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On functional equations of local zeta functions of prehomogeneous vector spaces

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Introduction

In 1960's, M.Sato introduced the notion of prehomogeneous vector spaces and proved the functional equations of zeta functions associated with prehomogeneous vector spaces defined over $\mathbb{R}$ or $\mathbb{C}$

$$(|f(x)|_i^s)^\wedge = \sum_j \gamma_{ij}(s)|f(x)|_j^{-n/d-s} ,$$

where $f(x)$ is the relative invariant of the prehomogeneous vector space V, $\gamma_{ij}(s)$ are meromorphic functions on $\mathbb{C}$, n=dim V, d is the degree of $f(x)$, $\wedge$ means the Fourier transform, and

$$|f(x)|_j^{-n/d-s} := \begin{cases} |f(x)|^{-n/d-s} & x\in V_j \\ 0 & \text{otherwise} \end{cases}$$

where V_j are G_K-orbits in $V-S$ (for detail, see [SS]).

Similar functional equations associated with regular prehomogeneous vector spaces defined over p-adic fields have been proved by J.Igusa and F.Sato assuming some conditions of its singular set ([S],[I]).

But even when a prehomogeneous vector space (G,ρ,V) satisfies the sufficient conditions of F.Sato which assure the functional equations of zeta functions, the prehomogeneous vector space $(\tilde{G},\tilde{\rho},\tilde{V})$, which is obtained by the castling transform of (G,ρ,V), does not necessarily satisfy them(cf.[S],[SK]).

Thus even if the functional equations of zeta functions of (G,ρ,V) hold, we do not know whether the functional equations of zeta functions of $(\tilde{G},\tilde{\rho},\tilde{V})$ hold or not. Since the castling transform is a standard procedure of constructing new prehomogeneous vector spaces, it is natural to ask the existence of the functional equations of $(\tilde{G},\tilde{\rho},\tilde{V})$ when the functional equations of (G,ρ,V) hold.

In this paper we prove the following theorem using the results of [SO]:

Theorem *If the functional equations of zeta functions of (G,ρ,V) hold, then the functional equations of zeta functions of $(\tilde{G},\tilde{\rho},\tilde{V})$ hold and vice versa* (see §4).

§1 Preliminaries

First we recall some basic notions in the theory of prehomogeneous vector spaces over a $\mathfrak{p}$-adic field and give some definitions following [SO].

Let k be a $\mathfrak{p}$-adic field and denote by $\bar{k}$ (resp.$\mathcal{O}_k$)its algebraic closure (resp.maximal order). Let G be a connected linear algebraic group defined over k and V be a finite dimentional $\bar{k}$-vector space with k-structure V_k. Let $\rho: G \longrightarrow GL(V)$ be a k-rational representation of G on V. Then the triple (G,ρ,V) is called a *prehomogeneous vector space* if there exists a proper algebraic subset S of V such that $V-S$ is a single $\rho(G)$-orbit. The algebraic set S is called the *singular set*. It is known that $V-S$ is a single $\rho(G)$-orbit implies that the number of $\rho(G_k)$-orbits in V_k-S_k is finite. A nonzero rational function $P(v)$ is called a *relative invariant* of (G,ρ,V) if there exists a rational character $\chi(g)$ of G such that

$$P(\rho(g)v) = \chi(g)P(v) \qquad (g\in G,\ v\in V).$$

Let S_i $(1\le i\le l)$ be the k-irreducible hypersurfaces contained in S. For each i $(1\le i\le l)$, take a k-irreducible polynomial $P_i(v)$ defining S_i. Then it is known that $P_i(v)$ are relative invariants and any relative invariant $P(v)$ in $k[V]$ is written uniquely as

$$P(v)= c\prod_{i=1}^{l} P_i(v)^{\nu_i} \qquad (c\in k^\times,\ \ \nu_1,\dots,\nu_l\in \mathbf{Z}).$$

The polynomials $P_1,\dots,P_l$ are called the ***basic relative invariants*** of (G,ρ,V), and l the ***k-rank*** of (G,ρ,V).

