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Flat structure for the simply elliptic singularity and Jacobi form

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§1. Introduction.

There are three types of hypersurface simply elliptic singularity. They are represented in $\mathbf{C}^3$ as follows:

$$\tilde{E}_6 : x(x-z)(x-\lambda z) - zy^2 = 0,$$

$$\tilde{E}_7 : xy(x-y)(x-\lambda y) - z^2 = 0 \ (\lambda \neq 0, 1),$$

$$\tilde{E}_8 : y(y-x^2)(y-\lambda x^2) - z^2 = 0.$$

Let $\pi : X \to S$ be its (global) semi-universal deformation. Let $S' \to S$ be the universal covering of S. Let $\pi' : X' \to S'$ be the pull back of $\pi : X \to S$.

The period mapping $P : S' \setminus D \to \tilde{\mathbf{E}}$ (D is a discriminant set, $\tilde{\mathbf{E}}$ is a $\mathbf{C}$ affine half space, and P is a multi-valued holomorphic mapping) is defined by Kyoji Saito by use of the theory of *the primitive form.*

Let $\tilde{W}$ be the monodromy group which describes the multi-valuedness of P. Then $\tilde{W}$ acts on $\tilde{\mathbf{E}}$ properly discontinuous and there exists uniquely the holomorphic isomorphism:

$$\varphi : \tilde{\mathbf{E}}/\tilde{W} \to S' \setminus D'$$

which commutes the following diagram:

$$\begin{array}{ccccccc}
X & \leftarrow & X' & & & & \tilde{\mathbf{E}} \\
\downarrow & & \downarrow & & & \nearrow & \downarrow \\
S & \leftarrow & S' & \supset & S' \setminus D & & \tilde{\mathbf{E}}/\tilde{W} \\
 & & & & \cap & \swarrow & \\
 & & & & S' \setminus D' & &
\end{array}$$

where $D' := \{z \in S' | \pi^{-1}(z)$ has a simply elliptic singularity$\}$.

Problem: Describe the map φ. In other words, describe the deformation parameters (the coordinates of S') as the $\tilde{W}$ invariants on $\tilde{\mathbf{E}}$.

We call this problem the "Jacobi's inversion problem". We remark that $\tilde{W}$ contains the discrete Heisenberg group, thus $\tilde{W}$ invariants can be described by some theta functions.

To solve this problem, we recall the "flat structure" defined by K.Saito.

First, we embed the space S' into the $\mathbf{C}$ linear space V_1 with inner product. This embedding is constructed by the theory of the primitive form.

Secondly we construct the space $\tilde{\mathbf{E}}$ and the group $\tilde{W}$ by use of the extended affine root system, and we embed $\tilde{\mathbf{E}}/\tilde{W}$ into the $\mathbf{C}$ linear space V_2 with inner product. This embedding is constructed by the theory of Coxeter transformation for the extended affine root system.

Then the map φ is induced from the linear isomorphism $\varphi : V_2 \to V_1$ which preserves the inner products:

$$\begin{array}{ccc} S' & \leftarrow & \tilde{\mathbf{E}}/\tilde{W} \\ \cap & & \cap \\ V_1 & \leftarrow & V_2. \end{array}$$

We call these linear structures for $S'; \tilde{\mathbf{E}}/\tilde{W}$, *the flat structures.*

Therefore, if we know both the linear functions on S' and ones on $\tilde{\mathbf{E}}/\tilde{W}$, we can solve the Jacobi's inversion problem up to linear isomorphism.

The linear function on S' was already described by M.Noumi. Thus we reduce the problem to the following one.

Problem: Describe the linear functions on $\tilde{\mathbf{E}}/\tilde{W}$. In other words, describe the special $\tilde{W}$ invariants on $\tilde{\mathbf{E}}$ which corresponds to the linear functions.

In this article, we shall explain our approach to this problem by use of the generalized Jacobi form defined by Wirthmüller. Also we give the explicit calculation of the flat structure for the extended affine root system of type G_2. If we can calculate the cases of type E_l, then they give an answer to the above problem.

