数理解析研究所講究録
第 810 巻 1992 年 65-78

W ALGEBRA, TWISTOR, AND NONLINEAR INTEGRABLE SYSTEMS

高崎金久

KANEHISA TAKASAKI[♭]

INSTITUTE OF MATHEMATICS, YOSHIDA COLLEGE, KYOTO UNIVERSITY

Yoshida-Nihonmatsu-cho, Sakyo-ku, Kyoto 606, Japan

Abstract. W algebras arise in the study of various nonlinear integrable systems such as: self-dual gravity, the KP and Toda hierarchies, their quasi-classical (or dispersionless) limit, etc. Twistor theory provides a geometric background for these algebras. Present state of these topics is overviewed. A few ideas on possible deformations of self-dual gravity (including quantum deformations) are presented.

1. INTRODUCTION

The dramatic progress of the 2D gravity/string theory in the last few years [1] has revealed a new relation of field theory to integrable hierarchies of KdV, KP and Toda lattice type. The theory of nonlinear integrable systems has thus again proven its usefulness in physics. It is Virasoro and W symmetries of these integrable hierarchies that plays a central role in characterizing these models of gravity and strings as special solutions of an integrable hierarchy.

The structure of Virasoro and W symmetries arises in several different (but actually equivalent) forms. Firstly, the partition function Z (or its square root $\sqrt{Z}$) of these

[♭]E-mail: takasaki@jpnyitp.bitnet, takasaki@kurims.kyoto-u.ac.jp

models can be identified with the tau function τ of an integrable hierarchy, and satisfies a set of linear constraints (Virasoro constraints) of the form

$$\mathcal{L}_n \tau = 0, \quad n \geq -1 \tag{1}$$

for a set of Virasoro symmetry generators $\mathcal{L}_n$; these constraints can be further generalized to W algebraic analogues (W constraints)[2]. Secondly, the same model of 2D gravity can be reproduced from the canonical commutation relation (Douglas equation) [3]

$$[P, Q] = 1 \tag{2}$$

of two ordinary differential operators in one variable. A third expression is due to the Schwinger-Dyson equation (or loop equations) for loop correlation functions [4]. Although less obvious, the latter two expressions of 2D gravity, too, stem from Virasoro and W symmetries.

W symmetries also exist in the self-dual vacuum Einstein equation [5], a higher dimensional nonlinear integrable system. Not only being an integrable model of 4D gravity (self-dual gravity), this equation (as well as its hyper-Kähler versions in 4k dimensions) has also been extensively studied in the context of supersymmetric nonlinear sigma models [6], relativistic membranes [7], SU(∞) Toda fields [8] etc., and very recently, as an effective theory of $N = 2$ strings [9]. These diverse models of field theory may be thought of as higher dimensional counterparts of the above mentioned models of the 2D gravity/string theory.

Our basic standpoint is that W symmetries (in particular, $W_{1+\infty}$, $w_{1+\infty}$ and their variations [10]) provide us with a unified framework for understanding these nonlinear integrable systems. We start with a brief review of W algebraic structures in self-dual gravity, then turn to similar results on the KP and Toda hierarchies and their quasi-classical limits. In the final section, we shall present a few ideas on deformations of self-dual gravity and associated W algebraic structures.

2. Self-dual gravity

W algebraic structures of self-dual gravity can be deduced from Penrose's twistor theoretical approach (nonlinear graviton construction) [11]. To see this, it is convenient to start from the Plebanski equation [12]

$$\Omega_{,p\bar{p}}\Omega_{,q\bar{q}} - \Omega_{,p\bar{q}}\Omega_{,q\bar{p}} = 1 \tag{3}$$

where Ω is a Kähler potential and $p, q, \bar{p}, \bar{q}$ are suitably chosen complex coordinates. This equation (actually known to mathematicians before Plebanski) represents Ricci-flatness of a Kähler metric. We now introduce a new variable λ (known in the theory of nonlinear integrable systems as "spectral parameter") and, following Gindikin [13], make a linear combination

$$\omega(\lambda) \underset{\text{def}}{=} d\bar{p} \wedge d\bar{q} + \lambda\omega + \lambda^2 dp \wedge dq \tag{4}$$

of the holomorphic 2-form $dp \wedge dq$, the anti-holomorphic 2-form $d\bar{p} \wedge d\bar{q}$ and the Kähler form $\omega =_{\text{def}} \Omega_{,p^i\bar{p}^j}$, $p^i = (p, q)$, $\bar{p}^i = (\bar{p}, \bar{q})$. The Plebanski equation can be now cast into the exterior differential equations

$$d\omega(\lambda) = 0, \quad \omega(\lambda) \wedge \omega(\lambda) = 0 \quad (d\lambda = 0), \tag{5}$$

where d stands for total differential in $(p, q, \bar{p}, \bar{q})$ viewing λ a constant.