A relative invariant $P(v)$ is called ***nondegenerate*** if the Hessian $\det\left(\frac{\partial^2 P}{\partial v_i \partial v_j}\right)$ does not vanish identically. A prehomogeneous vector space (G,ρ,V) ia called ***regular*** if there exists a nondegenerate relative invariant; and then one can find a nondegenerate relative invariant in $k[V]$.

Let V^* be the vector space dual to V and $\rho^* : G \longrightarrow GL(V^*)$ the rational representation of G contragredient to ρ. The vector space V^* has a k-structure canonically defined by the k-structure of V. Then the representation ρ^* is defined over k.

Let m and n be positive integers with $m>n\ge 1$. We consider a rational representation $\rho_0 : H \longrightarrow GL(m)$ of a connected linear algebraic group H. We assume that H and ρ_0 are defined over the field k. Put $G=H\times GL(n)$ and $V=M(m,n)$. Also put $\tilde{G}= H \times GL(m-n)$ and $\tilde{V} = M(m,m-n)$. Let $\rho : G \longrightarrow GL(V)$ (resp. $\tilde{\rho} : \tilde{G} \longrightarrow GL(\tilde{V})$) be a rational representation of G on V (resp. $\tilde{G}$ on $\tilde{V}$) defined by

$$\rho(h,g_n)v = \rho_0(h)vg_n^{-1} \qquad ((h,g_n)\in G = H\times GL(n))$$

$$\text{(resp. } \tilde{\rho}(h,g_{m-n})w = {}^t\rho_0(h)^{-1}w\,{}^tg_{m-n} \quad ((h,g_{m-n})\in\tilde{G} = H\times GL(m-n))).$$

The triple $(\tilde{G},\tilde{\rho},\tilde{V})$ is called the *castling transform* of (G,ρ,V) and vice versa. Then we have the following lemma:

Lemma 1 (Sato-Kimura [SK]) *The triplet (G,ρ,V) is a prehomogeneous vector space if and only if so is its castling transform $(\tilde{G},\tilde{\rho},\tilde{V})$.*

In the following, we assume that (G,ρ,V) and $(\tilde{G},\tilde{\rho},\tilde{V})$ are prehomogeneous vector spaces with k-structure

$$G_k = H_k \times GL(n;k), \qquad V_k = M(m,n;k),$$
$$\tilde{G}_k = H_k \times GL(m-n;k), \qquad \tilde{V}_k = M(m,m-n;k).$$

Put $N = \binom{m}{n}$ and let $\Delta_1(v),\ldots,\Delta_N(v)$ (resp. $\tilde{\Delta}_1(w),\ldots,\tilde{\Delta}_N(w)$) be the minor determinants of $v\in V$ (resp. $w\in\tilde{V}$) of size n (resp. $m-n$).

Let V_0 be the vector space of column vectors of m entries and V_0^* the vector space dual to V_0. We identify V (resp. $\tilde{V}$) with the direct product of n (resp. $m-n$) copies of V_0 (resp. V_0^*) in the standard manner. Let $\Delta : V \longrightarrow \overset{n}{\wedge} V_0$ and $\tilde{\Delta} : \tilde{V} \longrightarrow \overset{m-n}{\wedge} V_0^*$ be the mapping defined by

$$\Delta(v) = \Delta(v_1,\ldots,v_n) = v_1\wedge\cdots\wedge v_n \quad (v_1,\ldots,v_n \in V_0)$$

and

$$\Delta(w) = \Delta(w_1,\ldots,w_{m-n}) = w_1\wedge\cdots\wedge w_{m-n} \quad (w_1,\ldots,w_{m-n} \in V_0^*),$$

respectively. We identify $\overset{n}{\wedge} V_0$ with $\overset{m-n}{\wedge} V_0^*$ via the canonical isomorphism:

$$\overset{n}{\wedge} V_0 \xrightarrow{\simeq} (\overset{m-n}{\wedge} V_0)^* \xrightarrow{\simeq} \overset{m-n}{\wedge} V_0^* .$$