§2. Flat structure.

We introduce some notations.

$\mathfrak{h}$: a Cartan subalgebra for a simple Lie algebra.

$\mathfrak{h}_{\mathbf{C}}:=\mathfrak{h}\otimes_{\mathbf{R}}\mathbf{C}$.

$\mathfrak{h}^*_{\mathbf{C}}:=Hom_{\mathbf{C}}(\mathfrak{h}_{\mathbf{C}},\mathbf{C})$.

$R(\subset\mathfrak{h}^*_{\mathbf{C}})$: set of roots.

$I:\mathfrak{h}_{\mathbf{C}}\times\mathfrak{h}_{\mathbf{C}}\to\mathbf{C}$ Killing form.

$Q(R)$:$\mathbf{Z}-$ span of the image of R by the isomorphism:

$$\mathfrak{h}^*_{\mathbf{C}}\to\mathfrak{h}_{\mathbf{C}}$$

induced from the Killing form.

$\tilde{\mathbf{E}}:=\mathbf{H}\times\mathfrak{h}_{\mathbf{C}}\times\mathbf{C}$. The symbol $e(x)$ denotes $exp(2\pi\sqrt{-1}x)$.

Definition 2.1. *A holomorphic function φ on $\mathbf{H}\times\mathfrak{h}_{\mathbf{C}}\times\mathbf{C}$ is an element of S_m, if it satisfies the conditions:*

1) $\varphi(\tau,z+\lambda+\mu\tau,t+I(\lambda,\lambda)\tau+2I(\lambda,z))=\varphi(\tau,z,t)\ \ \forall\lambda,\ \mu\in Q(R)$,

2) $\varphi(\tau,w(z),t)=\varphi(\tau,z,t)\ \ \forall w\in W$,

3) $\varphi(\tau,z,t+\alpha)=e(-m\alpha)\varphi(\tau,z,t)\ \ \forall\alpha\in\mathbf{C}$.

Put

$$S:=\bigoplus_{m=0}^{\infty}S_m. \tag{2.1}$$

Naturally S is a $\Gamma(\mathbf{H},\mathcal{O}_{\mathbf{H}})$-graded algebra, and the grading is defined by m.

Theorem 2.2. *([B-S1][B-S2][L][K-P])*

S is a polynomial algebra over $\Gamma(\mathbf{H},\mathcal{O}_{\mathbf{H}})$, freely generated by $l+1$ homogeneous elements $\Theta_0,\ldots\Theta_l$ of degree $m_i(i=0,\ldots,l)$ $(m_0\le m_1\le\ldots\le m_l)$.

Put,

$$Der_S:=\text{ the module of }\mathbf{C}\text{-derivations of the algebra }S, \tag{2.2}$$

$$\Omega^1_S:=\text{ the module of 1-forms for the algebra }S. \tag{2.3}$$

They are dual S-free modules by the natural pairing: $<,>$ with the dual basis:

$$Der_S = S\frac{\partial}{\partial\tau} \oplus \bigoplus_{i=0}^{l} S\frac{\partial}{\partial\Theta_i}, \tag{2.4}$$

$$\Omega_S^1 = Sd\tau \oplus \bigoplus_{i=0}^{l} Sd\Theta_i, \tag{2.5}$$

using a generator system Θ_i's of Theorem 2.2. Der_S and Ω_S^1 have the graded S-module structure in a natural way. There is a natural lifting map:

$$\Omega_S^1 \to \Omega_{\tilde{\mathbf{E}}}^1, \tag{2.6}$$

so that the form $\tilde{I}$ induces a S-bilinear form:

$$\tilde{I}_W : \Omega_S^1 \times \Omega_S^1 \to S. \tag{2.7}$$

The values of $\tilde{I}_W$ lie in S, since the form $\tilde{I}$ is invariant with respect to the group action in Def. 2.1. We remark that this symmetric tensor $\tilde{I}_W \in Der_S \otimes Der_S$ is degree 0.

In the rest of this paper, we assume that $m_{l-1} < m_l$.