By a classical theorem of Darboux, one can find two "Darboux coordinates" $P(\lambda)$ and $Q(\lambda)$ as

$$\omega(\lambda) = dP(\lambda) \wedge dQ(\lambda) \quad (d\lambda = 0). \tag{6}$$

In particular, these Darboux coordinates give a canonical conjugate pair

$$\{P(\lambda), Q(\lambda)\}_{\bar{p},\bar{q}} = 1 \tag{7}$$

for the Poisson bracket $\{F, G\}_{\bar{p},\bar{q}} =_{\text{def}} F_{,\bar{p}}G_{,\bar{q}} - F_{,\bar{q}}G_{,\bar{p}}$. Actually, these Darboux coordinates are not unique, but allow transformations

$$P(\lambda), Q(\lambda) \longrightarrow f\big(\lambda, P(\lambda), Q(\lambda)\big), g\big(\lambda, P(\lambda), Q(\lambda)\big) \tag{8}$$

by a two-dimensional symplectic (i.e, area-preserving) diffeomorphism depending also on λ; f and g are required to have a unit Jacobian for the second and third variables, hence defines an area-preserving diffeomorphism with parameter λ. The relevance of a W algebra (in this case, $w_{1+\infty}$) is already manifest.

Penrose's idea is to consider two special pairs of Darboux coordinates, say $\mathcal{U}(\lambda), \mathcal{V}(\lambda)$ and $\hat{\mathcal{U}}(\lambda), \hat{\mathcal{V}}(\lambda)$, with different complex analytic properties with respect to λ. These Darboux coordinate systems are then linked with each other by a symplectic mixing as in (8):

$$f(\lambda, \mathcal{U}(\lambda), \mathcal{V}(\lambda)) = \hat{\mathcal{U}}(\lambda), \quad g(\lambda, \mathcal{U}(\lambda), \mathcal{V}(\lambda)) = \hat{\mathcal{V}}(\lambda), \tag{9}$$

and this gives a Riemann-Hilbert problem in the group SDiff(2) of area preserving diffeomorphisms. The data (f, g), which then becomes an element of the loop group $\mathcal{L}$SDiff(2) of SDiff(2), is exactly Penrose's twistor data, and conversely, solving the above Riemann-Hilbert problem (which is generally a hard task though) for a given data give rise to all (local) solutions of self-dual gravity .

The existence of a large set of symmetries is now an obvious consequence of the $\mathcal{L}$SDiff(2) group structure in the twistor data (f, g): the action of this loop group on itself (from left or right) gives transformations of the corresponding solution of self-dual gravity via the Riemann-Hilbert problem. Infinitesimal symmetries accordingly have the structure of the loop algebra of $w_{1+\infty}$.

Algebraic structures found in self-dual gravity are thus more or less reminiscent of 2D gravity as well as W gravity [14], in which W algebras (both quantum and quasi-classical) give basic symmetries. This is also the case for dimensionally reduced models of 4D self-dual gravity [15]. Note, however, that the full symmetry algebra of 4D self-dual gravity is the loop algebra of $w_{1+\infty}$, far larger than $w_{1+\infty}$ itself. This reflects a higher dimensional characteristic of self-dual gravity, and suggests a possible direction of extending the notion of W algebras. We shall return to this issue in the final section.

3. KP HIERARCHY AND CANONICAL CONJUGATE PAIR

We have seen that a canonical conjugate pair, $\mathcal{U}$ and $\mathcal{V}$, takes place in the description of general solutions of self-dual gravity. The KP hierarchy, too, turns out to have a similar pair (of pseudo-differential operators).

The KP hierarchy, by definition, describes a commuting set of isospectral flows

$$\frac{\partial L}{\partial t_n} = [B_n, L], \quad B_n \underset{\text{def}}{=} (L^n)_{\geq 0}, \quad n = 1, 2, \ldots, \tag{10}$$

of a one-dimensional pseudo-differential operator

$$L \underset{\text{def}}{=} \partial + \sum_{n=1}^{\infty} u_{n+1}\partial^{-n}, \quad \partial \underset{\text{def}}{=} \partial/\partial x, \tag{11}$$

where $(\quad)_{\geq 0}$ stands for dropping negative powers of ∂ to obtain a differential operator. This is the ordinary Lax formalism of the KP hierarchy.