By taking the standard basis, we may identify $\overset{n}{\wedge} V_0$ and $\overset{m-n}{\wedge} V_0^*$ with $\bar{k}^N$, so that the mappings Δ and $\tilde{\Delta}$ are given by

$$\Delta(v) = (\Delta_1(v), \ldots, \Delta_N(v))$$
$$\widetilde{\Delta}(w) = (\widetilde{\Delta}_1(w), \ldots, \widetilde{\Delta}_N(w)).$$

Here the minor determinants are indexed such that

$$\det(v,w) = \sum_{i=1}^{N} \Delta_i(v)\widetilde{\Delta}_i(w).$$

Then it is easy to see that

$$\Delta(\rho(h,g_n)v) = \det g_n^{-1}\cdot(\overset{n}{\wedge}\rho_0(h))(\Delta(v))$$

and

$$\widetilde{\Delta}(\widetilde{\rho}(h,g_{m-n})w) = (\det g_{m-n}/\det \rho_0(h))\cdot(\overset{n}{\wedge}\rho_0(h))(\widetilde{\Delta}(w)).$$

Thus we consider that G(resp.$\widetilde{G}$) operates on $\Delta(V)$(resp.$\Delta(\widetilde{V})$).

Now put

$$V'=\{v\in V\,|\,\text{rank } v=n\} \quad (\text{resp. } \widetilde{V}'=\{w\in\widetilde{V}\,|\,\text{rank } w=m-n\})$$
$$V'_k=V'\cap V_k \quad (\text{resp. } \widetilde{V}'_k=\widetilde{V}'\cap\widetilde{V}_k).$$

Then we have

$$Y=\Delta(V')=\widetilde{\Delta}(\widetilde{V}')\subset\overline{k}^N$$
$$Y_k=\Delta(V'_k)=\widetilde{\Delta}(\widetilde{V}'_k)\subset k^N.$$

Moreover we have the following lemma:

Lemma 2 (cf.[SO] Lemma 1.2)

(1) *The k-rank of (G,ρ,V) is equal to the k-rank of $(\widetilde{G},\widetilde{\rho},\widetilde{V})$.*

(2) *There exist irreducible homogeneous polynomials $Q_1,\ldots,Q_l \in k[y_1,\ldots,y_n]$ (l=the k-rank of (G,ρ,V)) such that*

$$P_1(v)=Q_1(\Delta(v)) \;,\ldots,\; P_l(v)=Q_l(\Delta(v))$$

are the basic relative invariants of (G,ρ,V) over k and

$$\widetilde{P}_1(w)=Q_1(\widetilde{\Delta}(w)) \;,\ldots,\; \widetilde{P}_l(w)=Q_l(\widetilde{\Delta}(w))$$

are the basic relative invariants of $(\widetilde{G},\widetilde{\rho},\widetilde{V})$ over k.

(3) *Put d_i=deg Q_i. Then there exist k-rational characters $\psi_1,\ldots,\psi_l$ of H such that*

$$P_i(\rho(h,g_n)v)=\chi_i(h,g_n)P_i(v),\quad \chi_i(h,g_n)=(\det g_n)^{-d_i}\cdot\psi_i(h)$$
$$\tilde{P}_i(\tilde{\rho}(h,g_{m-n})w)=\tilde{\chi}_i(h,g_{m-n})\tilde{P}_i(w),$$
$$\tilde{\chi}_i(h,g_{m-n})=(\det g_{m-n}/\det \rho_0(h))^{d_i}\cdot\psi_i(h).$$

Let X be a subset of V_k (resp. $\tilde{V}_k$) to which G_k (resp. $\tilde{G}_k$) operates. Denote by $\mathcal{S}(X)$ the Schwartz-Bruhat space of the topological space X and by $\mathcal{S}'(X)$ its dual space as a $\mathbb{C}$-vector space. We call an element of $\mathcal{S}'(X)$ a *distribution* on X. And for a subset X' of X, we denote the characteristic function of X' by $Ch(X')$. For $f\in\mathcal{S}(X)$, $T\in\mathcal{S}'(X)$, $g\in G_k$, we define $f^g(x)\in\mathcal{S}(X)$ and $gT\in\mathcal{S}'(X)$ as follows:

$$f^g(x)=f(\rho(g)x),$$
$$gT(f(x))=T(f^g(x))=T(f(\rho(g)x)).$$

In this paper, we always mean by a character ω of G_k a character of G_k of the form

$$\omega=\phi(\chi),$$

where $\chi:G\longrightarrow k^\times$ is a k-rational character of G, and $\phi:k^\times\longrightarrow\mathbb{C}^\times$ is a continuous homomorphism.

Now for a group G_k and a character $\omega:G_k\longrightarrow\mathbb{C}^\times$, define $\xi(X,G_k,\omega)$ as follows:

$$\xi(X,G_k,\omega) = \{T\in\mathcal{S}'(X)\,|\,gT=\omega(g)^{-1}T \text{ for all } g\in G_k\}.$$

And for a subgroup H of G_k we define the ***modular function*** $\delta(H)$ by $d(h_0^{-1}hh_0)=\delta(H)(h_0)\cdot dh$, where dh is a left invariant measure on H.

Now we quote the following lemma which will be used in the next chapter:

Lemma 3([I] p.1015) *Let G be a linear algebraic group defined over* *k, G_k be the set of its k-rational points and H_k be a closed subgroup of G_k. Let $x=G_k/H_k$, and ω as above; then $\xi(x,G_k,\omega)\neq 0$ if and only if*

$$\omega\cdot\delta(G_k)|_{H_k} = \delta(H_k),$$

and in that case $\dim_{\mathbb{C}}\xi(x,G_k,\omega)=1$

Put

$$S(1)=\{v\in V_k|\ \text{rank}\ \ v<n\}.$$

Then we have

$$\rho(G_k)S(1)=S(1).$$

Thanks to results of [SO], we know that our theorem is valid if we restrict ourselves to the case where $x\in V_k\setminus S(1)$. Therefore we study the distributions $\xi(S(1),G_k,\omega)$ in §2 and prove the main theorem in §3.

§2 We keep the notations in §1. The aim of this section is to prove the following proposition:

Proposition 1

If $\xi(S(1),G_k,\omega)\neq 0$, there exist a finite number of subgroups H_i $1\leq i\leq q$ of $GL(n;k)$ such that

$$\omega|_{H_i} = \delta(H_i) \qquad\qquad \textit{for some } i,$$

and $\delta(H_i)$ is not identically equal to 1 if $H_i\neq GL(n;k)$.

Let V_k be as before and G' the set of k-rational points of a connected algebraic group which acts on V_k by a k-rational representation ρ.

Let A be a G'-stable subset of V_k such that for any $x,y\in A$, the isotropy subgroups G'_x and G'_y are conjugate in G'. Then there exists a complete system V_A of representatives of G'-orbits in A such that $G'_x=G'_y$ for any $x,y\in V_A$. We put $G'_A=G'_x$ for $x\in V_A$.
Then we have a bijection

$$\Phi:\ G'/G'_A \times V_A \longrightarrow A$$

defined by $\Phi(g\cdot G'_A\ ,x)=\rho(g)x$.
Now we assume that

$$\Phi \ \text{ is a homeomorphism.}$$

Lemma 4 *Under the conditions above, we have*

$$\xi(A,G',\omega)\neq 0 \quad \textit{if and only if} \quad \omega\cdot\delta(G')|_{G'_{iso}}=\delta(G'_{iso}).$$