Then in the S- graded module Der_S, the lowest degree vector fields become a free $\Gamma(\mathbf{H}, \mathcal{O}_{\mathbf{H}})$- module of rank 1 generated by $\frac{\partial}{\partial\Theta_l}$.

Multiplying a function $h \in \Gamma(\mathbf{H}, \mathcal{O}_{\mathbf{H}}^*)$, if necessary, we can take a highest degree generator Θ_l which satisfies

$$\frac{\partial^2}{\partial\Theta_l^2}\tilde{I}_W(d\Theta_l, d\Theta_l) = 0. \tag{2.8}$$

By (2.8), $\frac{\partial}{\partial\Theta_l}$ is normalized up to a constant factor.

Hereafter we fix $\Theta_0, \cdots, \Theta_l$ such that Θ_l satisfies the condition (2.8). We can define

$$T := \left\{ f \in S | \frac{\partial}{\partial\Theta_l} f = 0 \right\}, \tag{2.9}$$

$$\mathcal{F} := \left\{ \omega \in \Omega_S^1 | L_{\frac{\partial}{\partial\Theta_l}} \omega = 0 \right\}, \tag{2.10}$$

where $L_{\frac{\partial}{\partial \Theta_l}}$ means the Lie derivative with respect to the vector field $\frac{\partial}{\partial \Theta_l}$. By the above generators $\tau, \Theta_0, \cdots, \Theta_l$, we can represent $T, \mathcal{F}$ as follows:

$$T = \Gamma(\mathbf{H}, \mathcal{O}_{\mathbf{H}})[\Theta_0, \cdots, \Theta_{l-1}], \tag{2.11}$$

$$\mathcal{F} = T d\tau \oplus \bigoplus_{i=0}^{l} T d\Theta_i. \tag{2.12}$$

We define a T bilinear form,

$$\begin{array}{rcc} J^* : \mathcal{F} \times \mathcal{F} & \longrightarrow & T, \\ \omega_1 \times \omega_2 & \longmapsto & \frac{\partial}{\partial \Theta_l} \tilde{I}(\omega_1, \omega_2). \end{array} \tag{2.13}$$

The value $\frac{\partial}{\partial \Theta_l} \tilde{I}(\omega_1, \omega_2)$ belongs to T by the condition (2.8). Then the next important fact was shown by the Coxeter transformation theory for the extended affine root system.

Proposition 2.3. *(Saito [S])*

The T bilinear form J^* is non-degenerate and integrable.

This means that there exist generators $\tilde{\Theta}_i \in S_{m_i}$ of the algebra S over $\Gamma(\mathbf{H}, \mathcal{O}_{\mathbf{H}})$ which satisfy the following equations:

$$J^*(d\tilde{\Theta}_i, d\tilde{\Theta}_j) = const, J^*(d\tau, d\tilde{\Theta}_k) = const \quad (i, j, k = 0, \cdots, l). \tag{2.14}$$

The vector space $V = \mathbf{C}\tau \oplus \bigoplus_{i=0}^{l} \mathbf{C}\Theta_i$ has an intrinsic meaning and $Hom(V, \mathbf{C})$ is just the flat structure introduced in §1. We call these new generators $\tilde{\Theta}_i \in S_{m_i}$ of the algebra S, *the flat theta invariants.*

§3. Jacobi form.

Definition 3.1. *A Jacobi form of weight k, index m ($k, m \in \mathbf{Z}$) is a holomorphic function $\varphi : \mathbf{H} \times \mathfrak{h}_{\mathbf{C}} \times \mathbf{C} \to \mathbf{C}$ satisfying*

1) $\varphi(\tau, z + \lambda + \mu\tau, t + I(\lambda, \lambda)\tau + 2I(\lambda, z)) = \varphi(\tau, z, t) \ \ \forall\lambda, \ \mu \in Q(R)$,
2) $\varphi(\tau, w(z), t) = \varphi(\tau, z, t) \ \ \forall w \in W$,
3) $\varphi(\tau, z, t + \alpha) = e(-m\alpha)\varphi(\tau, z, t) \ \ \forall\alpha \in \mathbf{C}$,
4) $\varphi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}, t + \frac{c<z,z>}{2(c\tau+d)}\right) = (c\tau + d)^k \varphi(\tau, z, t)$ *forall* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z})$,
5) *φ has a Fourier development of the form*

$$e(-mt) \sum c(n) q^n \phi_n(z) \ \ (\ q = e(\tau))$$

with $c(n) = 0$ if $n < 0$.

