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Some Results on Jacobi Forms of Higher Degree

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Abstract

In this article, the author gives some of his results on Jacobi forms of higher degree without proof. The proof can be found in the references [Y1] and [Y2].

1 Jacobi Forms

First of all, we introduce the notations. We denote by Z, R and C the ring of integers, the field of real numbers and the field of complex numbers respectively. We denote by Z^+ the set of all positive integers. $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commuatative ring F. For any $M \in F^{(k,l)}$, ${}^tM$ denotes the transpose matrix of M. For $A \in F^{(k,l)}$, $\sigma(A)$ denotes the trace of A. For $A \in F^{(k,l)}$ and $B \in F^{(k,k)}$, we set $B[A] = {}^tABA$. E_n denotes the identity matrix of degree n. For any positive integer $g \in Z^+$, we let

$$H_g := \{\, Z \in C^{(g,g)} \mid Z = {}^tZ, \;\; Im\, Z > 0 \,\}$$

the Siegel upper half plane of degree g. Let $Sp(g, R)$ and $Sp(g, Z)$ be the real symplectic group of degree g and the Siegel modular group of degree g respectively.

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Let

$$(1.1) \qquad O_g(R^+) := \{M \in R^{(2g,2g)} \mid {}^tMJ_gM = \nu J_g \ \ for\ some\ \nu > 0\ \}$$

be the group of *similitudes* of degree g, where

$$J_g := \begin{pmatrix} 0 & E_g \\ -E_g & 0 \end{pmatrix}.$$

Let $M \in O_g(R^+)$. If ${}^tMJ_gM = \nu J_g$, we write $\nu = \nu(M)$. It is easy to see that $O_g(R^+)$ acts on H_g transitively by

$$M < Z >:= (AZ + B)(CZ + D)^{-1},$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in O_g(R^+)$ and $Z \in H_g$.
For $l \in Z^+$, we define

$$(1.2) \qquad O_g(l) := \{\, M \in Z^{(2g,2g)} \mid {}^tMJ_gM = lJ_g\ \}.$$

We observe that $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in O_g(l)$ is equivalent to the conditions

$$(1.3) \qquad {}^tAC = {}^tCA, \ \ {}^tBD = {}^tDB, \ \ {}^tAD - {}^tCB = lE_g$$

or

$$(1.4) \qquad A\,{}^tB = B\,{}^tA, \ \ C\,{}^tD = D\,{}^tC, \ \ A\,{}^tD - B\,{}^tC = lE_g.$$

For two positive integers g and h, we consider the *Heisenberg group*

$$H_R^{(g,h)} := \{\, [(\lambda,\mu),\kappa] \mid \lambda,\, \mu \in R^{(h,g)}, \ \ \kappa \in R^{(h,h)}, \ \ \kappa + \mu\,{}^t\lambda \ symmetric\,\}$$

endowed with the following multiplication law

$$[(\lambda,\mu),\kappa] \circ [(\lambda',\mu'),\kappa'] \ := \ [(\lambda+\lambda',\mu+\mu'),\kappa+\kappa'+\lambda\,{}^t\mu' - \mu\,{}^t\lambda'].$$

We define the semidirect product of $O_g(R^+)$ and $H_R^{(g,h)}$

$$O_R^{(g,h)} =: O_g(R^+) \ltimes H_R^{(g,h)} \tag{1.5}$$

endowed with the following multiplication law

$$\begin{aligned}(1.6)\qquad & (M,[(\lambda,\mu),\kappa])\cdot(M',[(\lambda',\mu'),\kappa']) \\ := \quad & (MM',[(\nu(M')^{-1}\tilde{\lambda}+\lambda',\nu(M')^{-1}\tilde{\mu}+\mu'),\nu(M')^{-1}\kappa+\kappa'+\nu(M')^{-1}(\tilde{\lambda}\,{}^t\mu'-\tilde{\mu}\,{}^t\lambda')]),\end{aligned}$$

with $M, M' \in O_g(R^+)$ and $(\tilde{\lambda},\tilde{\mu}) := (\lambda,\mu)M'$. Clearly the *Jacobi group* $G_R^{(g,h)} := Sp(g,R) \ltimes H_R^{(g,h)}$ is a normal subgroup of $O_R^{(g,h)}$. It is easy to see that $O_g(R^+)$ acts on $H_g \times C^{(h,g)}$ transitively by

$$(1.7)\ (M,[(\lambda,\mu),\kappa])\cdot(Z,W) := (M < Z >, \nu(W+\lambda Z+\mu)(CZ+D)^{-1}),$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in O_g(R^+)$, $\nu = \nu(M)$, $(Z,W) \in H_g \times C^{(h,g)}$.