Proof. It is easy to see that if there exists a G'-orbit $\mathcal{O}$ in A such that $\xi(\mathcal{O},G',\omega)\neq 0$, then we have $\xi(A,G',\omega)\neq 0$. Therefore "if" part is trivial from Lemma 3. Now we prove "only if" part.
Notice that any compact open subset of totally disconnected topological space $G'/G'_A \times V_A$ can be written as a disjoint union of compact open subsets of the form $U_o \times U_v$, where U_o (resp.U_v) is a compact open subset of G'/G'_A (resp. V_A).
Thus, if $\xi(A,G',\omega)\neq 0$, then there exists

$$T\in\xi(A,G',\omega) \quad \text{such that} \quad T(Ch(\Phi(U)))\neq 0$$

where $U=U_o \times U_v$. We fix these U_o and U_v. Now for each compact open subset $U(o)$ of G'/G'_A, we define a mapping τ by

$$\tau: U(o) \longrightarrow \Phi(U(o) \times U_v).$$

Then we have

$$\tau(U_o)=\Phi(U).$$

It is easy to see that τ is G'-*admissible* i.e.,

$$\tau(g\cdot U(o))=\Phi(g\cdot U(o) \times U_v)=g\cdot\Phi(U(o) \times U_v)=g\cdot\tau(U(o)). \qquad (1)$$

Now τ induces a linear mapping

$$\bar{\tau} : \mathcal{S}(G'/G'_A) \longrightarrow \mathcal{S}(A).$$

Using (1), we have

$$g\cdot(\bar{\tau}(f(x)))=\bar{\tau}(f(\rho(g)x))$$

for all $f\in\mathcal{S}(x)$. Now we define $F\in\mathcal{S}'(G'/G'_A)$ by

$$F(f)=T(\bar{\tau}(f)).$$

Then we have

$$g\cdot F(f)=T(\bar{\tau}(f^g))=T((\bar{\tau}(f))^g)$$

$$=g\cdot T(\bar{\tau}(f))=\omega^{-1}(g)T(\bar{\tau}(f))=\omega(g)^{-1}F(f),$$

where $g\in G'$.

Therefore we have $F\in\xi(G'/G'_A,G',\omega)$. Moreover we have

$$F(f)=T(Ch(U(o)))\neq 0$$

for $f=Ch(U(o))$, which implies

$$\xi(G'/G'_A,G',\omega)\neq 0.$$

Thus, from Lemma 3, we have

$$\omega\cdot\delta(G')|_{G'_A} = \delta(G'_A). \qquad \square$$

Now put

$$S_r=\{v\in S(1)\mid \text{rank } v = r\} \qquad \text{for } 0\leq r\leq n-1.$$

Then $\rho(1,GL(n;k))S_r=S_r$ for all $0\leq r\leq n-1$ and

$$S(1) = S_{n-1} \cup \cdots \cup S_0 \quad \text{(disjoint union)}.$$

Now the following lemma holds.

Lemma 5

Each S_r *can be decomposed as follows:*

$$S_r=\rho(1,GL)S_{r1} \cup \cdots \cup \rho(1,GL)S_{r\ell} \quad \textit{(disjoint union)},$$

where $\ell=\binom{m}{r}$, $GL=GL(n;k)$, *and by putting* $G'=GL$, $A=\rho(1,GL)S_{rh}$ *and* $V_A=S_{rh}$ *, these* G', A, and V_A *satisfy the conditions* (*a*),(*b*), *and* (*c*) *above for all* $1\leq h\leq \ell$.

Proof. For a matrix $x=(x_{ij})$ and $j=1,\ldots,n$, we denote by $I(j,x)$ the smallest i for which $x_{ij}\neq 0$. We denote by $Rep(r)$ the set of matrices of the form

$$x=\begin{pmatrix} w \mid 0\end{pmatrix}\in M(m,n), \quad w=(w_{ij})\in M(m,r)$$

with the condition that

$w_{I(j,x)h}=\delta_{jh}$ (Kronecker's symbol) for all $1\leq j\leq r$, $1\leq h\leq r$ and $I(j,x)\leq I(j+1,x)-1$ for all $1\leq j\leq r-1$.