The vector space of all functions φ is denoted by $J_{k,m}$. Put

$$J_{**} = \bigoplus_{k,m \in \mathbf{Z}} J_{k,m}, \quad M_* = \bigoplus_{k \in \mathbf{Z}} J_{k,0}.$$

Theorem 3.2. *(Wirthmüller)*

J_{**} is a polynomial algebra over M_*, freely generated by $l+1$ Jacobi forms $\varphi_0, \cdots, \varphi_l$, where φ_i is of weight k_i, index m_i.

Proposition 3.3.

For $\varphi \in J_{k,m}$; $\phi \in J_{k',m'}$,

$$\tilde{I}_W(d(\eta^{-2k}\varphi), d(\eta^{-2k'}\phi))/\eta^{-2k-2k'} \in J_{k+k'+2, m+m'},$$

where $\eta(\tau) := q^{1/24} \prod_{n=1}^{\infty}(1 - q^n), \quad (q = e(\tau))$.

We take $\varphi = \phi = \varphi_l$. Then

$$\phi := \tilde{I}_W(d(\eta^{-2k_l}\varphi_l), d(\eta^{-2k_l}\varphi_l))/\eta^{-2k_l-2k_l}$$

is a Jacobi form of weight $2k_l+2$, index $2m_l$. Since $J_{2,0}=\{0\}$ by the basic theory of modular forms, any function multiplied by φ_l^2 does not appear when ϕ is represented by the polynomial of $\varphi_0,\cdots,\varphi_l$ over M_*. This means that we can give the normalized lowest degree vector field $\frac{\partial}{\partial\Theta_l}$ introduced in (2.8) as

$$\frac{\partial}{\partial(\eta^{-2k_l}\varphi_l)}.$$

By this fact and Prop 3.2, we can calculate the tensor J^* represented by

$$\tau, \eta^{-2k_0}\varphi_0, \eta^{-2k_1}\varphi_1, \cdots, \eta^{-2k_l}\varphi_l.$$

In §4, we give the explicit calculation for the type G_2. If we calculate the cases of the type E_l, then this gives an answer to the problem in §1.

§4. Jacobi form of type G_2

We define a $\mathbf{C}$-bilinear form $<,>$ on $\mathbf{C}^2$ by

$$<z,w>=2z_1w_1+2z_2w_2-z_1w_2-z_2w_1$$

where $z=(z_1,z_2), w=(w_1,w_2)\in\mathbf{C}^2$.

Definition 4.1. *A Jacobi form of weight k, index m ($k,m\in\mathbf{Z}$) and type G_2 is a holomorphic function $\varphi:\mathbf{H}\times\mathbf{C}^2\times\mathbf{C}\to\mathbf{C}$ satisfying*

1) $\varphi(\tau,z+\lambda+\mu\tau,t+<\lambda,\lambda>\tau+2<\lambda,z>)=\varphi(\tau,z,t)\ \ \forall\lambda,\ \mu\in\mathbf{Z}^2$,

2) $\varphi\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d},t+\frac{c<z,z>}{2(c\tau+d)}\right)=(c\tau+d)^k\varphi(\tau,z,t)$,

3) $\varphi(\tau,z,t+\alpha)=e(-m\alpha)\varphi(\tau,z,t)\ \ \forall\alpha\in\mathbf{C}$,

4) $\varphi(\tau,-z_1-z_2,z_2,t)=\varphi(\tau,z_1,z_2,t)$,

$\varphi(\tau,z_1,-z_1-z_2,t)=\varphi(\tau,z_1,z_2,t)$,

$\varphi(\tau,-z_1,-z_2,t)=\varphi(\tau,z_1,z_2,t)$,

5) *φ has a Fourier development of the form*

$$e(-mt)\sum c(n,r_1,r_2)q^n\zeta_1^{r_1}\zeta_2^{r_2}\quad(\ q=e(\tau),\zeta_1=e(z_1),\zeta_2=e(z_2)\ )$$

with $c(n, r_1, r_2) = 0$ *if* $n < 0$.