Let ρ be a rational representation of $GL(g,C)$ on a finite dimensional complex vector space V_ρ. Let $\mathcal{M} \in R^{(h,h)}$ be a symmetric half integral matrix of degree h. We define

$$\begin{aligned}(1.8)\qquad & (f|_{\rho,\mathcal{M}}[(M,[(\lambda,\mu),\kappa])])(Z,W) \\ := \quad & \exp\{-2\pi\nu i\sigma(\mathcal{M}[W+\lambda Z+\mu](CZ+D)^{-1}C)\} \\ & \times\exp\{2\pi\nu i\rho(\mathcal{M}(\lambda Z{}^t\lambda+2\lambda{}^tW+(\kappa+\mu{}^t\lambda)))\} \\ & \times\sigma(CZ+D)^{-1}f(M<Z>,\nu(W+\lambda Z+\mu)(CZ+D)^{-1}),\end{aligned}$$

where $\nu = \nu(M)$.

Lemma 1.1. Let $g_i = (M_i,[(\lambda_i,\mu_i),\kappa_i]) \in O_R^{(g,h)}$ $(i=1,2)$. For any $f \in C^\infty(H_g \times C^{(h,g)}, V_\rho)$, we have

$$(f|_{\rho,\mathcal{M}}[g_1])|_{\rho,\nu(M_1)\mathcal{M}}[g_2] = f|_{\rho,\mathcal{M}}[g_1g_2]. \tag{1.9}$$

Definition 1.2. Let ρ and $\mathcal{M}$ be as above. Let

$$H_Z^{(g,h)} := \{\, [(\lambda,\mu),\kappa] \in H_R^{(g,h)} \mid \lambda,\mu \in Z^{(h,g)},\ \ \kappa \in Z^{(h,h)} \,\}.$$

A *Jacobi form* of index $\mathcal{M}$ with respect to ρ is a holomorphic function $f \in C^\infty(H_g \times C^{(h,g)}, V_\rho)$ satisfying the following conditions (A) and (B):

(A) $f|_{\rho,\mathcal{M}}[\tilde{\gamma}] = f$ for all $\tilde{\gamma} \in \Gamma_g^J := Sp(g,Z) \ltimes H_Z^{(g,h)}$.

(B) f has a Fourier expansion of the following form :

$$f(Z,W) = \sum_{\substack{T \geq 0 \\ \text{half-integral}}} \sum_{R \in Z^{(g,h)}} C(T,R) \exp(2\pi i \sigma(TZ + RW))$$

with $c(T,R) \neq 0$ only if $\begin{pmatrix} T & \frac{1}{2}R \\ \frac{1}{2}\,{}^tR & \mathcal{M} \end{pmatrix} \geq 0$.

If $g \leq 2$, the condition (B) is superfluous by Koecher principle(see [Z] Lemma 1.6). We denote by $J_{\rho,\mathcal{M}}(\Gamma_g)$ the vector space of all Jacobi forms of index $\mathcal{M}$ with respect to ρ. In the special case $V_\rho = C,\ \rho(A) = (det\, A)^k\ (k \in Z,\ A \in GL(g,C))$, we write $J_{k,\mathcal{M}}(\Gamma_g)$ instead of $J_{\rho,\mathcal{M}}(\Gamma_g)$ and call k the *weight* of a Jacobi form $f \in J_{k,\mathcal{M}}(\Gamma_g)$.

Ziegler([Zi] Theorem 1.8 or [E-Z] Theorem 1.1) proves that the vector space $J_{\rho,\mathcal{M}}(\Gamma_g)$ is finite dimensional.