It is trivial that $Rep(r)$ is a complete system of representatives of $\rho(1,GL)$-orbits in S_r. Also define $\alpha(x)\in \mathbf{Z}^r$ by

$$\alpha(x)=(I(1,x),\ldots,I(r,x)).$$

Then, for $x\in Rep(r)$, we have

$$1\leq I(1,x)<\cdots<I(r,x)\leq m.$$

Now put

$$I=\{(i_1,\ldots,i_r)\mid 1\leq i_1<\cdots<i_r\leq m\}.$$

It is clear that $|I|=\binom{m}{r}$, and we number the elements of the set I in an arbitrary order and write I as follows:

$$I=\{I_i \ (1\leq i\leq \binom{m}{r})\}.$$

For any $I_i=(I(1),\dots,I(r))\in I$ we define S_{ri} by

$$S_{ri}=\alpha^{-1}(I_i).$$

Now S_{ri} is homeomorphic to the k-vector space of dimension

$$(m-I(r))\times r+\sum_{j=1}^{r-1}(I(j+1)-I(j)-1)\times j,$$

and it is easy to see that if we put $G'=GL$, $A=\rho(1,GL)S_{rh}$, $V_A=S_{rh}$, these G', A, V_A satisfy the conditions (a),(b), and(c). Moreover the condition

$$S_r=\rho(1,GL)S_{r1}\cup\cdots\cup\rho(1,GL)S_{r\ell}\quad\text{(disjoint union)}$$

is now trivial. □

Now we can prove Proposition 1.

Proof of Proposition 1. We keep the notation in Lemma 5.
Since $\xi(S(1),G_k,\omega)\neq 0$ implies $\xi(S(1),\mathrm{GL}(n;k),\omega)\neq 0$, we have only to prove that the conditions in our proposition are necessary if $\xi(S(1),\mathrm{GL}(n;k),\omega)\neq 0$.

The isotropy subgroup of $GL(n;k)$ for $x\in\mathrm{Rep}(r)$, which we denote by H_x, is given by

$H_x=\{\left(\begin{array}{c|c}\mathrm{I}_r & 0\\ \hline * & *\end{array}\right)\in GL(n;k)\}$, where I_r is the identity matrix of size r.

Therefore $\delta(H_x)$ is not identically equal to 1 if $H_x\neq GL(n;k)$ ($H_x=GL(n;k)$ if and only if $x=0$). Thus using Lemma 4 and Lemma 5, $S(1)$ is decomposed as follows:

$$S(1)=\cup S_{rh}\quad\text{(disjoint union)},$$

where the union runs through $0\leq r\leq n-1$, $1\leq h\leq\binom{m}{r}$, and $\rho(1,GL(n;k))S_{rh}=S_{rh}$ for all $1\leq r\leq n-1$, $1\leq h\leq\binom{m}{r}$, and if $\xi(S_{rh},\mathrm{GL}(n;k),\omega)\neq 0$, then there exist $H_{rh}(=H_x$ for $x\in Rep(r))\subset GL(n;k)$

such that

$$\delta(GL(n;k))\cdot\omega|_{H_{rh}}=\delta(H_{rh})$$

and $\delta(H_{rh})$ is not identically equal to 1 if $H_{rh}\neq GL(n;k)$. Now since GL$(n;k)$ is unimodular, we have

$$\omega|_{H_{rh}}=\delta(H_{rh}).$$

Thus we have proved the proposition. □

§3 Main result

For $s=(s_1,\ldots,s_l)\in\mathbb{C}^l$, we define the character ω_s of G_k by

$$\omega_s=(\phi_1\omega_{s_1}(\chi_1),\ldots,\phi_l\omega_{s_l}(\chi_l)),$$

where $\omega_{s_i}(\alpha)=|\alpha|_k^{s_i}$ $(\alpha\in k^\times,1\leq i\leq l)$, ϕ_i $(1\leq i\leq l)$ are dual of $\mathcal{O}_k^\times$ and l is the *k-rank* of (G,ρ,V). Also define $\tilde{\omega}_s$ for $s=(s_1,\ldots,s_l)$ by

$$\tilde{\omega}_s=(\phi_1\omega_{s_1}(\tilde{\chi}_1),\ldots,\phi_l\omega_{s_l}(\tilde{\chi}_l)),$$

and put $\Omega=\{s\in\mathbb{C}^l|\ \xi(S,G_k,\omega_s)\neq 0\}$.

Now we recall some facts in the theory of prehomogeneous vector spaces roughly.

The foundamental theorem in the theory of regular prehomogeneous vector spaces states that there exist u$\in$GL$(n;\mathbb{Z})$, $\lambda\in\mathbb{C}^l$, and meromorphic functions $\gamma_{ij}(s)$ such that

$$F(x,s)_i=(|P(x)|_{k,i}^{s})^\wedge-\sum_j^{\nu}\gamma_{ij}(s)\cdot|P^*(x)|_{k,j}^{\lambda+\mathrm{u}s}$$

vanishes for all $x\in V_k$, $s\in\mathbb{C}^l$, and $1\leq i,j\leq\nu$, i.e.,

$$(|P(x)|_{k,i}^{s})^\wedge=\sum_j^{\nu}\gamma_{ij}(s)\cdot|P^*(x)|_{k,j}^{\lambda+\mathrm{u}s}, \tag{2}$$

where $^\wedge$ means the Fourier transform, ν is the number of

$\rho(G_k)$-orbits in V_k-S_k, and $|P(x)|^{s}_{k,i}=\prod_{h=1}^{l}|P_h(x)|^{s_h}_{k,i}$ (functional equations of zeta functions, see [S]).

For $x\in V_k-S$, the theorem above is proved by the uniqueness of relatively invariant distributions of homogeneous space (cf. Lemma 3) Moreover, thanks to results of [SO], we know that if the functional equations of zeta functions of (G,ρ,V) hold for $x\in V_k-S(1)$, then the functional equations of zeta functions of $(\tilde{G},\tilde{\rho},\tilde{V})$ hold for $x\in\tilde{V}_k-\tilde{S}(1)$. On the other hand, it is known that $F(x,s)_i\neq 0$ for some $x\in S(1)$ implies that there exists non-zero $T\in\xi(S(1),G_k,\omega_s)$ such that T is meromorphic with respect to $s\in\mathbb{C}^l$. Therefore we can prove $F(x,s)_i=0$ for all $x\in S(1)$, $s\in\mathbb{C}^l$ by showing that $\mathbb{C}^l\setminus\Omega$ is dence in $\mathbb{C}^l$. Hence, for the proof of the functional equations, it is enough to show that $\mathbb{C}^l\setminus\Omega$ is dense in $\mathbb{C}^l$.

Now we prove the main theorem.

Theorem *If the functional equations of zeta functions (2) of (G,ρ,V) hold, then the functional equations of zeta functions (2) of $(\tilde{G},\tilde{\rho},\tilde{V})$ hold and vice versa.*

Proof. Put $\Omega(1)=\{s\in\mathbb{C}^l\,|\,\xi(S(1),\mathrm{GL}(n;k),\omega_s)\neq 0\}$, $\tilde{\Omega}(1)=\{s\in\mathbb{C}^l\,|\,\xi(\tilde{S}(1),\mathrm{GL}(m-n;k),\tilde{\omega}_s)\neq 0\}$.
For a subset Ω' of $\mathbb{C}^l$, we denote $\mathbb{C}^l\setminus\Omega$ by $C(\Omega)$.
Now from Proposition 1 of §2 and the fact that ω_s is not trivial on $\mathrm{GL}(n;k)$, we know that $C(\Omega(1))$ and $C(\tilde{\Omega}(1))$ are dense subsets of $\mathbb{C}^l$.
Now using the remarks above, we have proved the theorem. □

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（注） 以上は 投稿予定の論文を、若干省略したものです。 応用等は省きましたが 例えば既約概均質ベクトル空間については、その少なくとも一つの k-form においては関数等式が成たつことがわかります。

また、以前 preprint を お渡ししたかたには、(G,ρ,V) の関数等式から その castling transform について示すのでなく、(G,ρ,V) において [S] の sufficient condition が成立するという仮定から その castling transform について 関数等式の成立を示していた事を 記しておきます（既約 p.v.については これで十分ですが）。