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A Study on Nash-Implementability

B.Nakano
Department of System Science, Tokyo Institute of Technology

I. Introduction

Organizations are consisted of interacted multiple agents who have their own goals, tastes, values, properties and/or technologies. Profile of these members' properties is usually called a characteristic or environment in Economic Theory. Organization members interact with each other cooperatively or non-cooperatively. As a result of interaction, some social state is attained as an equilibrium point. Organization may or may not have its objectives. If it has a clear goal in advance, organization members will try to attain its goal as an equilibrium point cooperatively or non-cooperatively. However, even if it has no goals, we may suppose there's a mild agreement which social states are much better than others. Pareto-optimality is such an example of social agreement or criterion.So we can always assume that there are some criteria or system requirements to achieve for any organization. The set of deirable social states,we can assume its existence conceptualy, are given in advance for any profile. If we adopt Pareto-optimality as a system requirement, the set of desirable social states becomes the set of Pareto-optimal social states for a given profile. Assuming the existence of the set is different from getting it concretely by some methods. So to find a solution we need to construct a device,i.e. a decentralized mechanism. This is Implementation problem.

This paper presents several results for Nash-Implementability. Implementation Theory was started at the end of 1970's in the areas of Mathematical Economics and Social Choice Theory. Traditionally, competitive mechanism has been well studied in General Equilibrium Theory and Welfare Economics. It has been showed that the mechanism has many nice properties such as pareto-efficiency, informational efficiency and unbiasedness etc.

However,in those theories the agents of the mechanism have been assumed to be price-takers and to behave honestly. The agents in the real economic markets usually behave strategically as we can suppose easily.Recent main interests in Mathematical Economics and Social Choice Theory are

Incentive-Compatibility and Implementability.
Incentive-Compatibility means that the agent who makes an unhonest decision in the incentive compatible mechanism looses his payoffs. That is, only the agents who behave honestly can get success in the incentive compatible mechanism.
On the other hand, the aim of Implementation Theory is to design a mechanism which realizes the given desirable requirements such as pareto-optimality, individual rationality and others.
To design a Nash-Implementable mechanism which satsifies some properties means the followings:

【Problem】 Can we construct a game form just like as pure competitive market mechanism such that the Nash-equilibrium solution of the game should satisfy the given properties when the game is played non-cooperatively? .

It is well known that the competitive mechanism is neither Nash-Implementable nor Incentive-Compatible in spite of having other nice properties[6]. So the researchers started to search for another mechanisms which are Nash-Implementable or Incentive-Compatible.Many able researchers have participated in studying this areas and produced many results.
In this paper,we discuss mainly around Nash-implementability condition. At first Nash-implementation theory started in private and public economic theory. Later the concepts and frameworks were extended to include Voting Theory. So in this paper we discuss the problems with a mind to involve both cases of Voting Theory and Economic Problem.
In Section Ⅱ, we give preliminary notations and definitions. Several results are given in Section Ⅲ.The conclusions are given in Section Ⅳ.

Ⅱ. Symbols, Notations and Definitions

As we stated at the end of the above introduction, we adopt the framework which can represent Social Chice Theory and Economic Theory.

Let A be an arbitrary set of **alternatives** or **economic states(allocations)**. The cardinality of A may be finite or infinite. In Section Ⅲ, we suppose A be a finite set with cardinality $|A|=m$ basically.And we will suppose A be a subset of finite dimensional Euclid Space in Sec. 3.2.

N is the set of **agents** who are the members of an organization. Every agent $i \in N$ has a **preference ordering** R_i on A. The set of possible pre ference order R_i for member i is denoted by $\mathbb{R}_i$. Set $\mathbb{R}=\mathbb{R}_1\times\cdots\times\mathbb{R}_n$. We call $R^-=(R_1,\cdots,R_n)\in\mathbb{R}$ **profile** of preference or **environment**.

In economic framework, agent's **characteristic** include other than prefe rence relation, i.e. initial endowment as well as utility function. In those case, we may use notations Θ_i in stead of $\mathbb{R}_i$ in order to show generality.Of cource, main component of $\theta_i\in\Theta_i$ is preference relation. Similarly, we set $\Theta=\Theta_1\times\cdots\times\Theta_n$. We can use $\mathbb{R}$ and Θ interchangeably depending on the context.

We call any map $f:\mathbb{R}\rightarrow\mathcal{P}(A)\backslash\{\phi\}$ **Social Choice Correspondence(SCC)** or **System Requirement**, where $\mathcal{P}(A)$ is the set of all subsets of A. To show explicitly f is correspondence,we denote this $f:\mathbb{R}\Rightarrow A$ instead of $f:\mathbb{R}\rightarrow\mathcal{P}(A)\backslash\{\phi\}$. Especially, if $f:\mathbb{R}\Rightarrow A$ is a mapping then we denote it as $f:\mathbb{R}\rightarrow A$, and we call it **Social Choice Function(SCF)**. For a given profile $R^-=(R_1,\cdots,R_n)$, $f(R^-)$ is the set of desirable alternatives in some sense under the profile R^-. We can also say $f(R^-)$ the solution set for R^- instead. How f should be given is the problem of selection of social value criterion, so we do not pursuit this problem anymore.

Before giving important definitions, we introduce several preliminary definitions.

Definition 1:Reflexivity, Antisymmetricity, Transitivity, Comparability

Preference ordering $R\subseteq A\times A$ is **reflexive** $\Leftrightarrow$ $(\forall x\in A)[(x,x)\in R]$.

Preference ordering $R\subseteq A\times A$ is **antisymmetric**

$\Leftrightarrow$ $(\forall x,y\in A)[(x,y)\in R\ \&\ (y,x)\in R\ \Rightarrow\ x=y]$.

Preference ordering $R\subseteq A\times A$ is **transitive**

$\Leftrightarrow$ $(\forall x,y,z\in A)[(x,y)\in R\ \&\ (y,z)\in R\ \Rightarrow\ (x,z)\in R]$.

Preference ordering $R\subseteq A\times A$ is **comparable**

$\Leftrightarrow$ $(\forall x,y\in A)[(x,y)\in R$ or $(y,x)\in R]$.

Definition 2:Weak Order and Strong Order

Preference ordering $R\subseteq A\times A$ is **weak order**

$\Leftrightarrow$ R is reflexive, transitive and comparative.

Preference ordering $R\subseteq A\times A$ is **strong order**

$\Leftrightarrow$ R is antisymmetric, transitive and comparative.