The vector space of all functions φ is denoted by $J_{k,m}$. Put

$$J_{**} = \bigoplus_{k,m \in \mathbf{Z}} J_{k,m}, \quad M_* = \bigoplus_{k \in \mathbf{Z}} J_{k,0}.$$

Theorem 4.2. *(Wirthmüller) The ring* J_{**} *is a polynomial algebra over* M_* *on three generators*

$$a_0 \in J_{0,1}, \quad a_2 \in J_{-2,1}, \quad a_6 \in J_{-6,2}.$$

These generators are unique up to constant multiplication. We can calculate the leading term of the Fourier development of these generators.

Proposition 4.3. *The leading term of the Fourier development of the above generators is as follows:*

$$a_0 = e(-t)[18 + f + g] \ (\ mod\ q\mathbf{C}\{q, \zeta_1, \zeta_2\}) \tag{4.1}$$

$$a_2 = e(-t)[-6 + f + g] \ (\ mod\ q\mathbf{C}\{q, \zeta_1, \zeta_2\}) \tag{4.2}$$

$$a_6 = e(-2t)[(f - g)^2] \ (\ mod\ q\mathbf{C}\{q, \zeta_1, \zeta_2\}) \tag{4.3}$$

where

$$f := \zeta_1^{-1} + \zeta_1\zeta_2^{-1} + \zeta_2 \tag{4.4}$$

$$g := \zeta_2^{-1} + \zeta_2\zeta_1^{-1} + \zeta_1 \tag{4.5}$$

Remark. *The above functions* f, g *coincides with the characters of the highest weight representation of the* A_2 *type simple Lie algebra.*

Theorem 4.4. *The relationship between the flat theta invariants and the Jacobi forms is as follows:*

$$\tilde{\Theta}_0 = P_{11}(\tau)a_0 + P_{12}(\tau)a_2, \tag{4.6}$$

$$\tilde{\Theta}_1 = P_{21}(\tau)a_0 + P_{22}(\tau)a_2, \tag{4.7}$$

$$\tilde{\Theta}_2 = P_{31}(\tau)\tilde{\Theta}_0^2 + P_{32}(\tau)\tilde{\Theta}_0\tilde{\Theta}_1 + P_{33}(\tau)\tilde{\Theta}_1^2 + \frac{1}{-4\pi\sqrt{-1}}\eta^{12}a_6, \tag{4.8}$$

where

$$P_{11}(\tau) := \frac{1}{2^5\pi^3}\eta(\tau)^{-2}Q_1(\tau), \tag{4.9}$$

$$P_{12}(\tau) := -\frac{3}{2^6\pi^5}\eta(\tau)^2 Q_2(\tau)^2, \tag{4.10}$$

$$P_{21}(\tau) := -\frac{1}{2^5\pi^3}\eta(\tau)^{-2}Q_2(\tau), \tag{4.11}$$

$$P_{22}(\tau) := \frac{3}{2^6\pi^5}\eta(\tau)^2 Q_1(\tau)^2, \tag{4.12}$$

$$P_{31}(\tau) := \frac{\sqrt{-1}}{8\pi}\eta(\tau)^4 Q_1(\tau), \tag{4.13}$$

$$P_{32}(\tau) := 2\frac{d\eta(\tau)}{d\tau}/\eta(\tau), \tag{4.14}$$

$$P_{33}(\tau) := \frac{\sqrt{-1}}{8\pi}\eta(\tau)^4 Q_2(\tau), \tag{4.15}$$

and

$$\eta(\tau) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \quad (q = e(\tau)) \tag{4.16}$$

$$g_2(\tau) := 60 \sum_{m,n\in\mathbf{Z},(m,n)\neq(0,0)} \frac{1}{(m\tau+n)^4}, \tag{4.17}$$

$$g_3(\tau) := 140 \sum_{m,n\in\mathbf{Z},(m,n)\neq(0,0)} \frac{1}{(m\tau+n)^6}, \tag{4.18}$$

$$Q_1(\tau) := \left[\frac{g_3(\tau)}{\eta(\tau)^{12}} + \sqrt{\frac{1}{27}}\sqrt{-1}(2\pi)^6\right]^{\frac{1}{3}}, \tag{4.19}$$

$$Q_2(\tau) := \left[\frac{g_3(\tau)}{\eta(\tau)^{12}} - \sqrt{\frac{1}{27}}\sqrt{-1}(2\pi)^6\right]^{\frac{1}{3}}, \tag{4.20}$$

such that

$$Q_1(\tau)Q_2(\tau) = \frac{g_2(\tau)}{3\eta(\tau)^8}, \tag{4.21}$$

$$Q_1(\tau)Q_1(\tau+1) = -\frac{g_2(\tau)}{3\eta(\tau)^8}, \tag{4.22}$$

$Q_1(\tau), Q_2(\tau)$ is determined uniquely by the condition (4.21),(4.22).

Proof. We can calculate the following differential relations:

$$I(da_0, da_0) = -\frac{1}{2\pi^2}g_2(\tau)a_0a_2 - \frac{9}{2\pi^4}g_3(\tau)a_2^2 - \frac{3}{8\pi^6}g_2(\tau)^2a_6, \tag{4.23}$$

$$I(da_0, d(\eta(\tau)^4 a_2)) = -\frac{1}{\pi^2} g_2(\tau)\eta(\tau)^4 a_2^2 - \frac{9}{4\pi^4} g_3(\tau)\eta(\tau)^4 a_6, \tag{4.24}$$

$$I(da_0, d(\eta(\tau)^{12} a_6)) = -\frac{5}{2\pi^2} g_2(\tau)\eta(\tau)^{12} a_2 a_6, \tag{4.25}$$

$$I(d(\eta(\tau)^4 a_2), d(\eta(\tau)^4 a_2)) = -\frac{2\pi^2}{3}\eta(\tau)^8 a_0 a_2 - \frac{1}{2\pi^2} g_2(\tau)\eta(\tau)^8 a_6, \tag{4.26}$$

$$I(d(\eta(\tau)^4 a_2), d(\eta(\tau)^{12} a_6)) = -2\pi^2 \eta(\tau)^{16} a_0 a_6, \tag{4.27}$$

$$I(d(\eta(\tau)^{12} a_6), d(\eta(\tau)^{12} a_6)) = -\frac{8\pi^2}{3}\eta(\tau)^{24} a_2^2 a_6. \tag{4.28}$$

Thus we obtain the algebraic equations and differential equations satisfied by $P_{ij}(\tau)$. By use of the following differential relations:

$$\frac{d}{d\tau}\left\{\left(\frac{g_2(\tau)}{\eta(\tau)^8}\right)^n\right\} = -\frac{3n\sqrt{-1}}{\pi}\frac{g_3(\tau)}{g_2(\tau)}\left\{\left(\frac{g_2(\tau)}{\eta(\tau)^8}\right)^n\right\}, \tag{4.29}$$

$$\frac{d}{d\tau}\left\{\left(\frac{g_3(\tau)}{\eta(\tau)^{12}}\right)^n\right\} = -\frac{n\sqrt{-1}}{6\pi}\frac{g_2(\tau)^2}{g_3(\tau)}\left\{\left(\frac{g_3(\tau)}{\eta(\tau)^{12}}\right)^n\right\}, \tag{4.30}$$