2 Singular Jacobi Forms

In this section, we define the concept of singular Jacobi forms and characterize singular Jacobi forms.

Let $\mathcal{M}$ be a symmetric positive definite, half integral matrix of degree h. A Jacobi form $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$ admits a Fourier expansion (see Definition

1.2 (B))

$$(2.1)\qquad f(Z,W)=\sum_{T,R} c(T,R)e^{2\pi i\sigma(TZ)}\cdot e^{2\pi i\sigma(RW)},\quad Z\in H_g,\quad W\in C^{(h,g)}.$$

A Jacobi form $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$ is said to be *singular* if it admits a Fourier expansion such that the Fourier coefficient $c(T,R)$ is zero unless $det\,(4T - R\mathcal{M}^{-1}\,{}^tR) = 0$.

Example 2.1. Let $\mathcal{M} = {}^t\mathcal{M}$ be as above. Let $S \in Z^{(2k,2k)}$ be a symmetric positive definite integral matrix of degree $2k$ and $c \in Z^{(2k,h)}$. We consider the theta series

$$(2.2)\qquad \vartheta^{(g)}_{S,c}(Z,W) := \sum_{\lambda\in Z^{(2k,g)}} e^{\pi i\sigma(S[\lambda]Z+2S\lambda\,{}^t(cW))},\quad Z\in H_g,\quad W^{(h,g)}.$$

We assume that $2k < g + rank(\mathcal{M})$. Then $\vartheta_{S,c}(Z,W)$ is a singular Jacobi form in $J_{k,\mathcal{M}}(\Gamma_g)$, where $\mathcal{M} = \frac{1}{2}{}^tc\mathcal{M}c$. We note that if the Fourier coefficient $c(T,R)$ of $\vartheta^{(g)}_{S,c}$ is nonzero, there exists $\lambda \in Z^{(2k,g)}$ such that

$$\frac{1}{2}\,{}^t(\lambda,c)S(\lambda,c) = \begin{pmatrix} T & \frac{1}{2}R \\ \frac{1}{2}{}^tR & \mathcal{M}\end{pmatrix}.$$

Thus

$$rank\begin{pmatrix} T & \frac{1}{2}R \\ \frac{1}{2}{}^tR & \mathcal{M}\end{pmatrix} \le 2k < g + rank(\mathcal{M}).$$

Therefore $det\,(4T - R\mathcal{M}^{-1}\,{}^tR) = 0$.

The following natural question arises:

Problem: *Characterize the singular Jacobi forms.*

The author([Y1]) gives some answers for this problem. He characterizes singular Jacobi forms by the *differential equation* and the *weight* of the representation ρ.

Now we define a very important differential operator characterizing *singular Jacobi forms.* We let

$$(2.3) \qquad \mathcal{P}_g := \{\, Y \in R^{(g,g)} \mid Y = {}^tY > 0 \,\}$$

be the open convex cone in the Euclidean space $R^{\frac{g(g+1)}{2}}$. We define the differential operator operator $M_{g,h,\mathcal{M}}$ on $\mathcal{P}_g \times R^{(h,g)}$ defined by

$$(2.4) \qquad M_{g,h,\mathcal{M}} := det\,(Y) \cdot det\left(\frac{\partial}{\partial Y} + \frac{1}{8\pi}{}^t\!\left(\frac{\partial}{\partial V}\right)\mathcal{M}^{-1}\left(\frac{\partial}{\partial V}\right)\right),$$

where $\frac{\partial}{\partial Y} = \left(\frac{(1+\delta_{\mu\nu})}{2}\frac{\partial}{\partial y_{\mu\nu}}\right)$ and $\frac{\partial}{\partial V} = \left(\frac{\partial}{\partial v_{kl}}\right)$.

Definition 2.2. An irreducible finite dimensional representation ρ of $GL(g,C)$ is determined uniquely by its highest weight $(\lambda_1, \cdots, \lambda_g) \in Z^g$ with $\lambda_1 \leq \cdots \leq \lambda_g$. We denote this representation by $\rho = (\lambda_1, \cdots, \lambda_g)$. The number $k(\rho) := \lambda_g$ is called the *weight* of ρ.