Notation1:

For a given profile $R^-=(R_1,\cdots,R_n)\in\mathbb{R}$ and an alternative $a\in A$, the set $L(a,R_i)$ is defined as $\{a'\in A\mid aR_ia'\}$. When we consider the case of economics, we may also use the notation $L(a,\theta_i)$ similarly.

$L(a, R_i)$ is the set of alternatives which are not prefered to $a \in A$ w.r.t. preference R_i.

By using these symbols and notations, important concepts for SCC, mono tonicity, Non-veto power and Unanimity are defined as follows.

Definition 3:Monotonicity

SCC $f: \mathbb{R} \Rightarrow A$ is **monotonic** iff $(\forall R^-, R^{-\prime} \in \mathbb{R})(\forall a \in A)[a \in f(R^-)$ & $(\forall i \in N)(L(a, R_i) \subseteq L(a, R_i')) \Rightarrow a \in f(R^{-\prime})]$.

Where $R^{-\prime} = (R_1', \cdots, R_n')$.

Simply the above definition says that the alternative $a \in f(R^-)$ which is ranked up by someone by changing order from R_i to R_i',i.e.$L(a, R_i) \subseteq L(a, R_i')$ for all $i \in N$,is also included in the solution set $f(R^{-\prime})$.

Definition 4:Dictatorship

SCC $f: \mathbb{R} \Rightarrow A$ is **dictatorial**

$\Leftrightarrow (\exists j \in N)(\forall R^- \in \mathbb{R})(\forall a \in A)[a \in f(R^-) \Rightarrow L(a, R_i) = A]$.

Definition 5:Veto Power, No-Veto Power(NVP)

Agent $i \in N$ has **veto power** with respect to SCC $f: \mathbb{R} \Rightarrow A$

$\Leftrightarrow (\exists a \in A)(\exists R^- \in \mathbb{R})[(\forall j \neq i)(L(a, R_i) = A$ & $a \notin f(R^-)]$.

Agent $i \in N$ has **no-veto power** with respect to SCC $f: \mathbb{R} \Rightarrow A$

$\Leftrightarrow (\forall a \in A)(\forall R^- \in \mathbb{R})[(\forall j \neq i)(L(a, R_j) = A \Rightarrow a \in f(R^-)]$.

SCC $f: \mathbb{R} \Rightarrow A$ has **no-veto power** iff $(\forall i \in N)(\forall a \in A)(\forall R^- \in \mathbb{R})$ $[(\forall j \neq i)(L(a, R_i) = A \Rightarrow a \in f(R^-)]$.

Veto-power means that no person can reject the alternative which all the members except for him rank on the top. So no-veto power means that the alternative which is top-ranked by all the members exept for one per son cannot be rejected by him.

Definition 6:Unanimity

SCC $f: \mathbb{R} \Rightarrow A$ is **unanimous**

$\Leftrightarrow (\forall R^- \in \mathbb{R})(\forall a \in A)[(\forall i \in N)(L(a, R_i) = A) \Rightarrow a \in f(R^-)]$.

Unanimity means that the alternative which all the members rank on the top should be selected.

Definition 7:Largeness of Θ_i

Θ_i is large $\Leftrightarrow$ $(\forall a \in A)(\forall \theta_i \in \Theta_i)(\exists \theta_i' \in \Theta_i)$
$[\theta_i' \neq \theta_i \ \& \ L(a, \theta_i) \subseteq L(a, \theta_i')]$.

Definition 8:Game Form(Mechanism), Direct Mechanism, Outcome function

Let S_i be strategy set for member $i \in N$.We call a map $h: S_1 \times \cdots \times S_n \to A$ **outcome function**. The triple $M=(S, h, A)$ is called **game form** or **mechanism**, where $S = S_1 \times \cdots \times S_n$ and A is a set of alternatives. We may call h itself mechanism instead of M. When S coincides with $\mathbb{R}$ or Θ, $M=(\mathbb{R}, h, A)$ or $M=(\Theta, h, A)$ is called a **direct mechanism**.

Usually, SCC f and mechanism h are different from each other. f can be interpreted as a social requirement or criterion which should satisfy some given desired properties. For example, $f(R^-) \subseteq A$ is a set of Pare to-optimal alternatives w.r.t. a given profile R^-.

On the other hand, h is a function which maps joint strategy $s=(s_1, \cdots, s_n)$ into some outcome $h(s) \in A$. In other words, h is an abstruction of **message exchange processes** between members.Each member i will choose his strategy $s_i \in S_i$ which attains most desirable outcome $a=h(s) \in A$ for his own preference R_i,i.e. which satisfies $h(s) R_i h(s')$ for all $s' \in S$ if it exists. However the outcome which occurs actually depends on other members' strategies, so he cannot control the outcome completely.This is a non-cooperative game situation. There are many solution concepts for non-cooperative game.

We introduce several notations as follows before giving solution conce pts.

Notation 2:

$S_{-i} = S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_n$.

$s_{-i}=(s_1, \cdots, s_{i-1}, s_{i+1}, \cdots, s_n) \in S_{-i}$.

We often write (s_i, s_{-i}) instead of $s=(s_1, \cdots, s_n)$ for simplicity.
For a mechanism $h: S \to A$, $h(S_i, s_{-i})$ is defined as the set $\{a \in A \mid h(s_i, s_i)=a$ for some $s_i \in S_i\}$.

Definition 9:Dominant Strategy Solution, Nash-Equilibrium Solution

Let a profile $R^- \in \mathbb{R}$ and a mechanism $M=(S, h, A)$ be given. Then
$s_i \in S_i$ is **dominant strategy** for i with respect to M and R^-
$\Leftrightarrow$ $(\forall s_{-i}' \in S_{-i})(\forall s_i' \in S_i)[h(s_i, s_{-i}') R_i h(s_i', s_{-i}')]$.
$s=(s_1, \cdots, s_n) \in S$ is **dominant strategy** for M and R^-

$\Leftrightarrow$ for any $i \in N$ s_i is dominant strategy w.r.t. M and R^-.

$s=(s_1,\cdots,s_n) \in S$ is **Nash-Equilibrium strategy** for M and R^-

$\Leftrightarrow$ $(\forall i \in N)(\forall s_i' \in S_i)[h(s) R_i h(s_i', s_{-i})]$,

i.e. $(\forall i \in N)[h(S_i, s_{-i}) \subseteq L(h(s), R_i)]$.