$$\begin{aligned} &\frac{d}{d\tau}\left\{\left(\frac{g_3(\tau)+\sqrt{\frac{1}{27}}\sqrt{-1}(2\pi)^6\eta(\tau)^{12}}{g_3(\tau)-\sqrt{\frac{1}{27}}\sqrt{-1}(2\pi)^6\eta(\tau)^{12}}\right)^n\right\} \\ &= -2\sqrt{3}(2\pi)^5 n\frac{\eta(\tau)^{12}}{g_2(\tau)}\left\{\left(\frac{g_3(\tau)+\sqrt{\frac{1}{27}}\sqrt{-1}(2\pi)^6\eta(\tau)^{12}}{g_3(\tau)-\sqrt{\frac{1}{27}}\sqrt{-1}(2\pi)^6\eta(\tau)^{12}}\right)^n\right\}, \end{aligned} \tag{4.31}$$

we can solve the equations satisfied by P_{ij}. *Q.E.D.*

Corollary 4.5.

The modular property of the flat theta invariants is as follows:

$$\tilde{\Theta}_0(\tau+1, z, t) = e(-1/12)\tilde{\Theta}_1(\tau, z, t), \tag{4.32}$$

$$\tilde{\Theta}_1(\tau+1, z, t) = -e(1/12)\tilde{\Theta}_0(\tau, z, t), \tag{4.33}$$

$$\tilde{\Theta}_2(\tau+1, z, t) = -\tilde{\Theta}_2(\tau, z, t), \tag{4.34}$$

$$\tilde{\Theta}_0(-\frac{1}{\tau}, \frac{z}{\tau}, t + \frac{< z, z >}{2\tau}) = \frac{\sqrt{-1}e(1/3)}{\tau}\tilde{\Theta}_1(\tau, z, t), \tag{4.35}$$

$$\tilde{\Theta}_1(-\frac{1}{\tau}, \frac{z}{\tau}, t + \frac{< z, z >}{2\tau}) = \frac{\sqrt{-1}}{e(1/3)\tau}\tilde{\Theta}_0(\tau, z, t), \tag{4.36}$$

$$\tilde{\Theta}_2(-\frac{1}{\tau}, \frac{z}{\tau}, t + \frac{< z, z >}{2\tau}) = -\tilde{\Theta}_2 - \frac{1}{\tau}\tilde{\Theta}_0\tilde{\Theta}_1 \tag{4.37}$$

Proof. By the equations:

$$Q_1(\tau)Q_1(\tau+1) = -\frac{g_2(\tau)}{3\eta(\tau)^8}, \tag{4.38}$$

$$Q_1(\tau)Q_1(-\frac{1}{\tau}) = -e(1/3)\frac{g_2(\tau)}{3\eta(\tau)^8}, \tag{4.39}$$

$$Q_2(\tau)Q_2(\tau+1) = -e(-1/3)\frac{g_2(\tau)}{3\eta(\tau)^8}, \tag{4.40}$$

$$Q_2(\tau)Q_2(-\frac{1}{\tau}) = -e(-1/3)\frac{g_2(\tau)}{3\eta(\tau)^8}, \tag{4.41}$$

$$P_{32}(\tau+1) = P_{32}(\tau), \tag{4.42}$$

$$P_{32}(-\frac{1}{\tau}) = \tau^2 P_{32}(\tau) + \tau, \tag{4.43}$$

$$\eta(-\frac{1}{\tau}) = \left(\frac{\tau}{\sqrt{-1}}\right)^{\frac{1}{2}} \eta(\tau), \tag{4.44}$$

we have (4.32)-(4.37). $Q.E.D.$

Corollary 4.6. *Let* $\varphi(\tau) := \frac{d}{d\tau}\eta(\tau)/\eta(\tau)$*. Then* $\varphi(\tau)$ *satisfies the following differential equation:*

$$\frac{d\varphi}{d\tau}(\tau) = 2\varphi^2(\tau) + \frac{1}{2^5 3\pi^2} g_2(\tau). \tag{4.45}$$

Proof. Since the differential equations (2.14) are overdetermined, thus we obtain (4.45) as the integrable condition. $Q.E.D.$

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