We need some other variables to describe the $W_{1+\infty}$ symmetries explicitly. One way is to use the tau function to realize those symmetries as linear differential operators $W_n^{(s)}$, $s = 1, 2, \ldots$, $n \in \mathbf{Z}$, in the t's [16]. Another way is to introduce a pseudo-differential operator of the form

$$M = \sum_{n=1}^{\infty} n t_n L^{n-1} + O(\partial^{-1}) \tag{12}$$

that satisfy the Lax equations

$$\frac{\partial M}{\partial t_n} = [B_n, M], \quad n = 1, 2, \ldots, \tag{13}$$

and the canonical commutation relation

$$[L, M] = 1. \tag{14}$$

Such a second Lax operator does exists, and arises in the linear system of the so called Baker-Akhiezer function $\psi = \psi(x, t, \lambda)$ as:

$$L\psi = \lambda\psi, \quad M\psi = \frac{\partial \psi}{\partial \lambda}, \quad \frac{\partial \psi}{\partial t_n} = B_n\psi. \tag{15}$$

The $W_{1+\infty}$ symmetries of the KP hierarchy can be reformulated as symmetries acting on this (L, M) pair [17].

The above description of $W_{1+\infty}$ symmetries elucidates an origin of (P, Q) pairs in $d \leq 1$ string theory [18]. The Douglas pair for $d < 1$ strings, indeed, is given by a (noncommutative) canonical transformation

$$P = L^p, \quad Q = ML^{1-p}/p + h(L) \tag{16}$$

of the (L, M) pair under the constraints

$$P = (P)_{\geq 0}, \quad Q = (Q)_{\geq 0}, \tag{17}$$

where $h(L) = \sum h_n L^n$ with suitable constant coefficients h_n. At the (p,q) critical point, the time variables are restricted to so called "small phase space": $t_{p+q} = p/(p+q)$, $t_{p+q+1} = t_{p+q+2} = \cdots = 0$.

4. Quasi-classical limit of KP hierarchy

The KP hierarchy has a quasi-classical (or dispersionless) limit [19]. This is a system of Lax type,

$$\frac{\partial \mathcal{L}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{L}\}_{k,x}, \quad \mathcal{B}_n = (\mathcal{L}^n)_{\geq 0}, \quad n = 1, 2, \ldots, \tag{18}$$

where $\mathcal{L}$, a quasi-classical counterpart of L, is a Laurent series of the form

$$\mathcal{L} \underset{\text{def}}{=} k + \sum_{n=1}^{\infty} u_{n+1} k^{-n}, \tag{19}$$

k is a parameter like λ, $(\quad)_{\geq 0}$ now means dropping all negative powers of k, and $\{\ ,\ \}_{k,x}$ the Poisson bracket in (k,x): $\{F, G\}_{k,x} =_{\text{def}} F_{,k}G_{,x} - F_{,x}G_{,k}$. As in the case of the KP hierarchy, one can introduce a second Laurent series $\mathcal{M} = \sum_{n=1}^{\infty} n t_n \mathcal{L}^{n-1} + O(k^{-1})$ that obeys similar equations,

$$\frac{\partial \mathcal{M}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{M}\}_{k,x}, \quad \{\mathcal{L}, \mathcal{M}\}_{k,x} = 1. \tag{20}$$

The above hierarchy (dispersionless or semi-classical KP hierarchy) gives a quasi-classical limit of the KP hierarchy in the following sense. Introduce a Planck constant $\hbar$ into the KP hierarchy and the associated linear system by replacing

$$\partial = \frac{\partial}{\partial x} \to \hbar \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial t_n} \to \hbar \frac{\partial}{\partial t_n}, \quad \frac{\partial}{\partial \lambda} \to \hbar \frac{\partial}{\partial \lambda} \tag{21}$$

and assume a quasi-classical (WKB) asymptotic form of the the Baker-Akhiezer function,

$$\psi(\hbar, x, t, \lambda) \sim \exp \hbar^{-1} S(x, t, \lambda). \tag{22}$$

The linear system then gives rise to a set of eikonal (or Hamilton-Jacobi) equations for the phase function S, which after somewhat lengthy calculations turn out to be equivalent to the above equations of the $(\mathcal{L}, \mathcal{M})$ pair. Futher, the tau function has accordingly an asymptotic form

$$\tau \sim \exp[\hbar^{-2} F(x,t) + O(\hbar^{-1})]. \tag{23}$$

In view of the relation to matrix models of 2D gravity in large-N limit [1], the function F should be called the "free energy" of the semi-classical KP hierarchy. Its exponential $\exp F$ gives exactly the tau function introduced in Ref. 20. A set of $w_{1+\infty}$ (= SDiff(2)) symmetries are also constructed in the same paper.