Where $h(S_i, s_{-i}) = \{a \in A \mid a = h(s_i, s_{-i}), s_i \in S_i\}$.

Notation 3:D(R⁻,M),N(R⁻,M)

We denote the set of dominant strategies for $i \in N$ w.r.t. M and R^- as $D_i(R^-;M)$. Similarly we denote the set of dominant strategies and the set of Nash-Equilibrium strategies w.r.t. M and R^- as $D(R^-;M)$ and $N(R^-;M)$ respectively.

Definition 10:Strategy Choice Function

Let the set $\mathbb{R}$ of profiles of possible preference orderings on A and a mechanism $M=(S,h,A)$ be given. Any correspondence $\mu:\mathbb{R} \Rightarrow S$ is called **strategy choice function**, where $\mu=(\mu_1,\cdots,\mu_n)$ and $\mu_i:\mathbb{R} \Rightarrow S_i$. Especially, if μ_i is determined depending only on R_i, we call such μ_i **selfish** and denote it as $\mu_i;R_i \Rightarrow S_i$ in that case.

If we adopt dominant strategy or Nash-Equilibrium strategy as a solution concept of a mechanism M, $\mu(R^-)$ is $D(R^-;M)$ or $N(R^-;M)$ for any $R^- \in \mathbb{R}$, respectly.

Definition 11:Implimentability, Weak-Implementability Truthful-Implementability

Let a SCC $f:\mathbb{R} \Rightarrow A$ and a mechanism $M=(S,h,A)$ be given. Then

M is **weakly μ-implementable** for f

$\Leftrightarrow$ $(\forall R^- \in \mathbb{R})[h(\mu(R^-)) \subseteq f(R^-)]$.

M is **μ-implementable** for f

$\Leftrightarrow$ $(\forall R^- \in \mathbb{R})[h(\mu(R^-)) = f(R^-)]$.

If M is direct mechanism, i.e., $S = \mathbb{R}$, then

M is **truthfully μ-implementable** for f

$\Leftrightarrow$ $(\forall R^- \in \mathbb{R})[R^- \in \mu(R^-) \,\&\, h(R^-) \in f(R^-)]$.

Specifically, if we adopt dominant strategy or Nash-Equilibrium strategy as a strategy choice function μ for the mechanism M, we say it dominant implementable or Nash-implementable for f, respectively.

As for dominant-implementability, many results are given already(see [1,7]). In the follpowing sections, we discuss only Nash-implemenrability.

Here let's give an example of a mechanism in order give an understanding for the importance of Nash-implementability.

Illustrative Example of mechanism:

Consider the **pure exchange economy** $\theta=\{(X_i, u_i, w_i)_{i\in N}\}$, where $\theta_i=(X_i, u_i, w_i)$. $A=X_1\times\cdots\times X_n$. This economy consists of individual households (agents) $i\in N=\{1,\cdots,n\}$. They have their own utility functions $u_i: X_i\to R$ and initial endowments $w_i\in X_i$. For a given profile of all members' characteristics $\theta=(\theta_1,\cdots,\theta_n)$, we can define the set $f(\theta)$ of allocations which are Pareto-optimal and individually-rational. Of course, we can not list up the elements of $f(\theta)$ explicitly and concretely.

$(x^*_1,\cdots,x^*_n)\in A$ is **Pareto-optimal** $\Leftrightarrow$ $\Sigma_{i=1}^n x_i{}^*=\Sigma_{i=1}^n w_i$ and $(\forall x=(x_1,\cdots, x_n)\in A)[\Sigma_{i=1}^n x_i=\Sigma_{i=1}^n w_i$ & $(\forall i\in N)(u_i(x_i)\geqq u_i(x_i{}^*))$ $\Rightarrow$ $x^*=x]$ hold.

$(x^*_1,\cdots,x^*_n)\in A$ is **individually-rational** $\Leftrightarrow$ $(\forall i\in N)[u_i(x^*_i)\geqq u_i(w_i)]$ holds.

To find an element $x^*\in f(\theta)$ we need to solve a multi-objective problem. However we cannot solve it centrally because of the too big size of the problem and the social necessity of keeping the **freedom** and **privacy**.

Problem: Can we find Pareto-optimal and individually-rational allocation x^*, that is $x^*\in f(\theta)$, without solving the above muti-objective optimization problem centrally?

To solve this problem noncentrally means that we must design some kind of game or rule, which we call mechanism in this paper,and that we solve it decentrally or competitively, i.e. we must find Nash-equilibrium solution non-cooperatively.

Competitive mechanism is one of the most famous mechanisms as such. We can interpret the competitive mechanism as a game. The competitive equilibrium solution $x^*\in A$ of it can be considered as Nash-equilibrium solution of the game which is attained as a result of free competition. The problem is whether $x^*\in f(\theta)$ holds. In precise, the competitive mechanism can be interpreted as n+1 players game,one of which is Walrasian auctioner,i.e. price setter. (In precise, competitive equilibrium solution is defined as (p^*,x^*), where p^* is equilibrium price.)

However the rule of the game is; the players should play truthfully as a price-taker.

As you can imagine easily, the real economic agents manipulate their supplies and demands strategicaly. Agents may not reveal their demands unhonestly. If we permit the agents to play unhonestly, the competitive mechanism, unfortunately, fails to give a Pareto-optimal allocation[6]. We can call this situation a kind of market-falure. That is, saying correctly with using our framework, competitive mechanism does not Nash-implement f under the condition of privacy-respecting. The importance of Nash-implementability problem lies here.

Ⅲ. Results

3.1 Condition for Nash-implementability

First, we give a necessary condition for Nash-implementability.

Theorem 1:

If SCC $f: \mathbb{R} \Rightarrow A$ is Nash-implementable, then it is monotonic.

Proof: Let $M=(S, h, A)$ be Nash-implementable for f. Let's pick R^- and $R^{-\prime} \in \mathbb{R}$ arbitrarily and fix them. Suppose that for an arbitrary fixed $a \in f(R^-)$ $L(a, R_i) \subseteq L(a, R_i')$] holds for any $i \in N$. By definition of Nash-implementability, we have $h(N(R^-; M)) = f(R^-)$. Thus we get $a \in h(N(R^-; M))$.Consequently, $h(s)=a$ holds for some $s \in N(R^-; M)$. From this we have $h(S_i, s_{-i}) \subseteq L(a, R_i)$ for any $i \in N$. Thus by assumption we have $h(S_i, s_{-i}) \subseteq L(a, R_i')$ for any $i \in N$.So we get $s \in N(R^{-\prime}, M)$. Consequently, $a \in h(N(R^{-\prime}; M)) = f(R^{-\prime})$ holds. Thus f is monotonic. **Q.E.D.**

The above theorem tells us that monotonicity of SCC f is neccesary condition for Nash-implementability. Next we consider sufficiency condition. Maskin gave first a sufficient condition for Nash-implementability for f.

Theorem 2:

SCC $f: \mathbb{R} \Rightarrow A$ satisfies monotonicity and NVP, then it is Nash-implementable.

Proof: Omitted. See Maskin[1985].

Maskin gave a constructive proof for Theorem 2 by constructing a mechanism concretely. We give Maskin's mechanism below.

【Maskin's Mechanism】

○**Strategy set** for $i \in N$; $S_i = \mathbb{R} \times A$.

○**Outcome Function** $h: S \to A$;

① $h(s) = a$ if $a \in f(R^-)$ and $s_i = (R^-, a)$ holds.

② $h(S_i, s_{-i}) = L(a, R_i)$
if $(\forall i \in N)[(\forall j \neq i)(a \in f(R^-)$ and $s_j = (R^-, a))]$ holds.

③ $h(S_i, s_{-i}) = A$ if $(\forall i \in N)(\exists j, k \in N)[j \neq i \neq k$ & $[s_j \neq s_k$ or $(s_j = (R^-, a)$ & $a \notin f(R^-))]]$ holds.

Where $R^- = (R_1, \cdots, R_n) \in \mathbb{R}$.

The above theorem is a sufficient condition. However,NVP is not neccesary condition for Nash-implementability unfortunately. We haven't know yet the neccesary and sufficient condition of Nash-implementability for f.

In the above mechanism S_i is $\mathbb{R} \times A$ and $\mathbb{R} = \mathbb{R}_A{}^n$, thus S_i is a very large set. So each member i must not only reveal his own alternative $a \in A$ and his own preference R_i but also must forecast all the other members' preferences R_j. It is too heavy work for organization members to forecast them. However, what is important of this theorem is that a sufficient condition for Nash-implementability was given first anyway.

Then several researchers endeavoured to reduce the size of strategy set S_i(Saijho[2], McKelvy[3]). We also discovered another mechanism which has smaller size of strategy set than Mackelvy and others' and which also Nash-implements SCC f.This result is one of the main results of this report. Before introducing our mechanism, we must give another two definitions.

Definition 12; Bottom alternative

$a_B \in A$ is **bottom alternative** for SCC $f: \mathbb{R} \Rightarrow A$

$\Leftrightarrow (\forall R^- \in \mathbb{R})[a_B \notin f(R^-)$ & $(\forall i \in N)(L(a_B, R_i) = \{a_B\})]$

In the above definition, you may think that the condition SCC $f: \mathbb{R} \Rightarrow A$ has a bottom alternative is too restricted.However it is not so.If there does not exist a bottom alternative in A, we can add a special alternative a_B to A so that it becomes bottom alternative for SCC f in $A \cup \{a_B\}$. We can regard $A \cup \{a_B\}$ as the set of alternatives which are given in advance.

Definition 13;Anonymity

SCC $f: \mathbb{R} \Rightarrow A$ is **anonymous** iff $f(R_{\pi(1)},\cdots,R_{\pi(n)}) = f(R^{-})$ holds for any $R^{-} \in \mathbb{R}$ and any bijective map $\pi: N \to N$.

This definition means that f is symmetric w.r.t. members. Before giving Theorem 3, we introduce next assumption.

Assumption 1:
For all members, $\mathbb{R}_1 = \cdots = \mathbb{R}_n$ holds.

Theorem 3:
Suppose $n \geqq 3$, and SCC $f: \mathbb{R} \Rightarrow A$ has the bottom alternative and satisfies anonymity, momotonicity and NVP, then f is Nash-implementable.

Before giving a proof for the above theorem, we introduce a mechanism M_γ which Nash-implements f.

【Mechanism M_γ】
Let a_B be bottom alternative for f. $M_\gamma = (S, h, A)$ is defined as follows.

$S_i \equiv \{(a_i, R_i^i, m_i) | a_i \in A, R_i^i \in \mathbb{R}_i, m_i \in N\}$ for all $i \in N$.
① $h(s)=a^*$ if $(\exists a^* \in A)(\forall i \in N)[a_i=a^*$ & $a^* \in f(R_1^1,\cdots,R_n^n)]$ holds.
② $h(s)=a_B$ if $(\exists a^* \in A)(\forall i \in N)[a_i=a^*$ & $a^* \notin f(R_1^1,\cdots,R_n^n)]$ holds.
③When there exist $a^* \in A$ and $j \in N$ such that for all $i \in N \backslash \{j\}$ $a_i=a^* \neq a_j$ holds, we set
$h(s)=a_j$ if $a_j \in L(a^*, R_{H}^{H}) \backslash \{a_B\}$ or $a^*=a_B$,
$=a^*$ otherwise.
④When the above cases ①②③ do not hold, we set
$h(s)=a_t$, where $t=(\Sigma_{i=1}^{n} m_i)_{\mathrm{mod}\ n}$, $n+1 \equiv 1 (\mathrm{mod}\ n)$.

In this mechanism, each organizational member must send a message(strategy) $s_i = (a_i, R_i^i, m_i)$ to other members. ④ says that if the opinion of members differ from each other a_t is selected according to this game's rule. However t is calculated by $t=(\Sigma_{i=1}^{n} m_i)_{\mathrm{mod}\ n}$ which depends on the numbers $\{m_i\}_{i=1}^{n}$ selected by all members, so each member cannot control the selection of an alternative which he desires. We will prove the above mechanism M_γ Nash-implements f.

Proof of Theorem 3:
To prove Nash-implementability for f, we must prove that $f(R^{-}) \equiv h(N(R^{-}; M_\gamma))$ for all $R^{-} \in \mathbb{R}$.