Theorem A. Let $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$ be a Jacobi form of index $\mathcal{M}$ with respect to ρ. Then the following are equivalent:

(1) f is a *singular* Jacobi forms.

(2) f satisfies the *differential equation* $M_{g,h,\mathcal{M}} f = 0$.

Theorem B. Let $2\mathcal{M}$ be a symmetric positive definite, *unimodular* even matrix of degree h. Assume that ρ satisfies the following condition

$$(2.5) \qquad \rho(A) = \rho(-A) \quad for\ all\ A \in GL(g,C).$$

Then any nonvanishing Jacobi form in $J_{\rho,\mathcal{M}}(\Gamma_g)$ is *singular* if and only if $2k(\rho) < g + rank\,(\mathcal{M})$. Here $k(\rho)$ denotes the *weight* of ρ.

Conjecture. For general ρ and $\mathcal{M}$ without the above assumptions on them, *a nonvanishing Jacobi form $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$ is singular if and only if*

$2k(\rho) < g + rank\,(\mathcal{M})$.

REMARKS. If $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$ is a Jacobi form, we may write

$$(*)\qquad f(Z,W) = \sum_{a\in\mathcal{N}} f_a(Z)\cdot\vartheta_{2\mathcal{M},a,0}(Z,W),\quad Z\in H_g,\quad W\in C^{(h,g)},$$

where $\{\, f_a : H_g \longrightarrow V_\rho \,|\, a\in\mathcal{N}\,\}$ are uniquely determined holomorphic functions on H_g. A singular modular form of type ρ may be written as a finite sum of theta series $\vartheta_{S,P}(Z)$'s with pluriharmonic coefficients (cf. [F]). The following problem is quite interesting.

Problem. Describe the functions $\{f_a \,|\, a\in\mathcal{N}\}$ explicitely given by (*) when $f\in J_{\rho,\mathcal{M}}(\Gamma_g)$ is a *singular* Jacobi form.

3 The Siegel-Jacobi Operators

In this section, we investigate the Siegel-Jacobi operator and the action of Hecke operator on Jacobi forms. The Siegel-Jacobi operator

$$\Psi_{g,r} : J_{\rho,\mathcal{M}}(\Gamma_g) \longmapsto J_{\rho^{(r)},\mathcal{M}}(\Gamma_r)$$

is defined by

$$(\Psi_{g,r}f)(Z,W) := \lim_{t\to\infty} f\left(\begin{pmatrix} Z & 0 \\ 0 & itE_{g-r}\end{pmatrix}, (W,0)\right),\quad f\in J_{\rho,\mathcal{M}}(\Gamma_g),$$

$Z\in H_r$, $W\in C^{(h,r)}$ and $J_{\rho,\mathcal{M}}(\Gamma_g)$ denotes the space of all Jacobi forms of index $\mathcal{M}$ with respect to an irreducible rational finite dimesional representation ρ *of* $GL(g,C)$. We note that the above limit always exists because a Jacobi form f admits a Fourier expansion converging uniformly on any set of the form

$$\{(Z,W)\in H_g\times C^{(h,g)} \,|\, Im\,Z \geq Y_0 > 0,\ W\in K\subset C^{(h,g)}\ compact\}.$$

Here the representation $\rho^{(r)}$ of $GL(r,C)$ is defined as follows. Let $V_\rho^{(r)}$ be the subspace of V_ρ generated by $\{f(Z,W)\,|\,f\in J_{\rho,\mathcal{M}}(\Gamma_g),\,(Z,W)\in H_g\times C^{(h,g)}\}$. Then $V_\rho^{(r)}$ is invariant under

$$\left\{\begin{pmatrix} g & 0 \\ 0 & E_{g-r}\end{pmatrix} : g\in GL(r,C)\right\}.$$

Then we have a rational representation $\rho^{(r)}$ of $GL(r,C)$ on $V_\rho^{(r)}$ defined by

$$\rho^{(r)}(g)v := \rho\left(\begin{pmatrix} g & 0 \\ 0 & E_{g-r}\end{pmatrix}\right)v,\quad g\in GL(r,C),\quad v\in V_\rho^{(r)}.$$