A quasi-classical limit of the Douglas pair (P, Q) is given by

$$\mathcal{P} \underset{\text{def}}{=} \mathcal{L}^p, \quad \mathcal{Q} \underset{\text{def}}{=} \mathcal{M}\mathcal{L}^{1-p}/p + h(\mathcal{L}), \tag{24}$$

and constrained by

$$\mathcal{P} = (\mathcal{P})_{\geq 0}, \quad \mathcal{Q} = (\mathcal{Q})_{\geq 0}. \tag{25}$$

The "genus zero" part of 2D gravity [1] as well as "topological minimal models" [21] are included into this family of solutions.

5. Toda lattice and its quasi-classical limit

What we have seen in the previous two sections persists in the Toda lattice hierarchy. The relativistic Toda field theory is given by the equation of motion

$$\frac{\partial^2 \Phi_n}{\partial z \partial \bar{z}} + \exp(\Phi_{n+1} - \Phi_n) - \exp(\Phi_n - \Phi_{n-1}) = 0, \quad n \in \mathbf{Z}. \tag{26}$$

In quasi-classical limit, the discrete variable n, too, has to be scaled as $\hbar n = s$ [22], and one obtains the equation

$$\frac{\partial^2 \Phi}{\partial z \partial \bar{z}} + \frac{\partial}{\partial s} \exp \frac{\partial \Phi}{\partial s} = 0 \tag{27}$$

for a three-dimensional field $\Phi = \Phi(z, \bar{z}, s)$. Because of this correspondence, the above equation [also called SU(∞) Toda equation] has been studied in detail by the methods of conformal field theories [8] and nonlinear integrable systems [23]. The notion of tau function (or free energy) and $w_{1+\infty}$ symmetries, too, have been established [24].

Quite accidentally, the same equation describes a dimensional reduction of self-dual gravity by a rotational S^1 symmetry, as first pointed out by relativists [25]. In their interpretation, remarkably, the Φ field is nothing but the radial coordinate of a 4D cylindrical coordinate system; a Legendre-like transformation converts it into a dependent variable. This is somewhat reminiscent of the fact [26] that the Liouville mode in a subcritical string theory can be interpreted as a time-like coordinate in a critical string theory. The Φ field might be a kind of Liouville mode in higher dimensional strings or membranes (in suitable quantization, if necessary).

We have seen that the semi-classical version of the KP/Toda hierarchy has two distinct characteristics in itself: In one hand, it has a Lax formalism very similar to the ordinary KP/Toda hierarchy; on the other hand, it has a pair of canonical conjugate variables like those in self-dual gravity. The corresponding twistor theory is a kind of "minitwistor theory" [27] associated with a two (rather than three) complex dimensional twistor space. Naturally, one may imagine that a higher dimensional analogue of the KP/Toda hierarchy should exist and reproduce self-dual gravity as a kind of quasi-classical limit. We now turn to this issue.

6. Deformations of self-dual gravity

The ordinary twistor theoretical framework based on 3D twistor spaces already provides us with a wide range of deformations of self-dual gravity. Penrose's nonlinear graviton construction, indeed, covers all conformally self-dual spaces. An interesting subfamily of deformations describing an Einstein-Maxwell theory is proposed by Flaherty [28] and recently studied by Park [29]. It recently turned out that a group of volume-preserving diffeomorphisms, SDiff(3), underlies this family of deformations and plays the same role as the $\mathcal{L}$SDiff(2) group in self-dual gravity [30]. This kind of deformations associated with an SDiff(3) group deserve further study.

Another idea, which might lead to quantum deformations, is to generalize the correspondence between the KP hierarchy and its quasi-classical version to self-dual gravity. As already mentioned, the correspondence

$$\text{KP/Toda hierarchy} \underset{\hbar\to 0}{\longrightarrow} \text{quasi-classical KP/Toda hierarchy}$$

strongly suggests a higher dimensional analogue such as

$$? \quad \underset{\hbar\to 0}{\longrightarrow} \text{4D self-dual gravity}$$

At the place of "?" should come a kind of quantization of self-dual gravity and twistor theory. A symmetry algebra coming into the place of "?" should be a quantum deformation of the loop algebra $\mathcal{L}w_{1+\infty}$ of $w_{1+\infty}$.