[1]:First, we show $f(R^{-})\subseteq h(N(R^{-};M_{\gamma}))$: Choose $R^{-}=(R_1,\cdots,R_n)\in \mathbb{R}$ and $a\in f(R^{-})$ arbitrarily. Choose $s=(s_1,\cdots,s_n)\in S$ as $s_i=(a,R_{i+1},i)$ for all $i\in N$. From assumption 1 and anonimity condition, we have $R_{i+1}\in \mathbb{R}_i$ and $a\in f(R_2,R_3,\cdots,R_n,R_1)=f(R^{-})$. By applying rule ① of mechanism M_{γ}, we have $h(s)=a$.

Next we will show this s is a Nash equilibrium solution for M_{γ}. Fix member i arbitrarily. Choose an arbitrary strategy $s_i'=(a_i,R^i_i,m_i)$. Then

ⓐ If $a_i=a$, then by rule ① or ② we have $h(s_i',s_{-i})\in\{a,a_B\}$.

ⓑ If $a_i\neq a$, then from $a\neq a_B$ and rule ③ we have

$$h(s_i',s_{-i})=a_i \quad \text{if } a_i\in L(a,R_i)\backslash\{a_B\}$$
$$=a \quad \text{otherwise.}$$

Thus we get $h(s_i',s_{-i})\in L(a,R_i)$. Consequently we have $h(S_i,s_{-i})\subseteq L(a,R_i)$. So $s\in N(R^{-};M_{\gamma})$. Thus $a=h(s)\in h(N(R^{-};M_{\gamma}))$ holds.

[2]:Next, we will show the converse inclusion relation $h(N(R^{-};M_{\gamma}))\subseteq f(R^{-})$:Choose $R^{-}=(R_1,\cdots,R_n)\in \mathbb{R}$ and $a\in h(N(R^{-};M_{\gamma}))$ arbitrarily. Then there exists $s\in N(R^{-};M_{\gamma})$ such that $h(s)=a$. Where $s=(s_1,\cdots,s_n)$ and $s_i=(a_i,R^i_i,m_i)$ for all i. We will show $a\in f(R^{-})$. We can consider next four cases.

1)Case-1;There exists $a^*\in A$ such that $a_i=a^*$ for all $i\in A$.

2)Case-2;There exist $a^*\in A$ and $i\in N$ such that $a_i=a^*\neq a_j$ for all $j\neq i$.

3)Case-3;There exist $j,k\in N$ and $a^*\in A$ such that $j\neq k$ and $a_i=a^*\neq a_j,a_k$ for all $i\neq j,k$.

4)Case-4;Except for the above three cases.

We will show case-1 only. We omit the proof of other cases.For we can show exactly and similiarly, and we need many spaces for it.

<u>Proof for case-1</u>;When there exists $a^*\in A$ such that $a_i=a^*$ for all i, we can divide this case into two subcases.

<u>Case-1-1</u>;When $a^*\in f(R^1_1,\cdots,R^n_n)$ holds;

From $h(s)=a$ and rule ①, we have $a^*=a$. Choose $i\in N$ arbitrarily and fix it. Choose $a'\in L(a,R^{i-1}_{i-1})\backslash\{a_B\}$ arbitralily.

ⓐ If $a'=a$, we have $a'\in h(S_i,s_{-i})$ by $h(s)=a=a'$.

ⓑ If $a'\neq a$, we define $s_i'\in S_i$ as $s_i'=(a',R^i_i,m_i)$.

By rule ③, we have $h(s_i',s_{-i})=a'$. Thus $a'\in h(S_i,s_{-i})$.

Consequently, we have $L(a,R^{i-1}_{i-1})\backslash\{a_B\}\subseteq h(S_i,s_{-i})$.

On the other hand, by assumption $s\in N(R^{-},M_{\gamma})$ we have $h(S_i,s_{-i})\subseteq L(a,R_i)$. Thus we get $L(a,R^{i-1}_{i-1})\backslash\{a_B\}\subseteq L(a,R_i)$. Moreover we have $L(a,R^{i-1}_{i-1})\subseteq L(a,R_i)$ for all i, since $a_B\in L(a,R_i)$ for all i.

Since $a\in f(R^n_n,R^1_1,\cdots,R^{n-1}_{n-1})$ holds by the assumption of anonimity, we have $a\in f(R^{-})$ from monotonicity.

Case-1-2;When $a^* \notin f(R^1_i,\cdots,R^n_n)$ holds;

From rule ②, $a=h(s)=a_B$ holds. Suppose $s_i{}'=(a',R^1_i,m_i)$ for i,where a' is an arbitrary element in $A\backslash\{a^*\}$. By rule ③, $h(s_i{}',s_{-i})\neq a_B$ holds. Thus $h(s_i{}',s_{-i})\notin\{a_B\}=L(a_B,R_i)=L(a,R_i)$.Consequently,$s\notin N(R^-,M_\gamma)$. This contradicts with the assumption. So we don't need consider Case-1-2.

Other cases can also be proved similarly. See [4] in detail. **Q.E.D.**

3.2 Public Good Economy Case

We don't have yet the necessary and sufficient condition for Nash-implementability for f. However, in a simple public goods economy model we can find the necessary and sufficient condition as shown later. Before giving the result, we must give several notations and definitions for public good model.

Notation 4:

R:the set of real numbers

Y:the set of provision level of **public good**. We assume $Y\subseteq R$.

X_i:the set of member i's **private good** consumption level.

We assume $X_i\subseteq R$.

$X=X_1\times\cdots\times X_n$

$w_i\in X_i$:**Initial endowment** for i.

$T_i\in X_i$; **tax** imposed on i. We asuume $T_i\leqq w_i$. If $T_i<0$,T_i means **subsidy**.

$A=\{(T_1,\cdots,T_n,y)\mid y\in Y,\Sigma^n_{i=1}T_i\geqq y\}$: the set of allocations.

We may write T instaed of $(T_1,\cdots,T_n)$.

$u_i(\cdot;\theta_i)$:$A\rightarrow R_e$; **utility function** for member i under θ_i.

$\theta_i=(X_i,u_i,w_i)\in\Theta_i$; We call the triple θ_i member i's **characteristic**.

$\theta=(\theta_1,\cdots,\theta_n)$: **profile** of member's characteristics.

$\Theta=\Theta_1\times\cdots\times\Theta_n$

$L((T,y),\theta_i)=\{(T',y')\in A\mid u_i(T,y)\geqq u_i(T',y')\}$.

