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COMPLETELY $\mathbb{Z}$ SYMMETRIC R MATRIX

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1. Introduction

In this paper, we shall introduce an infinite-dimensional R matrix related to the limiting case $n \to \infty$ of the completely $\mathbb{Z}_n$ symmetric R matrix. This is not the same as the R matrix of Gaudin [8], Gómez–Sierra [9], and Fateev–Zamolodchikov [5]. Of course, this R matrix satisfies the Yang-Baxter equation

$$(1.1) \qquad R_{12}(\lambda_1)R_{13}(\lambda_1+\lambda_2)R_{23}(\lambda_2) = R_{23}(\lambda_2)R_{13}(\lambda_1+\lambda_2)R_{12}(\lambda_1).$$

By means of the Fourier transformation, we shall give an R operator on $C^\infty(S^1 \times S^1)$. This R operator is also a solution of the Yang-Baxter equation (1.1). Moreover we shall apply the fusion procedure to the R operator, and shall construct a finite-dimensional R matrix from the R operator.

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The reason why we begin investigations of such an infinite-dimensional R matrix is as follows.

The quantum group is useful for the geometric interpretation of Macdonald's symmetric polynomials. In fact, Ueno–Takebayashi [17] and Noumi [13] proved that Macdonald's symmetric polynomials are the zonal spherical functions on some quantum homogeneous space. Although Macdonald's symmetric polynomials have two parameters, they dealed with the case of only one parameter. Further it is rather more natural to consider Macdonald's symmetric functions than Macdonald's symmetric polynomials. Roughly speaking, Macdonald's symmetric functions are the polynomials with infinite variables. Then we aim at defining the quantum group $U_{q,t}(gl_\infty)$, a deformation of $U(gl_\infty)$ with two parameters q and t for giving the geometric interpretation of Macdonald's symmetric functions. In view of Sklyanin algebra [16], it is important to find out the infinite-dimensional R matrix which can define the quantum group $U_{q,t}(gl_\infty)$.

On the other hand, Freund–Zabrodin [7] and Zabrodin [19] showed that Macdonald's symmetric functions are related to the limiting case $n \to \infty$ of the completely $\mathbb{Z}_n$ symmetric model.

The details will be discussed in a forthcoming paper [15].

2. Completely $\mathbb{Z}_n$ symmetric R matrix

Let us quickly review the completely $\mathbb{Z}_n$ symmetric R matrix. We denote by $R_{ij}^{k\ell}(\lambda)$ the Boltzmann weight for a single vertex with bond states $i, j, k, \ell \in \mathbb{Z}_n$. These Boltzmann weights $R_{ij}^{k\ell}(\lambda)$ define a matrix $R(\lambda)$ in the standard basis $\{e_i \otimes e_j; i, j \in \mathbb{Z}_n\}$ for $\mathbb{C}^n \otimes \mathbb{C}^n$. This matrix $R(\lambda)$ is said to be completely $\mathbb{Z}_n$ symmetric if $R_{ij}^{k\ell}(\lambda)$ satisfies the conditions below:

(1) $R_{ij}^{k\ell}(\lambda) = 0 \quad \text{unless } i + j = k + \ell \quad mod\, n,$

(2) $R_{i+p,j+p}^{k+p,\ell+p}(\lambda) = R_{ij}^{k\ell}(\lambda) \quad \text{for } \forall i, j, k, \ell \in \mathbb{Z}_n.$

Because of $\mathbb{Z}_n$ symmetry, there exists $S^{ab}(\lambda)$ $(a, b \in \mathbb{Z}_n)$ satisfying

$$R_{ij}^{k\ell}(\lambda) = \delta_{i+j,k+\ell} S^{k-i,\ell-i}(\lambda). \tag{2.1}$$

We define the Jacobi theta functions $\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z, \tau)$ of rational characteristics $a, b \in \frac{1}{n}\mathbb{Z}$ by

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z, \tau) = \sum_{m \in \mathbb{Z}} \exp[\pi\sqrt{-1}(m + a)^2\tau + 2\pi\sqrt{-1}(m + a)(z + b)], \tag{2.2}$$

and put

$$S^{ab}(\lambda) = \frac{\vartheta\begin{bmatrix} \frac{b-a}{n} + \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(\lambda + \kappa, n\tau)}{\vartheta\begin{bmatrix} -\frac{a}{n} + \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(\kappa, n\tau)\vartheta\begin{bmatrix} \frac{b}{n} + \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(\lambda, n\tau)}. \tag{2.3}$$

Here $Im\ \tau > 0$, and κ is a constant. These weights $S^{ab}(\lambda)$ give a solution of the Yang-Baxter equation (1.1). This completely $\mathbb{Z}_n$ symmetric R matrix has been studied by Belavin [2], Cherednik [3] [4], and Richey–Tracy [14].

The weight $S^{ab}(\lambda)$ is expressed as follows.

$$S^{ab}(\lambda)=q^{-\frac{n}{8}}\tilde{S}^{ab}(\lambda), \tag{2.4}$$

$$\begin{aligned}
&\tilde{S}^{ab}(\lambda)\\
&=q^{-\frac{ab}{n}}e^{2\pi\sqrt{-1}(\frac{b}{n}\kappa-\frac{a}{n}\lambda)}\\
&\times\frac{\prod_{m=1}^{\infty}(1-q^{n(m-1)-(b-a)}e^{-2\pi\sqrt{-1}(\lambda+\kappa)})(1-q^{nm+(b-a)}e^{2\pi\sqrt{-1}(\lambda+\kappa)})}{\prod_{m=1}^{\infty}(1-q^{n(m-1)+a}e^{-2\pi\sqrt{-1}\kappa})(1-q^{nm-a)}e^{2\pi\sqrt{-1}\kappa})}\\
&\times\frac{(1-q^{nm+b}e^{2\pi\sqrt{-1}\lambda})}{(1-q^{n(m-1)-b}e^{-2\pi\sqrt{-1}\lambda})}.
\end{aligned} \tag{2.5}$$

Here $q=e^{2\pi\sqrt{-1}\tau}$. Taking $Im\ \tau>0$ into account,

$$\lim_{n\to\infty}\tilde{S}^{ab}(\lambda)=\frac{1-q^{a-b}e^{-2\pi\sqrt{-1}(\lambda+\kappa)}}{(1-q^{a}e^{-2\pi\sqrt{-1}\kappa})(1-q^{-b}e^{-2\pi\sqrt{-1}\lambda})}. \tag{2.6}$$

3. Completely $\mathbb{Z}$ symmetric R matrix

Let us consider an infinite-dimensional R matrix,

$$R(\lambda)=\sum_{i,j,k,\ell\in\mathbb{Z}}R_{i,j}^{k,\ell}(\lambda)E_{ik}\otimes E_{j\ell}. \tag{3.1}$$

Here E_{ik} is the matrix unit in $\mathbb{Z}\times\mathbb{Z}$ matrix algebra. We impose a constraint of the completely $\mathbb{Z}$ symmetry on such an R matrix. Namely, we assume that there exists $S^{ab}(\lambda)$ $(a,b\in\mathbb{Z})$ such that

$$R_{i,j}^{k,\ell}(\lambda)=\delta_{i+j,k+\ell}S^{k-i,\ell-i}(\lambda). \tag{3.2}$$

Then the Yang-Baxter equation (1.1) reads, in terms of these weights $S^{ab}(\lambda)$ as follows:

$$
\begin{aligned}
&\sum_{k=-\infty}^{\infty} S^{k-i_1,i_2-k}(\lambda_1)S^{j_1-k,i_3-j_1}(\lambda_1+\lambda_2)S^{j_2-i_1-i_2+k,j_3-i_1-i_2+k}(\lambda_2)\\
(3.3)\qquad &= \sum_{k=-\infty}^{\infty} S^{j_1+j_2-i_2-k,j_3-i_1-i_2+k}(\lambda_2)S^{k-i_1,j_3-i_1}(\lambda_1+\lambda_2)S^{j_1-k,j_2-k}(\lambda_1),
\end{aligned}
$$

for all $i_1, i_2, i_3, j_1, j_2, j_3 \in \mathbb{Z}$ $s.t.$ $i_1+i_2+i_3 = j_1+j_2+j_3$.

In view of the limiting case $n \to \infty$ (2.6) of the completely $\mathbb{Z}_n$ symmetric R matrix, we find a solution of the Yang-Baxter equation.

Theorem 3.1. *We assume* $|q| < 1$, *and then*

$$
S^{ab}(\lambda) = -\frac{q^a}{e^{2\pi\sqrt{-1}\kappa} - q^a} + \frac{q^b}{e^{2\pi\sqrt{-1}\lambda} - q^b} \tag{3.4}
$$

is a solution of the equation (3.3). *Here* κ *is an arbitrary constant.*

Note that the both sides of equation (3.3) are absolutely convergent. We can prove the theorem above by residue calculus in a variable $u_1 \overset{\text{def}}{=} e^{2\pi\sqrt{-1}\lambda_1}$.

Remark 3.1. We put $q = e^{2\pi\sqrt{-1}\tau}$ $(Im\ \tau > 0)$, and replace κ with $\tau\kappa$ in (3.4). In the limiting case $q \to 1$,

$$
(q-1)S^{ab}(\tau\lambda)\Bigg|_{q\to 1} = -\frac{1}{\kappa - a} + \frac{1}{\lambda - b}. \tag{3.5}
$$

It is same as the weight S^{ab} given by Gaudin [8].

4. R OPERATOR

We shall realize the R matrix as an operator on some function space making use of the formula below (see [11] p.446):

$$\begin{aligned}(4.1)\qquad &\frac{1}{\pi}\frac{\vartheta_1'(0)\vartheta_1(x+y)}{\vartheta_1(x)\vartheta_1(y)}\\ &= \cot\pi x + \cot\pi y + 4\sum_{m,n=1}^{\infty} q^{mn}\sin 2\pi(mx+ny)\\ &\qquad\qquad |Im\, x| < Im\,\tau,\ |Im\, y| < Im\,\tau.\end{aligned}$$

Here $\vartheta_1(z)$ is the elliptic theta function,

$$\vartheta_1(z) = 2q^{\frac{1}{8}}\sin\pi z\prod_{m=1}^{\infty}(1-q^m)(1-2q^m\cos 2\pi z + q^{2m}). \tag{4.2}$$

Using the formula above, we can compute the Fourier transformation of the Boltzmann weight $R_{i,j}^{k,\ell}$ (cf. Gaudin [8]).

Theorem 4.1. *For $x,y\in\mathbb{R}$, $|Im\,\lambda| < Im\,\tau$, and $|Im\,\kappa| < Im\,\tau$,*

$$\begin{aligned}(4.3)\qquad &\sum_{i,j\in\mathbb{Z}} R_{i,j}^{k,\ell}(\lambda)e^{2\pi\sqrt{-1}(ix+jy)}\\ &= G(x-y:\lambda)e^{2\pi\sqrt{-1}(\ell x+ky)} - G(x-y:\kappa)e^{2\pi\sqrt{-1}(kx+\ell y)},\end{aligned}$$

where

$$G(x:\lambda) = \frac{1}{2\pi\sqrt{-1}}\frac{\vartheta_1'(0)\vartheta_1(\lambda+x)}{\vartheta_1(\lambda)\vartheta_1(x)}. \tag{4.4}$$

Let $\mathcal{V}$ be the set of C^∞-functions on the unit circle S^1, and let $\varphi_k(x) = e^{2\pi\sqrt{-1}kx}$. The set $\{\varphi_k; k\in\mathbb{Z}\}$ is a basis of $\mathcal{V}$. Then the theorem above says that the R matrix gives rise to a linear operator on $\mathcal{V} \otimes \mathcal{V}$ defined by

$$
\begin{aligned}
&(R(\lambda)(\varphi_k \otimes \varphi_\ell))(x,y) \\
\text{(4.5)} \qquad &\stackrel{\text{def}}{=} G(x-y:\lambda)(\varphi_k \otimes \varphi_\ell)(y,x) - G(x-y:\kappa)(\varphi_k \otimes \varphi_\ell)(x,y).
\end{aligned}
$$

In this case we call $R(\lambda)$ the R operator. Further the R operator can be regarded as a non-local operator on $\mathcal{V}\hat{\otimes}\mathcal{V} = C^\infty(S^1 \times S^1)$,

$$
\text{(4.6)} \qquad R(\lambda) = G(\lambda)\sigma - G(\kappa).
$$

Here $G(\lambda)$ and σ is an operator on $\mathcal{V}^{\hat{\otimes}2}$ defined by

$$
\text{(4.7)} \qquad (G(\lambda)\varphi)(x,y) = G(x-y:\lambda)\varphi(x,y),
$$

$$
\text{(4.8)} \qquad (\sigma\varphi)(x,y) = \varphi(y,x).
$$

We can see that the R operator actually belongs to $End(\mathcal{V}^{\hat{\otimes}2})$ i.e. $R(\lambda)\varphi \in \mathcal{V}^{\hat{\otimes}2}$ for $\varphi \in \mathcal{V}^{\hat{\otimes}2}$.

Now let us establish the Yang-Baxter equation for the R operator (4.6). For $N \geq 2$, we define the operator $R_{ij}(\lambda) \in End(\mathcal{V}^{\hat{\otimes}N}) \quad (1 \leq i,j \leq N, i \neq j)$.

$$
\text{(4.9)} \qquad R_{ij}(\lambda) = G_{ij}(\lambda)\sigma_{ij} - G_{ij}(\kappa),
$$

where for $\varphi \in \mathcal{V}^{\hat{\otimes}N}$

$$
\text{(4.10)} \qquad (G_{ij}(\lambda)\varphi)(x_1, x_2, \ldots, x_N) = G(x_i - x_j : \lambda)\varphi(x_1, x_2, \ldots, x_N),
$$

$$
\text{(4.11)} \qquad (\sigma_{ij}\varphi)(\ldots, x_i, \ldots, x_j, \ldots) = \varphi(\ldots, x_j, \ldots, x_i, \ldots).
$$

Theorem 4.2. *$R(\lambda)$ satisfies the Yang-Baxter equation* (1.1) *in End*$(\mathcal{V}^{\hat{\otimes}3})$. *Namely the operators defined by* (4.9) *for* $N = 3$ *satisfy*

$$R_{12}(\lambda_1)R_{13}(\lambda_1+\lambda_2)R_{23}(\lambda_2) = R_{23}(\lambda_2)R_{13}(\lambda_1+\lambda_2)R_{12}(\lambda_1). \tag{4.12}$$

This theorem can be verified in a slightly more general context.

Proposition 4.3. *If an analytic function $\theta(x)$ satisfies the three term equation*

$$\begin{aligned} &\theta(x+y)\theta(x-y)\theta(z+w)\theta(z-w) \\ &+\theta(x+z)\theta(x-z)\theta(w+y)\theta(w-y) \\ &+\theta(x+w)\theta(x-w)\theta(y+z)\theta(y-z) \\ &=0, \end{aligned} \tag{4.13}$$

then the operator

$$R(\lambda) \stackrel{\text{def}}{=} G(\lambda)\sigma - G(\kappa) \tag{4.14}$$

is a solution of the Yang-Baxter equation in the same sense as in Theorem 4.2. *Here*

$$(G(\lambda)\varphi)(x,y) = \frac{\theta'(0)\theta(\lambda+x-y)}{\theta(\lambda)\theta(x-y)}\varphi(x,y) \tag{4.15}$$

for a function $\varphi(x,y)$.

Remark 4.1. (1) The elliptic theta function $\vartheta_1(x)$ satisfies (4.13). It is actually Fay's trisecant formula (see [6] p.33–35). It is worthwhile noticing that the prime form on an elliptic curve is given by

$$E(x,y)\sqrt{dxdy} = \frac{\vartheta_1(y-x)}{\vartheta_1'(0)}. \tag{4.16}$$

(2) Analytic solutions of the three term equation (4.13) are given by

$$\theta(x) = \vartheta_1(x)\exp(\frac{1}{2}Ax^2 + B), \tag{4.17}$$

$$\theta(x) = \sin(\pi x)\exp(\frac{1}{2}Ax^2 + B), \tag{4.18}$$

$$\theta(x) = x\exp(\frac{1}{2}Ax^2 + B), \tag{4.19}$$

where A and B are arbitrary constants (see [18] p.461). In the situation of Proposition 4.3, we simply assume $\mathcal{V}$ to be a space of functions with one variable and $\mathcal{V}\hat{\otimes}\mathcal{V}$ to be a space of functions with two variables, respectively.

In what follows, we shall assume that the R operator is the generalized one in Proposition 4.3. We state the first inversion relation for R.

Proposition 4.4 (The first inversion relation).

$$R_{12}(\lambda)R_{21}(-\lambda) = \rho(\lambda)id, \tag{4.20}$$

where

$$\rho(\lambda) = \begin{cases} \vartheta_1'(0)^2\dfrac{\vartheta_1(\lambda+\kappa)\vartheta_1(\lambda-\kappa)}{\vartheta_1^2(\lambda)\vartheta_1^2(\kappa)} & \textit{in case } (4.17)\ , \\ \pi^2(\cot^2\pi\kappa - \cot^2\pi\lambda) & \textit{in case } (4.18)\ , \\ \dfrac{1}{\kappa^2} - \dfrac{1}{\lambda^2} & \textit{in case } (4.19)\ . \end{cases} \tag{4.21}$$

5. Fusion procedure

From the definition of the operator $R(\lambda)$ on $\mathcal{V}^{\hat{\otimes}2}$, we obtain

$$R(\kappa) = -2G(\kappa)P^{(-)}, \tag{5.1}$$

$$R(-\kappa) = -2P^{(+)}G(\kappa). \tag{5.2}$$

Here $P^{(+)} = \frac{1}{2}(1+\sigma)$ and $P^{(-)} = \frac{1}{2}(1-\sigma)$ are the projectors on $S^{(+)}(\mathcal{V}^{\hat{\otimes}2})$ the space of symmetric functions and $S^{(-)}(\mathcal{V}^{\hat{\otimes}2})$ the space of anti-symmetric functions, respectively. Specializing the spectral parameter in the Yang-Baxter equation (4.12), we get

$$\begin{aligned} R_{13}(\lambda+\kappa)R_{23}(\lambda)P^{(+)}_{12} &= P^{(+)}_{12}R_{13}(\lambda+\kappa)R_{23}(\lambda)P^{(+)}_{12} \\ &\qquad (\lambda_1=\kappa, \lambda_2=\lambda), \end{aligned} \tag{5.3}$$

$$\begin{aligned} R_{13}(\lambda)R_{12}(\lambda-\kappa)P^{(+)}_{23} &= P^{(+)}_{23}R_{13}(\lambda)R_{12}(\lambda-\kappa)P^{(+)}_{23} \\ &\qquad (\lambda_1=\lambda, \lambda_2=-\kappa). \end{aligned} \tag{5.4}$$

Taking the equations above into account, we can apply the fusion procedure for vertex models which was developed in [12] (see also [3] and [10]), to our case.

We define a product of the R operators on the function space $\mathcal{V}^{\hat{\otimes}L}\hat{\otimes}\mathcal{V}'^{\hat{\otimes}M} = \{\varphi(x_1,\dots,x_L : y_1,\dots,y_M)\}$:

$$\begin{cases} R_{\ell:1'\dots M'}(\lambda) = R_{\ell:M'}(\lambda)R_{\ell:M'-1}(\lambda-\kappa)\dots R_{\ell:1'}(\lambda-(M-1)\kappa), \\ R_{1\dots L:1'\dots M'}(\lambda) = R_{1:1'\dots M'}(\lambda+(L-1)\kappa)\dots R_{L:1'\dots M'}(\lambda), \end{cases} \tag{5.5}$$

where $\mathcal{V}'$ is a copy of $\mathcal{V}$. $R_{\ell:k'}(\lambda)$ indicates the action on the variables x_ℓ and y_k. Let

$S^{(+)}(\mathcal{V}^{\hat{\otimes} L})$ and $S^{(-)}(\mathcal{V}^{\hat{\otimes} L})$ be the spaces of symmetric functions and anti-symmetric functions, respectively, and let $P^{(\pm)}_{1\dots L}$ be the projector onto $S^{(\pm)}(\mathcal{V}^{\hat{\otimes} L})$,

$$P^{(\pm)}_{1\dots L} = \sum_{w \in \mathfrak{S}_L} (\pm 1)^{\ell(w)} w. \tag{5.6}$$

Here we denote by $\mathfrak{S}_L$ the symmetric group and by $\ell(w)$ the length of w. For $\varepsilon_1, \varepsilon_2 = (+), (-)$, define the operator $R^{\varepsilon_1\,\varepsilon_2}_{(L):(M')}(\lambda)$ by

$$R^{\varepsilon_1\,\varepsilon_2}_{(L):(M')}(\lambda) = P^{\varepsilon_1}_{1\dots L} P^{\varepsilon_2}_{1'\dots M'} R_{1\dots L:1'\dots M'}(\lambda) P^{\varepsilon_1}_{1\dots L} P^{\varepsilon_2}_{1'\dots M'}. \tag{5.7}$$

Theorem 5.1. *In* $End(S^{\varepsilon_1}(\mathcal{V}^{\hat{\otimes} L}) \hat{\otimes} S^{\varepsilon_2}(\mathcal{V}'^{\hat{\otimes} M}) \hat{\otimes} S^{\varepsilon_3}(\mathcal{V}''^{\hat{\otimes} N}))$,

$$\begin{aligned} & R^{\varepsilon_1\,\varepsilon_2}_{(L):(M')}(\lambda_1) R^{\varepsilon_1\,\varepsilon_3}_{(L):(N'')}(\lambda_1+\lambda_2) R^{\varepsilon_2\,\varepsilon_3}_{(M'):(N'')}(\lambda_2) \\ &= R^{\varepsilon_2\,\varepsilon_3}_{(M'):(N'')}(\lambda_2) R^{\varepsilon_1\,\varepsilon_3}_{(L):(N'')}(\lambda_1+\lambda_2) R^{\varepsilon_1\,\varepsilon_2}_{(L):(M')}(\lambda_1), \end{aligned} \tag{5.8}$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3 = (+), (-)$, *and* $\mathcal{V}''$ *is a copy of* $\mathcal{V}$.

In a forthcoming paper, we will discuss that the equation (5.8) corresponds to which functional equation of the elliptic theta function.

Example. We give an explicit formula for $R^{(+)}_{1:(M')}$, which is the case $L = 1$, $\varepsilon_2 = (+)$ in (5.7).

$$R^{(+)}_{1:(M')}(\lambda) = \frac{1}{M!} \sum_{j=1}^{M} \sigma^{j-1}_{1'\dots M'} G^{(+)}_{1:(M')} \sigma_{11'} + (-1)^M \prod_{j=1}^{M} G_{1j'}(\kappa), \tag{5.9}$$

where

$$(\sigma_{11'}\varphi)(x_1 : y_1, \dots, y_M) = \varphi(y_1 : x_1, y_2, \dots, y_M), \tag{5.10}$$

$$(\sigma_{1'\dots M'}\varphi)(x_1 : y_1, \dots, y_M) = \varphi(x_1 : y_2, \dots, y_M, y_1), \tag{5.11}$$

and $G^{(+)}_{1:(M')}$ is a multiplication operator defined inductively by

$$
\begin{aligned}
&G^{(+)}_{1:(M')}(x : y_1, \dots, y_M : \lambda) \\
&= \sum_{j=2}^{M} \{ G(x_1 - y_j : \lambda) G^{(+)}_{1:((M-1)')}(y_j : y_1, \dots, \hat{y}_j \dots, y_M : \lambda - \kappa) \\
&\qquad - G(x_1 - y_j : \kappa) G^{(+)}_{1:((M-1)')}(x_1 : y_1, \dots, \hat{y}_j \dots, y_M : \lambda - \kappa) \} \\
&+ (-1)^{M-1}(M-1)! G(x_1 - y_1 : \lambda) \prod_{j=2}^{M} G(y_1 - y_j : \kappa).
\end{aligned} \tag{5.12}
$$

6. Finite-dimensional representation of R operator

First we formulate the notion of finite-dimensional representations of the R operator. Let $\mathfrak{F}$ be a finite index set, and let $V^{\mathfrak{F}} = \oplus_{\alpha \in \mathfrak{F}} \mathbb{C} f_\alpha$ be a finite-dimensional subspace of $\mathcal{V}$ with a basis $\{f_\alpha : \alpha \in \mathfrak{F}\}$.

Definition 6.1. If the R operator preserves $V^{\mathfrak{F}} \otimes V^{\mathfrak{F}}$, then we call $V^{\mathfrak{F}} \otimes V^{\mathfrak{F}}$ a finite-dimensional representation of $R(\lambda)$, and define $R^{\mathfrak{F}}(\lambda)$ to be the matrix representation of $R(\lambda)|_{V^{\mathfrak{F}} \otimes V^{\mathfrak{F}}} \in End(V^{\mathfrak{F}} \otimes V^{\mathfrak{F}})$ with respect to the basis $\{f_\alpha \otimes f_\beta\}$.

We should remark that $R^{\mathfrak{F}}(\lambda)$ automatically becomes a solution of the Yang-Baxter equation (1.1).

Let us construct finite-dimensional representations of the trigonometric $R(\lambda)$ (4.18) in Proposition 4.3 with $A = B = 0$. (We can also obtain finite-dimensional representations of the rational $R(\lambda)$ (4.19) with $A = B = 0$.)

Proposition 6.1. *For $n \geq 1$, we set $\mathfrak{F} = \{0, 1, \dots, n-1\}$, $f_\alpha(x) = e^{2\pi\sqrt{-1}\alpha x}$ $(\alpha \in \mathfrak{F})$,* *and $V^{\mathfrak{F}} = \oplus_{\alpha\in\mathfrak{F}}\mathbb{C}f_\alpha$. Then $V^{\mathfrak{F}}\otimes V^{\mathfrak{F}}$ is a finite-dimensional representation.*

Proof. This proposition follows immediately from the action of $R(\lambda)$ on $V^{\mathfrak{F}}\otimes V^{\mathfrak{F}}$. Set $u = e^{2\pi\sqrt{-1}\lambda}$, $t = e^{2\pi\sqrt{-1}\kappa}$, and $R^{(n)}(\lambda) = \frac{1}{2\sqrt{-1}}(u^{\frac{1}{2}} - u^{-\frac{1}{2}})(t^{\frac{1}{2}} - t^{-\frac{1}{2}})R(\lambda)$, then

$$R^{(n)}(\lambda)(f_\alpha\otimes f_\beta)$$

$$(6.1)\qquad = \begin{cases} u^{-\frac{1}{2}}(t^{\frac{1}{2}} - t^{-\frac{1}{2}})f_\beta\otimes f_\alpha - t^{\frac{1}{2}}(u^{\frac{1}{2}} - u^{-\frac{1}{2}})f_\alpha\otimes f_\beta \\ +(u^{\frac{1}{2}} - u^{-\frac{1}{2}})(t^{\frac{1}{2}} - t^{-\frac{1}{2}})\sum_{\beta<\gamma<\alpha} f_{\alpha+\beta-\gamma}\otimes f_\gamma & (\alpha>\beta), \\ (u^{-\frac{1}{2}}t^{\frac{1}{2}} - u^{\frac{1}{2}}t^{-\frac{1}{2}})f_\alpha\otimes f_\alpha & (\alpha=\beta), \\ u^{\frac{1}{2}}(t^{\frac{1}{2}} - t^{-\frac{1}{2}})f_\beta\otimes f_\alpha - t^{-\frac{1}{2}}(u^{\frac{1}{2}} - u^{-\frac{1}{2}})f_\alpha\otimes f_\beta \\ +(u^{\frac{1}{2}} - u^{-\frac{1}{2}})(t^{\frac{1}{2}} - t^{-\frac{1}{2}})\sum_{\alpha<\gamma<\beta} f_\gamma\otimes f_{\alpha+\beta-\gamma} & (\alpha<\beta). \end{cases}$$

□

These finite-dimensional R matrices are new solutions of the Yang-Baxter equation. But these are not triangular. Now we discuss how to obtain the trigonometric, triangular R matrix from this matrix $R^{(n)}(\lambda)$.

Definition 6.2. Let $V = \oplus\mathbb{C}v_\alpha$ be a finite-dimensional vector space. For $T\in End(V^{\otimes N})$, we denote

$$(6.2)\qquad T(v_{\alpha_1}\otimes\cdots\otimes v_{\alpha_N}) = \sum_{\beta_1\ldots\beta_N} T^{\alpha_1\ldots\alpha_N}_{\beta_1\ldots\beta_N} v_{\beta_1}\otimes\cdots\otimes v_{\beta_N}.$$

Then we define $T_{(P)} \in End(V^{\otimes N})$ by

$$T_{(P)}(v_{\alpha_1} \otimes \cdots \otimes v_{\alpha_N}) = \sum_{w \in \mathfrak{S}_N} T^{\alpha_1 \ldots \alpha_N}_{\alpha_{w(1)} \ldots \alpha_{w(N)}} v_{\alpha_{w(1)}} \otimes \cdots \otimes v_{\alpha_{w(N)}}, \tag{6.3}$$

which is called the permutation part of T with respect to the basis $\{v_\alpha\}$.

Proposition 6.2. *The permutation part $R^{(n)}_{(P)}(\lambda)$ of $R^{(n)}(\lambda)$ with respect to the basis $\{f_\alpha\}$ satisfies the Yang-Baxter equation* (1.1).

Proof. From (1.1),

$$(R^{(n)}_{12}(\lambda_1)R^{(n)}_{13}(\lambda_1+\lambda_2)R^{(n)}_{23}(\lambda_2))_{(P)} = (R^{(n)}_{23}(\lambda_2)R^{(n)}_{13}(\lambda_1+\lambda_2)R^{(n)}_{12}(\lambda_1))_{(P)}. \tag{6.4}$$

By using (6.1), we easily see that

(6.5)

$$(R^{(n)}_{12}(\lambda_1)R^{(n)}_{13}(\lambda_1+\lambda_2)R^{(n)}_{23}(\lambda_2))_{(P)} = R^{(n)}_{(P)12}(\lambda_1)R^{(n)}_{(P)13}(\lambda_1+\lambda_2)R^{(n)}_{(P)23}(\lambda_2),$$

(6.6)

$$(R^{(n)}_{23}(\lambda_2)R^{(n)}_{13}(\lambda_1+\lambda_2)R^{(n)}_{12}(\lambda_1))_{(P)} = R^{(n)}_{(P)23}(\lambda_2)R^{(n)}_{(P)13}(\lambda_1+\lambda_2)R^{(n)}_{(P)12}(\lambda_1).$$

This completes the proof of Proposition 6.2. □

The permutation part $R^{(n)}_{(P)}(\lambda)$ acts on $V^{\mathfrak{F}} \otimes V^{\mathfrak{F}}$ in the following way:

$$\begin{aligned} &R^{(n)}_{(P)}(\lambda)(f_\alpha \otimes f_\beta) \\ &= \begin{cases} u^{-\frac{1}{2}}(t^{\frac{1}{2}} - t^{-\frac{1}{2}}) f_\beta \otimes f_\alpha - t^{\frac{1}{2}}(u^{\frac{1}{2}} - u^{-\frac{1}{2}}) f_\alpha \otimes f_\beta & (\alpha > \beta), \\ (u^{-\frac{1}{2}}t^{\frac{1}{2}} - u^{\frac{1}{2}}t^{-\frac{1}{2}}) f_\alpha \otimes f_\alpha & (\alpha = \beta), \\ u^{\frac{1}{2}}(t^{\frac{1}{2}} - t^{-\frac{1}{2}}) f_\beta \otimes f_\alpha - t^{-\frac{1}{2}}(u^{\frac{1}{2}} - u^{-\frac{1}{2}}) f_\alpha \otimes f_\beta & (\alpha < \beta). \end{cases} \end{aligned} \tag{6.7}$$

Therefore $R^{(n)}_{(P)}(\lambda)$ is a trigonometric, triangular R matrix, that is relavant to the one given in [1].

Remark 6.1. The idea of finite-dimensional representations of the R operator originates from the paper of Gaudin [8]. We reformulated his idea in an algebraic way.

7. Discussions

We propose some issues to be considered hereafter:

The first one is to calculate rigorously the free energy of our vertex model. This problem may be relavant to the c-functions of the Macdonald's symmetric functions (cf. [7] [19]). Our original motivations of this study is to construct a two-parameter deformation of $U(gl_\infty)$, so it is important to consider the algebra of the L operators associated to our R operator. The third problem concerns finite-dimensional representations of the R operator; namely, can we construct finite-dimensional representations in the elliptic case? Furthermore we think of it interesting to generalize the R operator to higher genus case.

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