In the Siegel case, we have the so-called Siegel $\Phi-$operator

$$\Phi=\Phi_{g,g-1}\,:\,[\Gamma_g,k]\longrightarrow[\Gamma_{g-1},k]$$

defined by

$$(\Phi f)(Z):=\lim_{t\to\infty} f\begin{pmatrix} Z & 0 \\ 0 & it\end{pmatrix},\quad f\in[\Gamma_g,k],\quad Z\in H_{g-1},$$

where $[\Gamma_g,k]$ denotes the vector space of all Siegel modular forms on H_g of weight k.

Here $[\Gamma_g,k]$ denotes the vector space of all Siegel modular forms on H_g of weight k.

The following properties of Φ are known :

(S1) If $k>2g$ and k is even, Φ is surjective.

(S2) If $2k<g$, then Φ is injective.

(S3) If $2k+1<g$, then Φ is bijective.

H. Maass([M1]) proved the statement (1) using Poincaré series. E. Freitag ([F2]) proved the statements (2) and (3) using the theory of singular modular forms.

The author([Y2]) proves the following theorems:

Theorem C. Let $2\mathcal{M} \in Z^{(h,h)}$ be a positive definite, unimodular symmetric even matrix of degree h. *We assume that* ρ *satisfies the condition (3.1)*:

$$\rho(A) = \rho(-A) \quad \textit{for all } A \in GL(g, C). \tag{3.1}$$

We also assume that ρ satisfies the condition $2k(\rho) < g + rank\,(\mathcal{M})$. Then the Siegel-Jacobi operator

$$\Psi_{g,g-1} : J_{\rho,\mathcal{M}}(\Gamma_g) \longrightarrow J_{\rho^{(g-1)},\mathcal{M}}(\Gamma_{g-1})$$

is injective. Here $k(\rho)$ denotes the *weight* of ρ.

Theorem D. Let $2\mathcal{M} \in Z^{(h,h)}$ be as above in Theorem A. *Assume that* ρ *satisfies the condition (3.1) and* $2k(\rho) + 1 < g + rank\,(\mathcal{M})$. Then The Siegel-Jacobi operator

$$\Psi_{g,g-1} : J_{\rho,\mathcal{M}}(\Gamma_g) \longrightarrow J_{\rho^{(g-1)},\mathcal{M}}(\Gamma_{g-1})$$

is an isomorphism.

Theorem E. Let $2\mathcal{M} \in Z^{(h,h)}$ be as above in Theorem A. *Assume that* $2k > 4g + rank\,(\mathcal{M})$ *and* $k \equiv 0 (mod\ 2)$. Then the Siegel-Jacobi operator

$$\Psi_{g,g-1} : J_{k,\mathcal{M}}(\Gamma_g) \longrightarrow J_{k,\mathcal{M}}(\Gamma_{g-1})$$

is surjecitve.

The proof of the above theorems is based on the important Shimura correspondence, the theory of singular modular forms and the result of H. Maass.

We recall

$$O_g(l) := \{ \, M \in Z^{(2g,2g)} \mid {}^tMJ_gM = lJ_g \, \}.$$

$O_g(l)$ is decomposed into finitely many double cosets *mod* Γ_g, i.e.,

$$O_g(l) = \cup_{j=1}^m \Gamma_g g_j \Gamma_g \quad (\textit{disjoint union}). \tag{3.2}$$

We define

$$T(l) := \sum_{j=1}^m \Gamma_g g_j \Gamma_g \in \mathcal{H}^{(g)}, \quad \textit{the Hecke algebra}. \tag{3.3}$$

Let $M \in O_g(l)$. For a Jacobi form $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$, we define

$$f|_{\rho,\mathcal{M}}(\Gamma_g M \Gamma_g) := l^{gk(\rho)-\frac{g(g+1)}{2}} \sum_i f|_{\rho,\mathcal{M}}[(M_i,[(0,0),0])], \tag{3.4}$$

where $\Gamma_g M \Gamma_g = \cup_i^m \Gamma_g M_i$ (finite disjoint union) and $k(\rho)$ denotes the weight of ρ.

Theorem F. Let $M \in O_g(l)$ and $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$. Then

$$f|_{\rho,\mathcal{M}}(\Gamma_g M \Gamma_g) \in J_{\rho,l\mathcal{M}}(\Gamma_g).$$

For a prime p, we define

$$O_{g,p} := \cup_{l=0}^\infty O_g(p^l). \tag{3.5}$$

Let $\check{\mathcal{L}}_{g,p}$ be the C-module generated by all left cosets $\Gamma_g M$, $M \in O_{g,p}$ and $\check{\mathcal{H}}_{g,p}$ the C-module generated by all double cosets $\Gamma_g M \Gamma_g$, $M \in O_{g,p}$. Then $\check{\mathcal{H}}_{g,p}$ is a commutative associative algebra. Since $j(\check{\mathcal{H}}_{g,p}) \subset \check{\mathcal{L}}_{g,p}$, we have a monomorphism $j : \check{\mathcal{H}}_{g,p} \longrightarrow \check{\mathcal{L}}_{g,p}$.

In a left coset $\Gamma_g M$, $M \in O_{g,p}$, we can choose a represntative M of the form

$$M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, \quad {}^tAD = p^{k_0}E_g, \quad {}^tBD = {}^tDB, \tag{3.6}$$

$$(3.7) \qquad A = \begin{pmatrix} a & \alpha \\ 0 & A^* \end{pmatrix}, \quad B = \begin{pmatrix} b & {}^t\beta_1 \\ \beta_2 & B^* \end{pmatrix}, \quad D = \begin{pmatrix} d & 0 \\ \delta & D^* \end{pmatrix},$$

where $\alpha,\ \beta_1,\ \beta_2,\ \delta \in Z^{g-1}$. Then we have

$$(3.8) \qquad M^* := \begin{pmatrix} A^* & B^* \\ 0 & D^* \end{pmatrix} \in O_{g-1,p}.$$

For any integer $r \in Z$, we define

$$(3.9) \qquad (\Gamma_g M)^* := \frac{1}{d^r}\Gamma_{g-1}M^*.$$

If $\Gamma_g M \Gamma_g = \cup_{j=1}^m \Gamma_g M_j$ (*disjoint union*), $M,\ M_j \in O_{g,p}$, then we define in a natural way

$$(3.10) \qquad (\Gamma_g M \Gamma_g)^* = \frac{1}{d^r}\sum_{j=1}^m \Gamma_{g-1}M_j^*.$$

We extend the above map (3.9) linearly on $\check{\mathcal{H}}_{g,p}$ and then we obtain an algebra homomorphism

$$(3.11) \qquad \begin{aligned} \check{\mathcal{H}}_{g,p} &\longrightarrow \check{\mathcal{H}}_{g-1,p} \\ T &\longmapsto T^* . \end{aligned}$$

It is known that the above map is a surjective map([ZH] Theorem 2).

Theorem G. Suppose we have

(a) a rational finite dimensional representation

$$\rho : GL(g, C) \longrightarrow GL(V_\rho),$$

(b) a rational finite dimensional representation

$$\rho_0 : GL(g-1, C) \longrightarrow GL(V_{\rho_0})$$

(c) a linear map $R : V_\rho \longrightarrow V_{\rho_0}$ satisfying the following properties (1) and (2):

(1) $R \circ \rho \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} = \rho_0(A) \circ R$ for all $A \in GL(g-1, C)$.

(2) $R \circ \rho \begin{pmatrix} a & 0 \\ 0 & E_{g-1} \end{pmatrix} = a^r R$ for some $a \in Z$.

Then for any $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$ and $T \in \check{\mathcal{H}}_{g,p}$, we have

$$(R \circ \Psi_{g,g-1})(f|T) = R(\Psi_{g,g-1} f)|T^*,$$

where T^* is an element in $\check{\mathcal{H}}_{g-1,p}$ defined by (3.11).

Corollary. The Siegel-Jacobi operator is compatible with the action of $T \mapsto T^*$. Precisely, we have the following commutative diagram:

$$\begin{array}{ccc} J_{\rho,\mathcal{M}}(\Gamma_g) & \xrightarrow{\psi_{g,g-1}} & J_{\rho(g-1),\mathcal{N}}(\Gamma_{g-1}) \\ \big\downarrow T & & \big\downarrow T^* \\ J_{\rho,\mathcal{N}}(\Gamma_g) & \xrightarrow{\psi_{g,g-1}} & J_{\rho(g-1),\mathcal{N}}(\Gamma_{g-1}) \end{array} .$$

Here $\mathcal{N}$ is a certain symmetric half integral semipositive matrix of degree h.

Definition 3.2. Let $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$ be a Jacobi form. Then we have a Fourier expansion given by (B) in Definition 1.2. A Jacobi form f is called a *cusp form* if $c(T, R) \neq 0$ implies $\begin{pmatrix} T & \frac{1}{2}R \\ \frac{1}{2}\,{}^tR & \mathcal{M} \end{pmatrix} > 0$. We denote by $J^{cusp}_{\rho,\mathcal{M}}(\Gamma_g)$ the vector space of all cusp forms in $J_{\rho,\mathcal{M}}(\Gamma_g)$.

Theorem H. Let $1 \leq r \leq g$. Assume $k(\rho) > g + r + rank\,(\mathcal{M}) + 1$ and $k(\rho)$ even. Then

$$J^{cusp}_{\rho,\mathcal{M}}(\Gamma_r) \subset \Psi_{g,r}(J_{\rho,\mathcal{M}}(\Gamma_g)).$$

4 Final Remarks

In this section we give some open problems which should be investigated and give some remarks.

Let

$$G_R^{(g,h)} := Sp(g,R) \ltimes H_R^{(g,h)}$$

be the *Jacobi group* of degree g. Let $\Gamma_g^J := Sp(g,Z) \ltimes H_Z^{(g,h)}$ be the discrete subgroup of $G_R^{(g,h)}$. For the case $g = h = 1$, the spectral theory for $L^2(\Gamma_1^J \backslash G_R^{(1,1)})$ had been investigated almost completely in [B1] and [B-B]. For general g and h, the spectral theory for $L^2(\Gamma_g^J \backslash G_R^{(g,h)})$ is not known yet.

Problem 1. Decompose the Hilbert space $L^2(\Gamma_g^J \backslash G_R^{(g,h)})$ into irreducible components of the Jacobi group $G_R^{(g,h)}$ for general g and h. In particular, classify all the irreducible unitary or admissible representations of the Jacobi group $G_R^{(g,h)}$ and establish the *Duality Theorem* for the Jacobi group $G_R^{(g,h)}$.

Problem 2. Give the *dimension formulae* for the vector space $J_{\rho,\mathcal{M}}(\Gamma_g)$ of Jacobi forms.

Problem 3. Construct Jacobi forms. Concerning this problem, discuss the *vanishing theorem* on the vector space $J_{\rho,\mathcal{M}}(\Gamma_g)$ of Jacobi forms.

Problem 4. Develope the theory of L-functions for the Jacobi group $G_R^{(g,h)}$. There are several attempts to establish L-functions in the context of the Jacobi group by Japanese mathematicians A. Mürase and T. Sugano using so-called the Whittaker-Shintani functions.

Problem 5. Give applications of Jacobi forms, for example in algebraic geometry and physics. In fact, Jacobi forms have found some applications in proving non-vanishing theorems for L-functions of modular forms [BFH], in the theory of Heeger points [GKS], in the theory of elliptic genera [Za] and in the string theory [C].

By a certain lifting, we may regard Jacobi forms as smooth functions on the Jacobi group $G_R^{(g,h)}$ which are invariant under the action of the discrete subgroup Γ_g^J and satisfy the differential equations and a certain growth condition.

Problem 6. Develope the theory of *automorphic forms* on the Jacobi group $G_R^{(g,h)}$. We observe that the Jacobi group is *not reductive.*

Finally for historical remarks on Jacobi forms, we refer to [B2].

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