Where $\theta_i=(X_i,u_i,w_i)$, $T=(T_1,\cdots,T_n)$ and $T'=(T_1',\cdots,T_n')$.■

$\Sigma^n_{i=1}T_i\geqq y$ means that public good cannot be provided greater than accumulated total tax.Most important component of characreristic θ_i is utility function $u_i(\cdot;\theta_i)$.We may simply write this u_i instead of $u_i(\cdot;\theta_i)$. Compared with the former Social Choice Theory, Θ_i corresponds to R_i. So we can apply the same concepts to public good model as to Social Choice Theory.

Assumption 2:
We assume $\Theta_1 = \cdots = \Theta_n$.

Definition 14;Public Good Economy
We call $(\theta_1, \cdots, \theta_n, Y)$ **public good economy.**

Definition 15;Selfishness
In the above public good economy model, $\theta_i = (X_i, u_i, w_i)$ is **selfish**
$\Leftrightarrow (^\forall y \in Y)(^\forall T, T' \in X)$
$[(T,y),(T',y) \in A \ \& \ T_i > T_i' \Rightarrow u_i(T',y) > u_i(T,y)]$.
Where $T=(T_1, \cdots, T_n)$ and $T'=(T_1', \cdots, T_n')$.

Under this model, we have next theorem.

Theorem 4:
In the above public good model, suppose $n \geqq 3$ and $\theta_i = (X_i, u_i, w_i)$ is selfish for all i. Then, $f: \Theta \Rightarrow A$ is monotonic iff f is Nash-implementable.

Proof;We have already proved if-part in Theorem 1 without condition. So we only need to prove only-if part. Suppose f is monotonic.
We introduce the following mechanism $M=(S, h, A)$. Then we will show the mechanism M Nash-implements f.

【Mechanism M】
○Strategy space S_i:
Let $\mathcal{A}_i(T,y) \equiv \{L((T,y), \theta_i) \mid \theta_i \in \Theta_i\}$ and $\mathcal{B}_i(T,y) \equiv \{L((T,y), \theta_{i+1}) \mid \theta_{i+1} \in \Theta_{i+1}\} \cup \{\phi\}$. S_i is defined as the set $\{((T^i, y^i), A_i, B_i) \mid (T^i, y^i) \in A, A_i \in \mathcal{A}_i(T^i, y^i), B_i \in \mathcal{B}_i(T^i, y^i)\}$.
○Outcome function $h: S \to A$:
For the sake of defining outcome function h, we introduce next definition.

[Definition]; f-consistency
Suppose $s=(s_1, \cdots, s_n) \in S$ and $j \in N$ be given.
s_{-j} is **f-consistent** iff there exist $(T', y') \in A$ and $\theta' \in \Theta$ such that
①$(T', y') \in f(\theta')$
②$(T^i, y^i) = (T', y')$ for all $i \neq j$ and
③$A_i = L((T', y'), \theta_i')$ & $B_i = L((T', y'), \theta_{i+1}')$ for all $i \neq j$ hold.
Where $s_i = ((T^i, y^i), A_i, B_i)$ for all i.■

$h: S \to A$ is defined as follows:

① When for some j s_{-j} is f-consistent for s,

$h(s)=(T^j, y^j)$ if $(T^j, y^j) \in B_{j-1}$ holds

$=(T^{j-1}, y^{j-1})$ otherwise.

② When we cannot apply ①,

$h(s)=(T^*, y^*)$.

Where $T^*=(T^*_1, \cdots, T^*_n)$, $T^*_i=(\Sigma_{k \in N} T^k_i - r^i)/n$, $y^*=\Sigma_{k \in N} y^k/n - \max r + \max\{r \backslash (\max r)\}$, $r=\{r^1, \cdots, r^n\}$ and $r^i=\Sigma_{k \in N} T^k_i - y^i$ for all i.

Since $n \geqq 3$ and $\Sigma_{i \in N} T^*_i - y^* = [\max r - \max\{r \backslash (\max r)\}] \geqq 0$ from assumption, h is well defined. End of definition of mechanism M.

We will show this mechanism M Nash-implements f.

[1]: First, we show that $f(\theta) \subseteq h(N(\theta; M))$ holds for all $\theta \in \Theta$.

Choose $\theta \in \Theta$ and $(T,y) \in f(\theta)$ arbitrarily. Choose $s_i \in S_i$ for all i as $s_i=((T,y), A_i, B_i)$. Where $A_i \in L((T,y), \theta)$ and $B_i \in L((T,y), \theta_{i+1})$. Then we have $h(s)=(T,y)$.

For an arbitrary j, if we choose $s_j'=((T^j, y^j), A_j, B_j)$, then s_{-j} is f-consistent. By definition of h, we have

$h(s_j', s_{-j})=(T^j, y^j)$ if $(T^j, y^j) \in L((T,y), \theta_j)$

$=(T,y)$ otherwise.

Consequently, we get $u_j(h(s); \theta_j) \geqq u_j(h(s_j', s_{-j}); \theta_j)$ for all $j \in N$ and $s_j' \in S_j$. Thus $s \in N(\theta; M)$ holds. Thus we have $(T,y)=h(s)=h(N(\theta; M))$.

[2]: Next, we show that $h(N(\theta; M)) \subseteq f(\theta)$ holds for all $\theta \in \Theta$.

First we show

P1: $L((T,y), \theta_j) \neq A$ for all $\theta \in \Theta$, $(T,y) \in A$ and $j \in N$.

[Proof of P1]: Set $\theta \in \Theta$, $(T,y) \in A$ and $j \in N$ arbtrarily. If we set $T'=(T_j - \varepsilon, T_{j+1} + \varepsilon, T_{-(j,j+1)})$ for some $\varepsilon > 0$, then we have $u_j((T,y); \theta_j) < u_j((T',y); \theta_j)$.

Thus $L((T,y), \theta_j) \neq A$ was proved. End of proof of P1.

Choose and fix $\theta \in \Theta$ and $(T,y) \in h(N(\theta; M))$ arbitrarily(where $T=(T_1, \cdots, T_n)$). Then there exists some $s \in N(\theta; M)$ such that $h(s)=(T,y)$ holds. Where $s=(s_1, \cdots, s_n)$, $s_i=((T^i, y^i), A_i, B_i)$ and $T^i=(T^i_1, \cdots, T^i_n)$. We will show the following P2 and P3 hold for this $s \in S$.

P2: s_{-j} is f-consistent for all j.

P3: $B_{j-1} \subseteq L((T,y), \theta_j)$ for all j.