A candidate of quantum deformations of $\mathcal{L}w_{1+\infty}$ is the loop algebra $\mathcal{L}W_{1+\infty}$ of $W_{1+\infty}$. We do not know what an associated deformation of self-dual gravity looks like. Recent diverse proposals for a 3D field formulation of $d = 1$ matrix models [31] are very suggestive in that respect. Also interesting are a family of nonlinear integrable systems recently presented by Hoppe et al [32]; Lax representations of these models exploit the $W_{1+\infty}$ algebra or the Moyal algebra [33] in a quite explicit way. This will also be related to the star-product membrane theory [34], a deformation of relativistic membrane theory with a Moyal bracket replacing a Poisson bracket in its Hamiltonian density. Such a possible link with membrane theory is very significant, because Ooguri and Vafa [9] point out that $N = 2$ strings look like membranes.

Another possible deformation of $\mathcal{L}w_{1+\infty}$ might be due to the notion of "quantized spectral parameters" [35]. A basic idea of this notion is, roughly, to replace a spectral (i.e., loop) parameter, say ζ, by an operator of the form

$$\hat{\zeta} = \zeta \exp(-\hbar\partial/\partial x),$$

where x is a space variable like that of the KP hierarchy. One can indeed derive such an operator from a reduction of the Toda lattice (or, rather, modified KP) hierarchy [36]. If the loop algebra $\mathcal{L}w_{1+\infty}$ can be deformed to a "quantized" loop algebra with such a "quantized spectral parameter," an associated deformation of self-dual gravity, if exists, will naturally include the extra variable x within its independent variables. This is a very interesting possibility, because the deformed self-dual gravity then will have a direct connection with a KP-type hierarchy, hence will offer a hint to unify self-dual gravity with KP-type hierarchies.

A similar idea of deformations of self-dual gravity can be found in a paper of Bakas and Kiritsis [37]. They first introduce an extension W_∞^N of W_∞ (or rather $W_{1+\infty}$) with U(N) inner symmetries, and point out that the large-N limit W_∞^∞ of W_∞^N will become isomorphic to the SpDiff(4) algebra of 4D infinitesimal symplectic diffeomorphisms. (In the current convention of W algebra, therefore, this algebra should rather be called w_∞^∞; no quantum deformation seems to have been constructed until now.) This algebra includes the loop algebra $\mathcal{L}w_{1+\infty}$ of $w_{1+\infty}$ (i.e., of the SDiff(2) algebra) as a subalgebra. On the basis of these observations, they argue that this algebra (and possible quantum deformations) should be related to a quantum deformation of self-dual gravity.

Note that the loop algebra $\mathcal{L}w_{1+\infty}$ is three dimensional in its nature, the three variables being, e.g., λ, $\bar{p}$ and $\bar{q}$. (There are some other choices of those variables [5].) The W_∞^∞ algebra should be accompanied with four variables, i.e., canonical coordinates of a 4D symplectic manifold. It is amusing to imagine that the extra variable x associated with the quantized spectral parameter is exactly the fourth one that should be added to the previous three variables.

This will suggest to consider an extended 4D twistor space rather than an ordinary 3D twistor space. The first two variables λ (or rather k) and x are fundamental ingredients of the quasi-classical KP hierarchy. A 4D symplectic manifold with four coordinates, say λ, x, y, z and a symplectic form $d\lambda \wedge dx + dy \wedge dz$ apparently look like a nice framework for the aforementioned unification program. Unfortunately, this program has not been successful due to unexpected difficulties. This is a quite technical issue and details are omitted here.

A more hopeful direction would be that of the ordinary formulation of nonchiral w_∞ gravity [14] and its hypothetical "topological" version [38]. In these theories, SpDiff(4) symmetries are rather living in a 4D space-time with a symplectic structure (typically, the cotangent bundle $T^*\Sigma$ of a Riemann surface Σ), or acting on a moduli space M_∞ of such manifolds. The $\mathcal{L}w_{1+\infty}$ subalgebra of SpDiff(4), in that picture, cannot be identified with the twistor theoretical symmetry algebra that stems from a 3D twistor space. Nevertheless, a link with self-dual gravity still persists as Hitchin conjectures [38]. This conjecture seems to have been verified in the context of the $N = 2$ string theory by Ooguri and Vafa [9]; they show a construction of a hyper-Kähler metric (i.e., a solution of self-dual gravity) on $T^*\Sigma$. It would be interesting to see how the SpDiff(4) algebra on M_∞ act on these solutions; those SpDiff(4) symmetries might give rise to "constraints" like the W constraints of 2D gravity. If this is true, we expect a new nonlinear integrable system to lie behind.

References

[1] For an overview, see: Kaku, M., Strings, conformal fields, and topology (Springer-Verlag, 1991), Chapters 13 and 14, and references cited therein.

[2] Fukuma, M., Kawai, H., and Nakayama, R., Commun. