing that S' defines a pairing

$$\widetilde{DR}(M,F) \otimes_{\mathbf{C}} \widetilde{DR}(M,F)(w) \to (\Omega_X^{\cdot}, F)[2d_X],$$

since

$$\widetilde{DR}(M,F) \cong (\Omega_X^{\cdot}, F) \otimes (L_{\mathcal{O}}, F)$$

where F on $\Omega_X^{\cdot}$ is the stupid filtration.

We are going to claim that a polarized VHS defines a polarized Hodge module by the above construction.

Theorem 2.4.2. *$\mathfrak{M} = (M,F,K)$ as above is a polarized Hodge module of weight w.*

We refer the detail of the proof to [S1,5.4.3] and give here an outline of the proof.

The proof is done by an induction on d_X and uses the stability by projective morphism 4.1.1. (The case $d_X = 0$ is trivial.)

One has to check the condition HM2) for a holomorphic function g.

First one reduces to the case when $g^{-1}(0)$ is normal-crossing. For this, take a resolution of $g^{-1}(0)$, a proper birational morphism $f : \tilde{X} \to X$ such that $f^{-1}g^{-1}(0)$ is normal-crossing. Note that $\widetilde{\mathfrak{M}} = f^*\mathfrak{M}$ corresponds to the polarized VHS $f^*(L_{\mathcal{O}}, F, L, S')$. By the canonical morphism

$$f^* : (M, K) \to f_*(\tilde{M}, \tilde{K}),$$

$\mathfrak{M}$ is a direct factor of $P_\ell {}^pH^0 f_*(\tilde{M}, F, \tilde{K}, \tilde{S})$ by the stability 4.1.1 & [S1,5.3.1].

Then one assumes that $g = x_1^{m_1} \cdots x_n^{m_n}$. One may assume in addition that $g = (x_1 \cdots x_n)^m$ by taking a ramified covering of X.

In this situation, one proves the following assertions.

0) (M, F) is q-r along g.

1) $\psi_{g,1}(M, F) \overset{can}{\to} \phi_{g,1}(M, F)$ is a strict epimorphism and one has

$$P_N Gr^W_{j-1} Gr^V_{-i/m} i_{g*}(M, F) \simeq a_{j*} a_j^!(M, F)[j] \otimes \epsilon^j,$$

where W is the N-filtration for $N = t\partial_t + i/m$ and ϵ is the orientation sheaf of the divisor with normal crossings $g^{-1}(0)$, cf.[S1,3.6.10].

2) There is an isomorphism

$$P_N Gr^W_{w-1+i} \psi_g(M, F) \simeq (\mathbb{C}^m) \otimes (a_{i+1})_*(a_{i+1})^*(M, F)(-i)[-i-1] \quad (i \geq 0)$$

compatibly with k-structure such that

$$Gr^W \psi_g S \cdot (id \otimes N^i) = S_m \otimes {}^p a_{i+1}^* S$$

on $P_N Gr^W_{w-1+i} \psi_g K$. ($S_m$ denotes the standard pairing on $\mathbb{Z}^m$.)

In 1),2), a_j denotes the normalization

$$a_j : \tilde{D}^{(j)} \to D^{(j)} \hookrightarrow X$$

of the union of intersections of j components of $g^{-1}(0)$.

From these assertions, the theorem follows immediately.

Note that this calculation using $\tilde{D}^{(j)}$ is central to the proof, cf.[S1,Introduction].

2.4.3 In 2) of 2.2.2, we saw that a Hodge module generically comes from a VHS. It means that a Hodge module is uniquely determined by the generic VHS if the underlying D-module is a minimal extension (i.e., the underlying perverse sheaf is a (twisted) intersection complex). The converse to this statement is true as we state below.

Let Y be a reduced divisor with normal crossings and $j : U = X \setminus Y \hookrightarrow X$ the inclusion. Given a filtered D-module with k-struture $\mathfrak{M} = (M, F, K) \in MF_h(D_X, k)$ such that $M|_U$ is $\mathcal{O}_U$-coherent and that M is the minimal extension of $M|_U$.

Put

$$(M, F)|_U = (\Omega_U^{dx}, F) \otimes (L, F)$$

so that one has $(M, F) = (j_{!*}(\Omega_U^{dx} \otimes L), F)$.

Assume that (L, F) underlies a VHS $\mathbb{H}$ polarized by S'.

Theorem 2.4.4. *One has $\mathfrak{M} \in MH_X(X,k;w)^p$. Its polarization S is induced from S'.*

Construction of S :

Use the following filtered quasi-isomorphism

$$DR(j_{!*}(M|_U), F) \to (\mathcal{L}_{(2)}(\mathbb{H}), F)[d_X]$$

in $D^bF^f(\mathcal{O}_X, Diff)$. constructed by Kashiwara and Kawai [KK]. The right complex is the L^2-complex for the Poincaré metric on U with coefficients in $\mathbb{H}$.

S' induces a pairing

$$(\mathcal{L}_{(2)}(\mathbb{H}), F) \otimes (\mathcal{L}_{(2)}(\mathbb{H}), F) \to (\mathcal{D}b^{\cdot}, F[-w])$$

which gives rise to the required pairing S. Here $\mathcal{D}b^{\cdot}$ is the complex of currents and its F is the stupid filtration.

Points of the proof :

The proof is by induction on d_X.

One checks the condition HM2) only for $\psi_{g,1}$, since $\psi_{g,1} \to \phi_{g,1}$ is strictly epimorphic.

Some reductions, cf.the proof of 2.4.2 : First one can assume that $g^{-1}(0)_{red} \cup Y$ is a divisor with normal crossings, by the stability 4.1.1. Thus one can assume that $g^{-1}(0)_{red} \subset Y$.

Then one can reduce to the case when M has unipotent monodromy and that $g = (x_1 \cdots x_n)^m$.

One proves the following two statements:

(1) $P_N Gr_j^W \psi_g \mathfrak{M}$ admits the strict support decomposition (compatibly with F) such that each Z-component corrseponds to a polarized VHS of weight $w - d_Z$ on a (Zariski) open set of Z and the pull-back of $\mathfrak{M}$ to Z is quasi-unipotent and regular of normal crossing type, cf.§4.

(2) For each point $x \in X$, consider the stalk of (M, F) at x (denote it by the same symbol). Then the monodromy filtration W on $\psi_g(M, F)$ exists in $MF(D_{\Delta^n})_{rncqu}$ and the primitive decomposition of $Gr_j^W \psi_g(M, F)$ is compatible with F.

These replace the calculation made in the proof of Theorem 2.4.2.

Then the induction proceeds.

§3 Mixed Hodge modules

§3.1 Specialization and prolongation

Mixed Hodge modules are defined as those extensions of polarizable Hodge modules which satisfy certain admissibility conditions discussed in 3.1. They are somehow defined to have nice behaviour under functorial operations. Finally they are shown to form a category as large as one expects.

3.1.1 We assume X to be smooth for simplicity. Once one has a suitable definition, the formalism in the sequel goes through in the same way in the singular case as in the smooth case, cf.Appendix.

Recall that $MHW(X)$ denotes the category of (W-)filtered objects in $MF(D_X)$ whose associated graduation belong to $MH(X,k)$. Then M has a V-filtration along g for $\mathfrak{M} = (M, F, K; W)$ and (M, F) is q-r along g. Put $\widetilde{\mathfrak{M}} = i_{g_*}\mathfrak{M}$.

Definition 3.1.2

1) Consider the following condition :

Sp1) *F, W, V on $\tilde{M}$ are compatible.*

If this condition is satisfied, put

$$\begin{aligned}\psi_g(M,F,K) =&(\oplus_{-1\leq\alpha<0} Gr^V_\alpha(\tilde{M}, F[1]), {}^p\psi_g K)\\ \phi_{g,1}(M,F,K) =&(Gr^V_0(\tilde{M},F), {}^p\phi_{g,1}K)\end{aligned}$$

We consider also the filtration induced from W on M :

$$\begin{aligned}L_i\psi_g(M,F,K) =&\psi_g W_{i+1}(M,F,K)\\ L_i\phi_{g,1}(M,F,K) =&\phi_g W_i(M,F,K)\end{aligned}$$

2) We say that the vanishing-cycle functors along g are well-defined for $\mathfrak{M}$ if the following condition is satisfied in addition to Sp1) :

Sp2) *There exists a relative monodromy filtration W on $({}^p\psi_g K, L), ({}^p\phi_{g,1}K, L)$ respectively.*

Then we define the nearby-cycle and vanishing-cycle functors as follows :

$$\begin{aligned}\psi_g\widetilde{\mathfrak{M}} =&(\psi_g(M,F,K);W)\\ \phi_{g,1}\widetilde{\mathfrak{M}} =&(\phi_{g,1}(M,F,K),W).\end{aligned}$$

Remark 3.1.3

The above condition Sp2) is equivalent to the condition that the relative monodromy filtration W is well-defined on the specialization $Sp_D(K,L)$ along the divisor defind by g. Hence the title of this subsection. We try not to use the specialization functor in this exposition and refer the reader to [S2,2.2].

Next we consider the prolongeability across the divisor $g^{-1}(0)$. Let

$$j : X \hookrightarrow \bar{X}, \quad \bar{X} \setminus X = g^{-1}(0)_{red}$$

the natural inclusion.

Definition 3.1.4

We say that the *direct image* j_* *(resp.* $j_!$*) is well-defined for* $\mathfrak{M} \in MHW(X)$ if there exists $\widetilde{\mathfrak{M}} \in MHW(\bar{X})$ such that

(1) $\widetilde{\mathfrak{M}}|_X = j^{-1}\widetilde{\mathfrak{M}} = \mathfrak{M}$.

(2) The vanishing-cycle functors along g are well-defined for $\mathfrak{M}$.

(3) $\tilde{K} = j_* K \quad$ (resp. $\tilde{K} = j_! K$).

Note that $j_*\mathfrak{M}$ (resp. $j_!\mathfrak{M}$) is unique (if it exists).

§3.2 Mixed Hodge modules

We define the category of mixed Hodge modules inductively as follows.

Definition 3.2.1

We say that $\mathfrak{M} \in MHW(X)$ *belongs to the category* $MHM(X)$ if the following conditions are satisfied :

MHM) *For any complex manifold* Y*, any open set* $U \subset X \times Y$*, any finite number of holomorphic functions on* U $g_1, g_2, \cdots$ *and an integer* $r \geq 1$*, the vanishing-cycle functors along* g_r *are well-defined for* $\mathfrak{M}_r$ *and* $j_{r*}, j_{r!}$ *are well-defined for* $j_r^{-1}\mathfrak{M}$.

Here $j_r : U \setminus g_r^{-1}(0) \hookrightarrow U$ *is the inclusion and* $\mathfrak{M}_r$ *is defined inductively as follows:*

$$\mathfrak{M}_1 := (\mathfrak{M} \boxtimes \mathbb{Q}_Y^H[d_Y])|_U$$

$$\mathfrak{M}_r := \text{any one of} \begin{cases} \psi_{g_{r-1}}(\mathfrak{M}_{r-1}), \quad \phi_{g_{r-1},1}(\mathfrak{M}_{r-1}), \\ j_{g_{r-1}*} j_{g_{r-1}}{}^{-1}(\mathfrak{M}_{r-1}), \; j_{g_{r-1}!} j_{g_{r-1}}{}^{-1}(\mathfrak{M}_{r-1}) \end{cases} \quad (r > 1).$$

$\mathbb{Q}_Y^H[d_Y]$ *denotes the following objects in* $MHW(X)$:

$$((\Omega_Y^{d_Y}, F), \mathbb{Q}_Y[d_Y]; W), \qquad Gr_{-i}^F = Gr_i^W = 0, \quad i \neq d_Y.$$

The above condition is local on X.

It is readily checked that for a closed immersion $i : X \hookrightarrow Y$ the functor

$$i_* : MHM(X) \to MHM_X(Y)$$

is an equivalence of categories, where $MHM_X(Y)$ denotes the full subcategory of $MHM(Y)$ consisting of the objects with support in X.

3.2.2 $\mathfrak{M} \in MHW(X)$ is called *(graded)-polarizable* if each $Gr_k^W \mathfrak{M}$ is a polarizable Hodge module.

There is a descrition of *Ext* groups in the category of mixed Hodge modules $MHM(X)$ [S6], which in principle allows us to see how mixed Hodge modules made of pure ones.

3.2.3 The following are the operations which can be easily defined.

(1) Tate twist (n), cf.2.1.3.
(2) Vanishing-cycle functors $\psi_g, \phi_{g,1}$.
(3) Extension $j_*j^{-1}, j_!j^{-1}$ over a locally principal divisor $X \setminus U$ ($j : U \hookrightarrow X$ the inclusion).
(4) Cohomological inverse images $f^!$ by a smooth morphism $f : X \to Y$:

$$\begin{aligned} H^{-\ell} f^! \mathfrak{M} = &\quad \omega_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}(M, F, W[-\ell]) \otimes \omega_Y^{-1})[\ell] \\ H^{\ell} f^* \mathfrak{M} = &\quad f^! \mathfrak{M}[-2\ell](-\ell) \ (\simeq \mathfrak{M} \boxtimes k_Z^H[d_Z] \text{ if } X \simeq Y \times Z) \end{aligned}$$

where $\ell = d_X - d_Y$.

Stability by these operations is almost trivial by the definition. But we need Theorem 5.2.2 ([S2,3.21]) to show that cohomological inverse imges by a smooth morphism preserve the polarizability.

§3.3 More about the extendability

3.3.1 We consider the conditions for extension over (locally principal) divisors. We suppose the following situation :

$$E_0 = X \times \{0\} \hookrightarrow E = X \times \mathbb{C} \hookleftarrow E^* = X \times \mathbb{C}^*$$

Then the following are necessary in order for $\mathfrak{M}' \in MHW(E^*)$ to extend to $\mathfrak{M} \in MHW(E)$ over E :

(1) j_*K' (or $j_!K'$) is constructible.
(2) $F_pV_{<0}M = V_{<0}M \cap j_*F_pM'$ is coherent over $\mathcal{O}_E$.
(3) F, V, W are compatible on $V_{<0}M$.
(4) The relative monodromy filtration W exists for $(\psi_t K, L)$.
(5) $Gr_i^W \mathfrak{M}'$ extends to a Hodge module on E.

Note that $(V_{<0}M; F, V, W)$ depends only on $j^{-1}\mathfrak{M} = \mathfrak{M}'$. Thus the above conditions are conditions on $\mathfrak{M}'$, which we call the *extendability conditions.*

We denote by $MHW(E^*)^{(p)}_{ex}$ the full subcategory of $MHW(E)^{(p)}$ consisting of those objects obeying the extendability conditions. Here we replace the condition 5) by a similar one $5)^p$ imposing polarizability on the Hodge module which extends $Gr_i^W \mathfrak{M}'$ when we consider $MHW(E^*)^p$.

We remark that one can extend the above consideration to the case when E, E_0 are replaced by a line bundle on X and its zero section, cf.[S2,2.10].

For an object $\mathfrak{M}'$ of $MHW(E^*)$, put

$$\psi_t\mathfrak{M} = \psi_t\mathfrak{M}' := (\oplus_{-1\leq\alpha<0}Gr_\alpha^V(M, F[1]), {}^p\psi_t K; W)$$
$$\psi_{t,1}\mathfrak{M}' := (Gr_{-1}^V(M, F[1]), {}^p\psi_{t,1}K; W).$$

3.3.2 "Gluing"

Let us consider the category $MHW(E^*, E_0)_{ex}^{(p)}$ of objects $(\mathfrak{M}', \mathfrak{M}''; u, v)$, with

$$\mathfrak{M}' \in MHW(E^*)_{ex}^{(p)}, \quad \mathfrak{M}'' \in MHW(E_0)^{(p)}$$
$$u : \psi_{t,1}\mathfrak{M}' \to \mathfrak{M}'', \quad v : \mathfrak{M}'' \to \psi_{t,1}\mathfrak{M}'(-1)$$
$$vu = N \quad (\text{on} \quad \psi_{t,1}\mathfrak{M}')$$

where u, v are morphisms in $MHW(E_0)^{(p)}$.

Consider also the category $MHW(E)_{sp}^{(p)}$ of those $\mathfrak{M}$ which are specializable, i.e. for which the vanishing cycle functors along E_0 are wel-defined.

Proposition ([S2,2.8]). *The following functor is an equivalence of categories :*

$$MHW(E)_{sp}^{(p)} \longrightarrow MHW(E^*, E_0)_{ex}^{(p)}$$
$$\mathfrak{M} \quad \mapsto \quad (j^{-1}\mathfrak{M}, \phi_{t,1}\mathfrak{M}; can, Var)$$

We briefly indicate the construction of the inverse functor.

Given a quadruple $(\mathfrak{M}', \mathfrak{M}''; u, v) \in MHW(E^*, E_0)_{ex}^{(p)}$. Then by the theory of gluing perverse sheaves by Deligne, MacPherson, Verdier, etc. we have a unique perverse sheaf K on E and a regular holonomic D-module M with $DR_E(M) \simeq K \otimes \mathbb{C}$ such that

$$j^{-1}(M, K) \cong (M', K'), \quad \phi_{t,1}(M, K) \cong (M'', K'')$$

Then the filtration F on V_0M is defined by the following :

$$m \in F_pV_0M \iff \begin{cases} j^{-1}m \in F_pM' \\ Gr_0^V m \in F_pM'' \end{cases}$$

c We extend it to the whole M by

$$F_pM = \sum_i (F_{p-i}V_0M)\partial_t^i.$$

To get W on (M, K), we define the filtrations L induced by W on $\mathfrak{M}', \mathfrak{M}''$ (inductively):

$$L_k\psi_{t,1}(M, K) := \psi_{t,1}W_k(M', K')$$
$$L_k\phi_{t,1}(M, K) := u(L_{k-1}\psi_{t,1}(M, K) + v^{-1}(L_{k-1}\psi_{t,1}(M, K) \cap W_k\phi_{t,1}(M, K)$$

Again, by the theory of gluing perverse sheaves, we obtain the unique filtration W on (M, K) such that

$$j^{-1}W_i(M, K) = W_i(M', K'), \quad \phi_{t,1}W_i(M, K) = L_i\phi_{t,1}(M, K).$$

The compatiblity of F, V, W is guaranteed by the following lemma :

Lemma. *Suppose that (M,F) is q-r along E_0 and that F,V,W on $V_{<0}$ are compatible for $(M,F,W)\in MFW(D_E)$.*

Then F,V,W are compatible on M and $Gr_i^W(M,F)$ is q-r along E_0.

3.3.3 As to the prolongation, we have

Proposition [S2,2.11]. *For $\mathfrak{M}'\in MHW(E^*)^{(p)}_{ex}$, one has a functorial prolongation $j_*\mathfrak{M}'$ (resp. $j_!\mathfrak{M}'$) $\in MHW(E)^{(p)}_{sp}$ (up to a canonical isomorphism).*

For $\mathfrak{M}\in MHW(E)^{(p)}_{sp}$, one has $j^{-1}\mathfrak{M}\in MHW(E)^{(p)}_{ex}$ and a functorial morphism

$$\mathfrak{M}\to j_*j^{-1}\mathfrak{M}\quad (resp. \mathfrak{M}\to j_!j^{-1}\mathfrak{M}).$$

We merely remark two things. First note that $j_*\mathfrak{M}'$ corresponds to $(\mathfrak{M}',\psi_{t,1}\mathfrak{M}'(-1),N,id)$, in the dictionary of 3.3.2.

Also $\mathfrak{M}\to j_*j^{-1}\mathfrak{M}$ corresponds to the following diagram :

$$\begin{array}{ccc}\psi_{t,1}\mathfrak{M} & \xrightarrow{id} & \psi_{t,1}\mathfrak{M}\\ can\downarrow\uparrow Var & & N\downarrow\uparrow id\\ \phi_{t,1}\mathfrak{M} & \xrightarrow[Var]{} & \psi_{t,1}\mathfrak{M}(-1)\end{array}$$

Secondly it is not easy to show the polarizability of $j_*\mathfrak{M}'$. Actually one needs Kashiwara's canonical splitting, cf.C.2

3.3.4 Gluing of mixed Hodge modules

We play the same game as in 3.3.2 in the case of mixed Hodge modules.

Let g be a holomorphic function on Y and consider the following morphisms :

$$\begin{array}{ccc} X=g^{-1}(0)_{red}\overset{i}{\hookrightarrow} Y \overset{j}{\hookleftarrow} & U=Y\setminus X\\ Y\quad \overset{i_g}{\hookrightarrow} Y\times\mathbb{C}\overset{j_g}{\hookleftarrow} & \{g\neq t\}\end{array}$$

Consider the following categories :

$$MHM(U)^{(p)}_Y=\{\text{MHM extendable to} Y\}$$

$$MHM(U,X)^{(p)}_{ex}=\left\{(\mathfrak{M}',\mathfrak{M}'';u,v);\quad \begin{array}{c}\mathfrak{M}'\in MHM(U)^{(p)}_Y,\mathfrak{M}''\in MHM(X)^{(p)}\\ u\in Hom(\psi_{g,1}\mathfrak{M}',\mathfrak{M}''),\\ v\in Hom(\mathfrak{M}'',\psi_{g,1}\mathfrak{M}'(-1)),\end{array} vu=N\right\}.$$

Then we have

Proposition [S2,2.28]. *The following functor is an equivalence of categories :*

$$\begin{array}{ccc} MHM(Y)^{(p)} & \longrightarrow & MHM(U,X)^{(p)}_{ex} \\ \mathfrak{M} & \mapsto & (j^{-1}\mathfrak{M}, \phi_{g,1}\mathfrak{M}; can, Var) \end{array}$$

For the construction of the inverse functor, one needs Beilinson's functor ξ_g explained below. Using this, the object which gives back $(\mathfrak{M}', \mathfrak{M}''; u, v)$ is the single complex associated to the following diagram :

$$\begin{array}{ccc} \mathfrak{M}'' & \xrightarrow{v} & \psi_{g,1}\mathfrak{M}'(-1) \\ u\uparrow & & \uparrow \\ \psi_{g,1}\mathfrak{M}' & \rightarrow & \xi_g j_* \mathfrak{M}' \end{array}$$

Remark Saito also proves gluing of Verdier type ([S2,2.31]) and that of MacPherson-Vilonen type ([S2,2.32]).

3.3.5 Beilinson's functor

The situation is the same as in 3.3.4. We define a functor

$$\xi_g : MHM(Y)^{(p)} \to MHM(Y)^{(p)}$$

by the formula :

$$\xi_g\mathfrak{M} := \psi_{t,1}(j_g)_!(j_g)^{-1}(\mathfrak{M} \boxtimes \mathbb{Q}^H_{\mathbb{C}}[1]).$$

Then we have the following excat sequences :

$$0 \to \psi_{g,1}\mathfrak{M} \to \xi_g\mathfrak{M} \to \mathfrak{M} \to 0$$

$$0 \to j_! j^{-1}\mathfrak{M} \to \xi_g\mathfrak{M} \to \phi_{g,1}\mathfrak{M} \to 0$$

As an application of Beilinson's functor, we have

Proposition [S2,2.23]. *The natural functor*

$$D^b MHM_X(Y) \to D^b_X MHM(Y)$$

is an equivalence of categories, whose quasi-inverse is given by $\phi_{g,1}$.

In fact, we have quasi-isomorphisms :

$$\mathfrak{M}^{\cdot} \leftarrow \xi_g\mathfrak{M}^{\cdot} \to \phi_{g,1}\mathfrak{M}^{\cdot}$$

for $\mathfrak{M}^{\cdot} \in D^b_X MHM(Y)$.

§3.4 Standard operations

We recall standard operations for mixed Hodge modules as in the sheaf theory, which are important for applicaitons. For more complete treatment using derived categories in algebraic context, we postpone it until 3.5.

3.4.1 Cohomological projective direct image

Let $f : X \to Y$ be a projective morphism between separated and reduced complex spaces. We then have the cohomological direct image functor when a suitable strictness condition is satisfied, cf.A.5.

Here we admit the most esssential result of stablility for polarizable (or pure) Hodge modules, which will be treated separately and in detail in §4, and explain the mixed case.

Theorem ([S2,2.14,2.15]). *Let $f : X \to Y$ be a projective morphism. For an object $\mathfrak{M} = ((M,F), K : W) \in MHW(X)^p$, the following hold.*
1) $f_*(M,F)$ is strict and

$$H^j f_*\mathfrak{M} := (H^j f_*(M,F), {}^pH^j f_*K; W[j]) \in MHW(Y)^p$$

where $W_i H^j f_*(M,F) = Im(H^j f_* W_i(M,F) \to H^j f_*(M,F))$.
2) Put $h = gf$ where G is a holomorphic function on Y. If $\psi_h, \phi_{h,1}$ are well-defined for $\mathfrak{M}$, then $\psi_g, \phi_{g,1}$ are well-defined for $H^j f_*\mathfrak{M}$ and one has

$$\psi_g H^j f_*\mathfrak{M} \cong H^j f_*(\psi_h \mathfrak{M}), \quad \phi_{g,1} H^j f_*\mathfrak{M} \cong H^j f_*(\phi_{h,1}\mathfrak{M}).$$

3) The spectral sequence

$$E_1^{-i,i+j} = H^j f_* Gr_i^W(M,F,K) \Longrightarrow H^j f_*(M,F,K)$$

degenerates at E_2-terms and d_1 is a morphism of Hodge modules.
Moreover $f_*(M,F), Decf_*W)$ is strict and one has

$$H^j(f_*(M,F), Decf_*W) \cong (H^j f_*(M,F), H^j f_* W[j]).$$

Comment ot the proof

As was mentioned above, the case $\mathfrak{M}$ is pure is treated in §4 separatedly and assumed here. (cf.[S2,5.3.1] for 1), [S2,5.3.4] for 2))

Using the E_2-degenaration in 3), 1) is deduced from the pure case.

For the validity of 3), the strictness of $f_* Gr_i^W(M,F)$ and the assumption $(H^j f_* Gr_i^W(M,F) \in MF_h(D_Y)$ and) $H^j f_* Gr_i^W \mathfrak{M} \in MH(Y, i+j)$ are sufficient.

3) is proved by the well-known argument of weight [TH II] (slightly extended in exact categories).

2) in the general case follows from the following lemma asserting that the monodromy weight filtration is preserved under projective direct image.

Put

$$\tilde{f} = f \times id : X \times \mathbb{C} \to Y \times \mathbb{C}$$

and consider $\tilde{M} = i_{g_*} M$.

Lemma ([S2,2.16]). *Assume that the following conditions hold for* $\tilde{M} \in MF_hW(D_{\tilde{X}}, k)$ *such that* $(\tilde{M}, F)$ *is q-r along* $X \times \{0\}$.

a) F, V, W *on* $\tilde{M}$ *are compatible.*

b) There exist relative monodromy filtrations $W = W(N)$ *on* $(\psi_t\tilde{M}, L)$, $(\phi_{t,1}\tilde{M}, L)$, *cf.3.1.2,2) for* L.

c) $f_*Gr_i^W Gr_k^L \psi_t(\tilde{M}, F), f_*Gr_i^W Gr_k^L \phi_{t,1}(\tilde{M}, F)$ *are strict and their* j*-th cohomologies* $H^j f_*$ *belong to* $MH(Y, i+j)$.

d) $H^j f_* W$ *on* $H^j f_* Gr_k^L \psi_t(\tilde{M}, K)$ *(resp.* $H^j f_* Gr_k^L \phi_{t,1}(\tilde{M}, K)$*) equals* $W(N)[k]$.

Then we have the following :

1) $\tilde{f}_* Gr_i^W(\tilde{M}, F)$ *and* $\tilde{f}_*\tilde{M}, F)$ *are strict on a neighborhood of* $Y \times \{0\}$ *and their cohomologies are q-r along* $Y \times \{0\}$.

2) $F, V, H^j \tilde{f}_* W$ *on* $H^j \tilde{f}_* \tilde{M}$ *are compatible.*

3) $H^j f_* W$ *equals* $W(N)$ *on* $(H^j f_* \psi_t(\tilde{M}, F, K), H^j f_* L[j])$ *(resp.* $(H^j f_* \phi_{t,1}(\tilde{M}, F, K), H^j f_* L[j])$*).*

4) $F, H^j f_* W, H^j f_* L$ *on* $H^j f_* \psi_t \tilde{M}$ *(resp.* $H^j f_* \phi_{t,1}\tilde{M}$*) are compatible.*

It might be worthwhile to mention that it is important to consider the filtration $Dec(f_*W)$ on $f_*\psi_t\tilde{M}$ (resp. $f_*\phi_{t,1}\tilde{M}$). One compares the spectral sequences associated to $Dec(f_*W)$ and $H^j f_* W[j]$ (which turn out to degenerate at E_2-terms). Note also that one uses Kashiwara's canonical splitting in a passage from Gr_k^L to the whole.

From the above theorem, cohomological projective direct image $H^j f_*\mathfrak{M}$ is well-defined. In fact, well-definedness of vanishing cycle functors follows from 2). Well-definedness of direct image $j_*, j_!$ for $H^j f_*\mathfrak{M}$ follows from 1) applied for $H^j f_*(j_*\mathfrak{M})$, $H^j f_*(j_!\mathfrak{M})$.

3.4.2 Cohomological direct image for projectively compactifiable morphisms

Definition A morphism $f : X \to Y$ is said to be *projectively compactifiable* if there exists a factorization $f = \bar{f} \cdot j$ such that $j : X \hookrightarrow \bar{X}$ is an open immersion with $\bar{X} \setminus X$ a divisor and $\bar{f} : \bar{X} \to Y$ is a projective morphism, cf.[SGA4,XVII].

Two projective compactifications are said to be equivalent if there exists a third one which dominates both.

The projective compactification $f = \bar{f} \cdot j$ is said to be admissible for $\mathfrak{M} \in MHM(X)$ if $\mathfrak{M}$ is extendable to $\bar{X}$. In this case, one puts

Definition $H^j f_*\mathfrak{M} := H^j \bar{f}_*(j_*\mathfrak{M}), \quad H^j f_!\mathfrak{M} := H^j \bar{f}_*(j_!\mathfrak{M}),$

which depends only on the equivalence class of projective compactifications of f.

3.4.3 Cohomological inverse image

CONSTRUCTION For any morphism $f : X \to Y$, there exist functors

$$H^j f^*, H^j f^! : MHM(Y) \longrightarrow MHM(X)$$

compatibly with the functors

$${}^pH^j f^*, {}^pH^j f^! : Perv(k_Y) \longrightarrow Perv(k_X)$$

cf.[S2,2.19].

The strategy is simple : factor f into $f = p \cdot i$ with p a smooth morphism of relative dimension ℓ, i a closed immersion.

Then the definition will be the following :

$$\begin{aligned} H^j f^* \mathfrak{M} &:= H^{j-\ell} i^* H^\ell p^* \mathfrak{M} \\ H^j f^! \mathfrak{M} &:= H^{j+\ell} i^* H^{-\ell} p^! \mathfrak{M} \end{aligned}$$

The smooth inverse image was recalled in 3.2.

The inverse image by a closed immersion is defined in the following way.

Assume $X = \cap_i g_i^{-1}(0)_{red}$ for holomorphic functions $g_1, \cdots, g_r$ on Y. Put

$$U_i = \{g_i \neq 0\}, \quad j_I : U_I = \cap_{i \in I} U_i \hookrightarrow Y.$$

Then one defines

$$\begin{aligned} H^j i^* \mathfrak{M} &:= H^j(\oplus_{\#I=-\cdot}(j_I)_! j_I^{-1} \mathfrak{M}) \\ H^j i^! \mathfrak{M} &:= H^j(\oplus_{\#I=\cdot}(j_I)_* j_I^{-1} \mathfrak{M}) \end{aligned}$$

mimicking the definition in the sheaf theory. Note that the complexes in the right hand side are those in $MHM(Y)$. The independence of factorizations is standard.

3.4.4 Direct image by a Zariski open immmersion

CONSTRUCTION For an open immersion $j : U \hookrightarrow Y$ such that $Y \setminus U = X$ a closed subspace of Y, there exist functors $H^k j_! j^{-1}, H^k j_* j^{-1} : MHM(Y) \to MHM(Y)$ compatibly with ${}^pH^k j_! j^{-1}, {}^pH^k j_* f^{-1} : Perv(k_Y) \to Perv(k_Y)$. Moreover one has the following long exact sequences :

$$\cdots \to H^k j_! j^{-1} \mathfrak{M} \to H^k \mathfrak{M} \to i_* H^k i^* \mathfrak{M} \to \quad H^{k+1} j_! j^{-1} \mathfrak{M} \to \cdots$$
$$\cdots \to i_* H^k i^! \mathfrak{M} \to H^k \mathfrak{M} \to H^k j_* j^{-1} \mathfrak{M} \to \quad i_* H^{k+1} i^! \mathfrak{M} \to \cdots$$

($H^0 j_! j^{-1}, H^0 j_* j^{-1}$ coinside with $j_! j^{-1}, j_* j^{-1}$ when X is a divisor).

The definition is, using the notation in 3.4.3,

$$H^k j_! j^{-1}\mathfrak{M} := H^k(\oplus_{\#I=1-\cdot}(j_I)_! j_I^{-1}\mathfrak{M})$$
$$H^k j_* j^{-1}\mathfrak{M} := H^k(\oplus_{\#I=1+\cdot}(j_I)_* j_I^{-1}\mathfrak{M})$$

These functors in 3.4.3,3.4.4 preserve the polarizability. To show this one needs the fact that polarizability is preserved under smooth inverse image, which in turn relies upon Theorem 5.??, cf.[S2,3.20].

§3.5 Derived category of mixed Hodge modules

3.5.1 We review the formalism of standard operations for mixed Hodge modules in parallel to that for constructible sheaves on an algebraic variety (=separated and reduced scheme of finite type over $\mathbb{C}$).

To work in algebraic category means to replace all notions in preceding sections by corresponding algebraic analogues.

We refer the reader to the [S2,§4] for precise change in their definitions and estrict ourselves to point out main differences.

For incetance, $Perv(k_X)$ is the full subcategory of $Perv(k_{X^{an}})$ with respect to algebraic stratifications. D_X is the ring of differential operators with coefficients in $\mathcal{O}_X$ when X is smooth. The regularity of a holonomic D_X-modules means regularity at infinity.

$MHM(X)$ is defined to be the full subcategory of $MHW(X)^p$ by imposing the stability under the operations $\psi_g, \phi_{g,1}, j_!, j_*, \boxtimes k_Y^H[d_Y]$, where Y is smooth and $j : U \hookrightarrow X$ an open immersion such that $X \setminus U$ is a divisor. Thus the graded pieces $Gr_\cdot^W \mathfrak{M}$ are assumed to be polarizable.

3.5.2 Cohomological operations

For *any* morphism f, one can define $H^j f_*, H^j f_!, H^j f^*, H^j f^!$ because of the existence of proper compactification for f and od Chow's lemma.

Beilinson's functor ξ_g (3.3.5) is available so that one has an equivalence of categories :

$$i_* : D^b MHM(X) \to D^b_X MHM(Y)$$

for a closed immersion $i : X \to Y$.

We have (by the calculation in §5)

$$MHM(pt) = \{\text{graded-polarizable MHS}\}$$

and also

$$MHM(X) \quad \text{is stable by external tensor products.}$$

Of course, $MHM(X)$ is an abelian category.

3.5.3 Derived category

Let $D^bMHM(X)$ be the derived category of bounded complexes in $MHM(X)$.

Then the (k-)rational part of $\mathfrak{M}^\cdot \in D^bMHM(X)$ is a bounded complex of perverse sheaves $K^\cdot$. Thanks to the realization functor [BBD,§3], we can obtain a usual complex :

$$rat : D^bMHM(X) \to D^b_c(k_X) \ ; \quad rat(\mathfrak{M}) = real(K^\cdot)$$

Then we can push the long arguments into the following simple :

Theorem. *We have the standard operations*

$$f_*, f_!, f^*, f^!, \mathbb{D}, \psi_g, \phi_{g,1}, \boxtimes, \otimes, Hom$$

on $D^bMHM(X)$ and these are compatible with similar operations on $D^b_c(k_X)$ through the functor rat.

CONSTRUCTION: Mimic the arguments in Beilinson [B1,§3].

For $f_*, f_!$, take affine open coverings of X and Y compatibly with f such that one can have acyclic resolutions of finitely many given objects. It is all right because $f_*, f_!$ are left or right derived functors of $H^0 f_*, H^0 f_!$ in the case X, Y are affine.

f^* (resp. $f^!$) are defined to be left (resp. right) adjoint functor of f_* (resp. $f_!$) and their existence is reduced to the case f is a closed immersion and to the case f is a projection $p : X \times Y \to Y$.

For the former case, cf. the complex used in 3.4.3. For the latter, one represent p^* by an external tensor product $\mathfrak{M}\boxtimes ?$ (for some $\mathfrak{M} \in MHM(X)$) (i.e. prove its naturality and existence).

Remark The technique (due to Beilinson) to construct standard functors between derived categories of certain nice abelian categories is axiomatized in [S12].

§ **Appendix** – Hodge modules on singular spaces

We explain how to treat D-modules and Hodge modules on a singular space in this §.

A.1 Let X be a reduced and separated complex analytic space. Consider a covering by open sets $X = \cup_i U_i$ which is locally finite and also a collection of local embeddings into smooth spaces $(U_i \hookrightarrow V_i)_i$ where $U_i \to V_i$ is a closed immersion into a smooth space V_i. For an index subset I, one has the following induced morphism

$$U_I := \cap_{i \in I} U_i \hookrightarrow V_I := \Pi_{i \in I} V_i,$$

and for $I \subset J$ one has the following commutative diagram :

$$\begin{array}{ccc} U_I & \hookrightarrow & V_I \\ \uparrow & \circlearrowleft & \uparrow pr_{IJ} \\ U_J & \hookrightarrow & V_J \end{array}$$

Note that pr_{IJ} is simply a projection.

A.2 Consider a collection of Hodge modules $\mathfrak{M} = \{M_I\}_I$ and morphisms u_{IJ} : $pr_{IJ*} M_J \simeq M_I$ on $V_J \setminus (U_I \setminus U_J)$ which satisfies the condition :

$$SuppM_I \subset U_I, \quad M_I \in MH(V_I, n)$$
$$u_{IK} = u_{IJ} \cdot (pr_{IJ})_* u_{JK} \text{ on } V_J \setminus (U_I \setminus U_K) \quad (I \subset J \subset K).$$

One can define an obvious equivalence of such data. An equivalence class has a meaning independent of the choice of local embeddings and open coverings by 2.2.2,1).

The category of such objects is denoted by $MH(X, n)$. $\{M_I\}_I$ is called a representant of a Hodge module $\mathfrak{M}$. When X is smooth, we can take $(U_i)_i = \{X\}$ and consider a usual Hodge module as a representant on X. Therefore the definition using representants generalizes the definition in 2.2.

Using this notion one can also consider the category $MHW(X)$.

A.3 For $\mathfrak{M} = \{M_I\}_I \in MH(X; n)$, one has a perverse sheaf $K \in Perv(k_X)$ which induces K_I underlying M_I for each I.

Similarly one has a globally defined (K, W) for an object of $MHW(X)$.

This enables us to define a polarization of a Hodge module $\mathfrak{M}$ on a singular space as a pairing

$$S : K \otimes K \to a_X^! k(-n)$$

which induces a polarization for each representant M_I by restriction. Similarly one has the notion of graded polarization for an object of $MHW(X)$.

A.4 We can consider the category $MF(D_X)$ of filtered D-modules on a singular space X similarly to $MH(X)$.

Let $f: X \to Y$ be a proper morphism between separated and reduced complex spaces. Then we have a functor

$$f_*: DF(D_X) \to DF(D_Y),$$

the definition of which we are going to explain.

First choose locally finite open coverings of X and Y : $X = \cup_i U_i, Y = \cup_i U_i'$ and closed embeddings into smooth spaces : $U_i \hookrightarrow V_i, U_i' \hookrightarrow V_i'$ such that there is a morphism $f_i : V_i \to V_i'$ making the following diagram commute :

$$\begin{array}{ccc} U_i & \xrightarrow{f|_{U_i}} & U_i' \\ \cap & \circlearrowleft & \cap \\ V_i & \xrightarrow{f_i} & V_i' \end{array}$$

Let $\{(M_I, F)\}$ be a representant of $(M, F) \in MF(D_X)$ (thus (M_I, F) belongs to $MF(D_{V_I})$ with support in U_I).

Put then

$$f_*(M,F) := \underset{=}{s}[\bigoplus_{\#I-1=-\cdot} DR^{-1} \cdot (f_I)_! \cdot \widetilde{DR}(\tau_{\leq 2d(I)+1} G(M_I, F))]$$
$$f_I = \Pi_{i \in I} f_i : \Pi_{i \in I} V_i \to \Pi_{i \in I} V_i'$$

Here G denotes Godement's canonical flabby resolution such that $Gr_p^F G(M_I)$ is flabby ($\forall p$) and $d(I) = min\{dim V_i; i \in I\}$. This is just to take a suitable Čech complex with respect to the "covering" $\{U_i \to V_i\}$ (and to take the associated simple complex $\underset{=}{s}$).

A.5 Using f_* in A4,we can consider cohomological direct image functor.

Let $f : X \to Y$ be as in A4 and let us take an object $(M, F, K) \in MF_h(D_X, k)$, whose definition is obvious and not recalled here.

If any local representant of $f_*(M, F)$ is strict, we say $f_*(M, F)$ is *strict* and define

$$H^j f_*(M,F,K) := (H^j f_*(M,F), {}^p H^j f_* K) \quad \in MF_h(D_Y, k)$$

the isomorphism $\alpha : DR(H^j f_* M) \cong \mathbb{C} \otimes {}^p H^j f_* K$ being induced from

$$\bigoplus_{\#I-1=-\cdot} (j_I)_! \widetilde{DR}(M_I) \cong \mathbb{C} \otimes K$$

(j_I is the inclusion $U_I = \cap_{i \in I} U_i \to X$).

§ **Complements** 1 – Strictness and compatible filtrations

We recollect here two key notions which are quite technical but crucial in the theory of mixed Hodge modules.

C.1 Let $\psi : (E_1^{\cdot}, F) \to (E_2^{\cdot}, F)$ be a morphism between filtered objects in an abelian category $\mathcal{A}$. Here we consider decreasing filtrations and use the standard convention : $F_p = F^{-p}$.

We say that ψ is *strict or strictly compatible with the filtrations* if the following holds :

$$\psi(F^p E_1) = F^p E_2 \cap Im\psi \quad (\forall p)$$

For a filtered complex $(K^{\cdot}, F)$, one has a spectral sequence :

$$E_0^{p,q} = Gr_F^p K^{p+q} \Longrightarrow H^{p+q}(K^{\cdot})$$
$$E_1^{p,q} = H^{p+q}(Gr_F^p K^{\cdot})$$

We say that $(K^{\cdot}, F)$ is *strict* if every $d^i : (K^i, F) \to (K^{i+1}, F)$ is strict. Then it is known that $(K^{\cdot}, F)$ is strict iff the above spectral sequence degenerates at E_1-terms [D1,II (1.3.2)].

C.2 It is necessary to extend the above notions to exact categories.

An additive category $\mathcal{C}$ is called *exact* if it is a full subcategory of an abelian category $\widetilde{\mathcal{C}}$ and is stable by extensions in $\widetilde{\mathcal{C}}$. The notion of exact category is independent of the embedding.

A morphism $\psi : E_1 \to E_2$ is said to be *strict* if $Ker\psi, Coker\psi$ belong to $\mathcal{C}$.

A decreasing filtration of an object E in $\mathcal{C}$ is, by definition, a family of strict monomorphisms $u_{ij} : F^j E \to F^i E (i < j)$ (i.e. $Coker u_{ij}$ in $\widetilde{\mathcal{C}} \in \mathcal{C}$) which satisfy $u_{ij} \cdot u_{jk} = u_{ik} (i < j < k)$.

Then it can be verified that the strictness of a morphism ψ of filtered objects in an abelian category $\mathcal{A}$ is equivalent to that of the morphism ψ in the exact category $F(\mathcal{A})$ of filtered objects in $\mathcal{A}$.

The definition of strictness of a filtered complex is extended to that of a complex $(E^{\cdot}, d)$ in $\mathcal{C}$; $(E^{\cdot}, d)$ is said to be *strict* if

(1) every $d^i : E^i \to E^{i+1}$ is strict, and

(2) $Imd^{i-1} \to Kerd^i$ is an strict monomorphism $(\forall i)$.

The second condition assures that the cohomology object $H^i(E^{\cdot})$ is in $\mathcal{C}$, in which case we say that $(E^{\cdot}, d)$ is *weakly strict relative to* $\widetilde{\mathcal{C}}$.

C.3 We should be careful when we treat several filtrations, for it is known that $Gr_{F_1}^{i_1} Gr_{F_2}^{i_2} Gr_{F_3}^{i_3}$ depends on the order of filtrations in general. It is convenient to introduce the following notion of a nice family of filtrations.

Let $F_I = (F_i)_{i \in I}$ be a family of filtrations indexed by a finite set I on an object $A \in \mathcal{A}$. The category of such an I-filtered object $E = (A, F_I)$ is denoted by $F_I(\mathcal{A})$. Put

$$F^{\nu} E := \{\cap_{j \in J} F_j^{\nu_j} A\}_{J \subset I} \quad (\nu = (\nu_i)_{i \in I} \in \mathbb{Z}^I),$$

where J runs through the subsets of I.

F_I is said to be *compatible* if $F^\nu E$ is "short exact" for $\forall\nu \in \mathbb{Z}^I$. ($F^\nu E \in S_I^c(\mathcal{A})$ in the notation of [S1,§1].) The condition "short exact" means that 3^n-lemma holds where n is the cardinality of I. If $n = 2$, $3^2 = 9$-lemma means the exactness of the following square($= 2$) diagram :

$$\begin{array}{ccccc}
F_1^{\nu_1} \cap F_2^{\nu_2}(A) & \longrightarrow & F_1^{\nu_1}(A) & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
F_2^{\nu_2}(A) & \longrightarrow & A & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & * & \longrightarrow & *
\end{array}$$

We leave to the reader the task of formulating 3^n-lemma for $n \geq 3$.

We extend the strictness to multi-filtered complexes ; ($E^\cdot \in CF_I(\mathcal{A})$ is said to be *strict* if the induced filtrations on $H^k((E^\cdot)$ are compatible, namely, $H^kF^\nu(E^\cdot)$ is "short exact".

Then that $\psi : E \to E'$ is strict is equivalent to the strictness of the complex $[E \xrightarrow{\psi} E']$ or to that $Ker\psi, Coker\psi$ are "short exact".

The following criterion is practically convenient [S1,§1].

Theorem. *Assume for simplicity that there exist inductive limits in $\mathcal{A}$. Let $E = (A, F_I)$ be an I-filtered object with the condition :*

$$\cup_p F_i^p A = A, F_i^p = 0 \quad (p >> 0).$$

Take an element $i \in I$ and put $I' = I \backslash \{i\}$.

(i) *E is compatible iff $Gr_{F_i}^p(A, F_{I'})$ are compatible for $\forall p$ and the following are surjective :*

$$F^\nu F_i^p(A, F_{I'}) \to F^\nu Gr_{F_i}^p(A, F_{I'}) \quad (\forall p, \forall \nu \in \mathbb{Z}^{I'}).$$

(ii) *Let $E^\cdot = (A^\cdot, F_I)$ be a complex of I-filtered objects and assume that F_I on each term satisfies the above condition and is compatible. Then $E^\cdot$ is strict iff $(A^\cdot, F_i)$ and $Gr_{F_i}^p(A, F_{I'})$ ($\forall p$) are strict.*

§ **Complements** 2 – Relative monodromy filtrations –

We recollect here basic notions and facts related to relative monodromy filtrations, cf.[SZ],[S1,1.3],[S2,§1],[K1],[Weil II,1.6],etc.

C.2.0 Recall the definiton of a N-filtration $W = W(N)$ for a nilpotent endomorphism N acting on an object E of an exact category $\mathcal{C}$,cf.C.1.1. It is a finite increasing filtration W such that

$$1) \quad NW_iE \subset W_{i-2}E, \quad 2) \quad N^i : Gr_i^W E \simeq Gr_{-i}^W E.$$

It is unique if it exists and it exists in an abelian category.

For such a filtration W, we put

$$PGr_i^W E = P_N Gr_i^W E := Ker(N^{i+1} : Gr_i^W E \to Gr_{-i-2}^W E)$$

for $i \geq 0$ if this belongs to $\mathcal{C}$. Then one has

$$Gr_i^W E \simeq \oplus_{m \geq max\{0,-i\}} PGr_{i+2m}^W E \quad (\forall i)$$

(the primitive decomposition).

C.2.1 Let L be a finite filtration on an object $E \in \mathcal{C}$. Assume that a nilpotent endomorphism N respects L.

Definition An N-filtration relative to L (or relative monodromy filtration or relative weight filtration) is a finite increasing filtration W on E such that

$$1) \quad NW_iE \subset W_{i-2}E, \quad 2) \quad N^k : Gr_{i+k}^W Gr_i^L E \simeq Gr_{i-k}^W Gr_i^L E.$$

A priori, this notion has nothing to do with weight or monodromy.

It is unique if it exists. One has an inductive formula (if it exists) :

$$1) \quad W_{-i+k}L_kE = W_{-i+k}L_{k-1}E + N^i W_{i+k}L_kE \quad (i > 0)$$
$$2) \quad W_{i+k}L_kE = Ker(N^{i+1} : L_kE \to L_k/W_{-i-2+k}L_kE \quad (i \geq 0)$$

If $Gr_i^L E = 0$ for $i \neq n$, then the N-filtration W for (E, L) exists and $W = W(N)[n]$.

Note that the construction of N-filtration is functorial and if $u : ((E_1, L), N) \to ((E_2, L), N)$ induces $\tilde{u} : (E_1, L, W) \to (E_1, L, W)$, then $\tilde{u}$ is strict.

C.2.2 Some results on relative N-filtrations

First recall the inductive criterion for the existence of relative N-filtration due to Steenbrink-Zucker :

Proposition [SZ,2.20]. *Assume that $\mathcal{C}$ is an abelian category and that the N-filtration for $(L_{k-1}E, L)$ exists.*

Then the N-filtration for (L_kE, L) exists if and only if

$$N^{i+1} : Ker(N^{i+1} : Gr_k^L E \to Gr_k^L E) \to L_{k-1}E/N^{i+1}L_{k-1}E + W_{k-i-2}L_{k-1}E$$

is zero for $i \geq 0$, i.e.

$$N^j L_k E \cap L_{k-1}E \subset N^j L_{k-1}E + W_{k-j-1}L_{k-1}E \quad (j \geq 1)$$

Another important fact on relative monodromy filtration is Kashiwara's canonical splitting :

Proposition [K,3.2.9] (cf.[S2,1.5]). *Assume that the N-filtration W for (E, L) exists and that the primitive parts $P_N Gr_{i+k}^W Gr_k^L E$ belong to $\mathcal{C}$.*

Then there exists a canonical splitting

$$Gr_i^W E \simeq \bigoplus_k Gr_i^W Gr_k^L E.$$

Moreover, if we write the section by $s_{i,k} : Gr_i^W Gr_k^L E \to Gr_i^W L_k E$ and denote by $s'_{i,k}$ its restriction to the primitive part $s_{i,k}|_{PGr_i^W Gr_k^L E}$, then we have the following :

$$1) \quad Im(s_{i+k,k}) = \sum_{0 \leq m \leq i+2m} N^m Im(s'_{i+k+2m,k})$$

$$2) \quad Im(s'_{i+k,k}) = pr(W_{i+k}L_kE \cap N^{-(i+1)}(\sum_{\ell} N^{\ell} Im(s'_{j+\ell,\ell}))$$

REFERENCES

[B1] A.A.Beilinson, *On the derived category of perverse sheaves*, K-theory, Arithmetic and Geometry, Lecture Notes in Math., vol. 1289, Springer-Verlag, 1987, pp. 27–41.

[B2] A.A.Beilinson, *How to glue perverse sheaves*, K-theory, Arithmetic and Geometry, Lecture Notes in Math., vol. 1289, Springer-Verlag, 1987, pp. 42–51.

[BBD] A.A.Beilinson, J.Bernstein, P.Deligne, *Faisceaux pervers*, Astérisque **100** (1983).

[Bo] A.Borel et al., *Algebraic D-modules, Perspectives in Math.*, vol. 2, Academic Press, 1987.

[CK] E.Cattani, A.Kaplan, *Polarized mixed Hodge structures and the local monodromy of a variation of Hodge structures*, Invent. Math. **67** (1982), 101–115.

[CKS] E.Catteni,A.Kaplan,W.Schmid, *L^2 and intersection cohomologies for a polarizable variation of Hodge structure*, Invent. Math. **87** (1987), 217–252.

[D] P.Deligne, *Critère de dégénérescence de suites spectrales et théorème de Lefschetz*, Publ. Math. IHES **35** (1968), 107–126.

[TH] P.Deligne, *Théorie de Hodge I*, Actes du Congrès intern. des math., Nice, vol. I, Gauthier-Villars, 1970, pp. 425–430; *II*, 1972, pp. 5–57; *III*, 1974, pp. 5–77.

[Weil II] P.Deligne, *La conjecture de Weil II*, Publ. Math. IHES **52** (1980), 137–252.

[GN-A] F.Guillen, V.Navarro-Aznar, *Sur le théorème local des cycles invariants*, Duke Math. J. **61** (1990), 133–155.

[H] Naofumi Honda, *Holonomic D-modules, perverse sheaves, and Riemann-Hilbert correspondence*, This volume.

[K1] M.Kashiwara, *A study of variation of mixed Hodge structure*, Publ. RIMS, Kyoto Univ. **22** (1986), 991–1024.

[K2] M.Kashiwara, *The asymptotic behavior of a variation of polarized Hodge structure*, Publ. RIMS, Kyoto Univ. **21** (1985), 853–875.

[KK] M.Kashiwara,T.Kawai, *The Poincaré lemma for a variation of Hodge structure*, Publ. RIMS, Kyoto Univ. **23** (1987), 345–407.

[L] E.Looijenga, *L^2-cohomology of locally symmetric varieties*, Compositio Math. **67** (1988), 3–20;E.Looijenga, M.Rapoport, *Weights in the local cohmology of a Baily-Borel compactification*, Complex Geometry and Lie theory, Proc. Symp. Pure Math., vol. 53, 1991, pp. 223–260.

[Sa] C.Sabbah, *D-modules et cycles évanescents (d'après B.Malgrange et M.Kashiwara)*, Conférence de La Rabida 1984 Vol.III, Hermann, Paris, 1987, pp. 53–98.

[Sx] Morihiko Saito.

[S1] —, *Modules de Hodge polarisables*, Publ. RIMS, Kyoto Univ. **24** (1988), 849–995.

[S1bis] —, *Hodge structure via filtered $\mathcal{D}$-modules*, Astérisque **130** (1985), 145–162.

[S2] —, *Mixed Hodge modules*, Publ. RIMS, Kyoto Univ. **26** (1990), 221–333.

[S2bis] —, *Mixed Hodge modules*, Proc. Japan Acad. **62A** (1986), 360–363.

[S2ter] —, *On the derived category of mixed Hodge modules*, Proc. Japan Acad. **62A** (1986), 364–366.

[S3] —, *Introduction to mixed Hodge modules*, Astérisque **179-180** (1989), 145–162.

[S4] —, *Duality for vanishing cycle functors*, Publ. RIMS, Kyoto Univ. **25** (1989), 889–921.

[S5] —, *Decomposition theorem for proper Kähler morphisms*, Tohoku Math. J. **42** (1990), 127–149.

[S6] —, *Extension of mixed Hodge modules*, Compositio Math. **74** (1990), 209–234.

[S7] —, *Induced $\mathcal{D}$-modules and differential complexes*, Bull. Soc. Math. France **117** (1989), 361–387.

[S8] —, *Exponents and Newton polyhedra of isolated singuralities*, Math. Ann. **281** (1988), 411–417.

[S9] —, *On the structure of Brieskorn lattices*, Preprint Grenoble (1983)/RIMS (1988).

[S10] —, *On Steenbrink's conjecture*, Math. Ann. **289** (1991), 703–716.

[S11] —, *D-modules on analytic spaces*, Publ.RIMS, Kyoto Univ. **27** (1991), 291–332.

[S12] —, *On the formalism of mixed sheaves*, preprint RIMS **784** (1991).

[S13] —, *Hodge conjecture and mixed motives I*, Complex Geometry and Lie theory, vol. 53, 1991, pp. 283–303; *II*, Algebraic Geometry, Proc.,Chicago 1989, vol. 1479, 1991.

[S13bis] —, *On the injectivity of cycle maps*, preprint RIMS **721** (1990).

[S13ter] —, *Some remrks on the Hodge type conjecture*, preprint RIMS **794** (1991).

[SaZ] Morihiko Saito, S.Zucker, *The kernel spectral sequence of vanishing cycles*, Duke Math. J. **61** (1990), 329–339.

[MaS] Masa-Hiko Saito, *Application of Hodge modules – Kollár conjecture and Kodaira vanishing –*, This volume.

[MuS] Mutsumi Saito, *A short course on b-functions and vanishing cyles*, This volume.

[Sc] W.Schmid, *Variation of Hodge structure; the singularities of the period mappings*, Invent. Math. **22** (1973), 211–319.

[St1] J.Steenbrink, *Limits of Hodge structures*, Invent. Math. **31** (1976), 229–257.

[St2] J.Steenbrink, *Mixed Hodge structure on the vanishing cohomology*, Real and Complex Singularities (Proc. Nordic Summer School, Oslo, 1976), Sijthoff and Noordfoff, 1977, pp. 525–563.

[SZ] J.Steenbrink, S.Zucker, *Variation of mixed Hodge structure I*, Invent. Math. **80** (1985), 489–542.

[T] T.Tanisaki, *Hodge modules, equivariant K-theory and Hecke algebras*, Publ. RIMS, Kyoto Univ. **23** (1987), 841–879.

[U] S.Usui, *Period maps and their extensions*, This volume.

[V1] J.L.Verdier, *Extension of a perverse sheaf over a closed subspace*, Astérisque **130** (1985), 210–217.

[V2] J.L.Verdier, *Prolongement des faisceaux pervers monodromiques*, Astérisque **130** (1985), 218–236.

[Z1] S.Zucker, *Hodge theory with degenerating coefficients : L^2-cohomology in the Poincaré metric*, Ann. of Math. **109** (1979), 415–476.

[Z2] S.Zucker, *Variation of mixed Hodge structure II*, Invent. Math. **80** (1985), 543–565.