[Proof of P2]: Suppose s_{-j} is not f-consistent. Set $(T',y') \in A$ arbitrarily. Where $T'=(T'_1,\cdots,T'_n)$).
Set $s_j'=((T^{j\prime},y^{j\prime}),A_j',B_j')$ as follows (where $T^{j\prime}=(T^{j\prime}_1,\cdots,T^{j\prime}_n)$);

$T^{j\prime}_i=\Sigma_{k\neq j}T^k_i+r^i-nT'_i$ if $i\neq j$

$=\Sigma_{k\neq j}T^k_i+q-nT'_i$ if $i=j$.

Where $q=r'+\max(r\setminus\{r^j\})$, $r'=\Sigma_{i\in N}T'_i-y'$, $r^i=\Sigma_{k\in N}T^k_i-y^i$.

$y^{j\prime}=n\Sigma_{i\in N}T'_i-\Sigma_{k\neq j}y^k$.

A_j' is arbitrary element in $\mathcal{A}_j(T^j,y^j)$.

$B_j'=\phi$.

In order to show $s_j'\in S_j$, it is sufficient to show $(T^{j\prime},y^{j\prime})\in A$.

$$\Sigma_{i\in N}T^{j\prime}_i-y^{j\prime}=\Sigma_{i\in N}\Sigma_{k\neq j}T^k_i+\Sigma_{k\neq j}r^k+q-n\Sigma_{i\in N}T'_i-(n\Sigma_{i\in N}T'_i-\Sigma_{k\neq j}y^k)$$
$$=\Sigma_{k\neq j}(y^k+r^k-\Sigma_{i\in N}T^k_i)+q$$
$$=q \quad \text{(because of } r^i=\Sigma_{k\in N}T^k_i-y^i \text{ for all i)}$$
$$\geqq 0$$

Thus $s_j'\in S_j$. Set $s'=(s_j',s_{-j})$. From the above relation $q=\Sigma_{i\in N}T^{j\prime}_i-y^{j\prime}$, we can write $q=r^{j\prime}$. Moreover, $q=\max\{r^{j\prime},r^{-j}\}$. Since s_{-i}' is not f-consistent for all i, we can apply rule ②. If we set $h(s')=(T^*,y^*)$, we have ;for all $i\neq j$, $T^*_i=(\Sigma_{k\neq j}T^k_i+T^{j\prime}_i-r^i)/n=(\Sigma_{k\neq j}T^k_i+nT'_i-\Sigma_{k\neq j}T^k_i+r^i-r^i)/n$
$=T'_i$,

for j, $T^*_j=(\Sigma_{k\neq j}T^k_i+T^{j\prime}_i-q)/n=(\Sigma_{k\neq j}T^k_i+nT'_i-\Sigma_{k\neq j}T^k_i+q-q)/n=T'_j$.

$y^*=(\Sigma_{k\neq j}y^k+y^{j\prime})/n-\max\{r^{j\prime},r^{-j}\}+\max(\{r^{j\prime},r^{-j}\}\setminus\{r^{j\prime}\})=y'$. Thus we have $h(s')=(T^*,y^*)=(T',y')$. On the other hand, since $s\in h(N(\theta;M))$ we have $u_i((T,y);\theta_i)\geqq u_i((T',y');\theta_i)$. Thus $(T',y')\in L((T,y),\theta_i)$ holds. Consequently, we have $L((T,y),\theta_i)=A$. This contradicts with P1. Thus P2 is proved. End of proof of P2.

[Proof of P3]:
Choose $j\in N$ arbitrarily. From P2 s_{-j} is f-consistent. We set $s_j'=((T^{j\prime}y^{j\prime}),A_j,B_j)$ against an arbitrary $(T',y')\in B_{j-1}$ as $(T^{j\prime},y^{j\prime})=(T',y')$. Where A_j and B_j are arbitrary element of $\mathcal{A}_j(T^{j\prime},y^{j\prime})$ and $\mathcal{B}_j(T^{j\prime},y^{j\prime})$ respectively. Since s_{-j} is f-consistent, there exists $(T^*,y^*)\in A$ such that $(T^i,y^i)=(T^*,y^*)$ fo all $i\neq j$. Since $(T^{j\prime},y^{j\prime})=(T',y')\in B_{j-1}$ from rule ①, $h(s_j',s_{-j})=(T',y')$. Thus we have $(T',y')\in h(S_j,s_{-j})$. Thus $B_{j-1}\subseteq h(S_j,s_{j-1})$. From $s\in N(\theta;M)$ $h(S_j,s_{-j})\subseteq L((T,y),\theta_j)$ holds. Consequently we have $B_{j-1}\subseteq L((T,y),\theta_j)$. End of proof of P3.

Using P2 and P3 we show the assertion of this theorem.
From P2, it is sufficient to consider the case s_{-j} is f-consistent for all j. Since $h(s)=(T,y)$, there exists $i\in N$ such that $(T^j,y^j)=(T,y)$. Since s_{-j} is f-consistent, there exist $(T^*,y^*)\in A$ and $\theta^*\in\Theta$ such that

:① $(T^*,y^*)\in f(\theta^*)$,② $(T^i,y^i)=(T^*,y^*)$ for all $i\neq j$ and ③ $A_i=L((T^*,y^*),\theta^*_i)$ & $B_i=L((T^*,y^*),\theta^*_{i+1})$.Thus we've $(T^i,y^i)=(T,y)$ and $A_{i+1}=B_i$. Consequently we have ① $(T,y)\in f(\theta^*)$, ② $(T^i,y^i)=(T^*,y^*)$ for all i and ③ $A_i=L((T,y),\theta^*_i)$ & $B_i=L((T,y),\theta^*_{i+1})$ hold. From P3 $L((T,y);\theta^*_i)=B_{i-1}\subseteq L((T,y),\theta_i)$. Since f is monotonic, we have $(T,y)\in f(\theta)$.
Q.E.D. of Theorem 4.

3.3 Difficulty of Nash-implementability

We tried to construct a mechanism concretely which Nash-implements SCC that is monotonic and NVP. However it's very difficult to find a mechanism.So we guessed that the class of Nash-implementable SCCs may be very small. Thus in this section we consider about the difficulties for Nash-implementability.

First, we consider the case the set of alternatives A is finite,i.e. $|A|=m$. Suppose $f:R\Rightarrow A$ be a SCF. Let's write $f:R\rightarrow A$ instead of $f:R\Rightarrow A$ as mentioned in Section II when f is SCF.

Theorem 5:
Suppose A be a finite set with cardinality greater than 3 and $R=R_A{}^n$. Then we have

$f:R\rightarrow A$ is onto mapping and Nash-implementable ⇔ f is dictatorial.

Where R_A is the set of all weak order on set A.

Proof: Omitted.(see[4])

This theorem tells us that if SCC is onto mapping then Nash-implementa blity has almost the same meaning with dictatorship. This situation is very similar to the famous Arrow's Impossibility Theorem of Social Welfare Function(SWF). One of causes of Arrow's Impossibility was that the domain of SWF $R_A{}^n$ is too wide(we call this domain as **universal domain**). The situation is similar to Arrow's Impossibility for Nash-implementabi lity. If we restrict the domain of f, we may have more affirmative results.

Next, we consider the case A is infinite set and is equipped with met ric d, i.e. A is metric space whose metric function is $d:A\times A\rightarrow R$. We often write this (A,d) to indicate that A is metric space. Here we also give another necessary definitions and notation.

Definition 16:Metric on $\mathcal{P}(A)\backslash\{\phi\}$

For arbitrary sets B and C in $\mathcal{P}(A)\backslash\{\phi\}$ we define ρ and δ as follows: $\rho(B,C)\equiv \sup_{y\in B}\inf_{z\in C} d(y,z)$

$\delta(B,C)\equiv \max\{\rho(B,C),\rho(C,B)\}$.

Notation 5:

$F(R,A)\equiv\{f \mid f:R\Rightarrow A\}$ $\quad$ $F(\Theta,A)\equiv\{f \mid f:\Theta\Rightarrow A\}$

$M_0(F(R,A))\equiv\{f\in F(R,A)\mid f$ is monotonic$\}$.

For simplicity, we may write F and Mo instead of $F(R,A)$ and $M_0(F(R,A))$ respectively, if there is no confusion from the context.

Definition 17:Distance and Open sphere

For arbitrary f and $f'\in F(R,A)$ and $\varepsilon>0$, we define D and B as follows: $D(f,f')\equiv\sup\{\delta(f(R),f'(R))\mid R\in R\}$

$B(f,\varepsilon)\equiv\{f'\in F(R,A)\mid D(f,f')<\varepsilon\}$.

The metric defined above satisfies the following properties:

① $D(f,f'')\leqq D(f,f')+D(f',f'')$ for any f, f' and $f''\in F(R,A)$

② $D(f,f')=D(f',f)$ for any f and $f'\in F(R,A)$ and

③ $D(f,f)=0$ for any $f\in F(R,A)$.

We can check easily these properties. This type of metric is often called extended pseudometric(Klein & Thompson[5]). We introduce topology to set $F(R,A)$ by using this pseudometric function D.

Definition 18:Open set

$G\subseteq F(R,A)$ is open $\Leftrightarrow$ $(\forall f\in G)(\exists\varepsilon>0)[B(f,\varepsilon)\subseteq G]$.

We denote the set of open sets T.

The space $(F(R,A),T)$ defined above becomes topologixal space(Klein & Thompson[5]).

With these definitions we can give next theorem.

Theorem 6:

Suppose (A,d) is a dense metric space and Θ_i is large for some i. Then $M_0(F(\Theta,A))^c$ is everywhere dense in $(F(R,A),T)$.

Where H^c is the complement of set H.

Proof: We show the Closure of $M_0{}^c$ coincides with F. Since Closure of $M_0{}^c$ is $\{f\in F\mid\forall\varepsilon>0, B(f,\varepsilon)\cap M_0{}^c\neq\phi\}$, it is enough to show the following.

"$(\forall f \in M_\emptyset)(\forall \varepsilon > 0)(\exists f' \in M_\emptyset^c)[D(f, f') < \varepsilon]$".

Choose $f \in M_\emptyset$ and $\varepsilon > 0$ arbitrarily. Choose any $\theta \in \Theta$ and fix it. Since $f(\theta) \neq \phi$, there exists some $x \in f(\theta)$. Since Θ_i is large,there exists $\theta_i' \neq \theta_i$ such that $L(x, \theta_i) \subseteq L(x, \theta_i')$. Moreover by the assumption of denseness of A there exists some $y \in A$ such that $y \neq x$ and $d(x,y) < \varepsilon$. Let define $f' \in \mathrm{F}$ as follows:

$$f'(\theta) = f(\theta) \quad \text{if } \theta \neq (\theta_i', \theta_{-i})$$
$$= f(\theta) \setminus \{x\} \cup \{y\} \quad \text{otherwise.}$$

From $D(f, f') < \varepsilon$, we have $f' \in B(f, \varepsilon)$. And clealy $f' \in M_\emptyset^c$ holds.
Q.E.D.

By combining Theorem 1 with Theorem 6, we have next corollarly.

Corollarly:

Suppose (A, d) be a dense metric space and Θ_i is large for some i. Then $\{f \in \mathrm{F} \mid f$ is Nash-implementable$\}^c$ is everywhere dense in $(\mathrm{F}(\Theta, A), T)$.

Proof: From Theorem 6 we have $\{f \in \mathrm{F} \mid f$ is Nash-implementable$\} \subseteq M_\emptyset$. Thus we get $M_\emptyset^c \subseteq \{f \in \mathrm{F} \setminus f$ is Nash-implementable$\}^c$. Consequently, Closure of $M_\emptyset^c$ is included in Closure of $\{f \in \mathrm{F} \mid f$ is Nash-implementable$\}^c$. Thus we have Closure of $\{f \in \mathrm{F} \mid f$ is Nash-implementable$\}^c = \mathrm{F}$. **Q.E.D.**

Theorem 6 and the above Corollarly are easily understandable and natural results. In precise they don't say that Nash-implementable SCCs are very scarce.

IV. Conclusions

We gave several new results on Nash-implementability in this paper. First, a necessry condition for Nash-implementability for general level was given in Section 3.1. We developed a smaller size mechanism than Maskin and others,i.e. we gave an alternative proof for it. We also gave a necessary and sufficient condition for Nash-implementability against a simple public good model in Section 3.2. The condition was monotonicity. In Section 3.3,we discussed how is the size of Nash-implementable class is, and almost all of SCC are not Nash-implementable. The last result is natural one. For the case the set of alternatives are finite set,we discussed that the class of Nash-implementable SCC is almost the same to that of dictatorial ones.

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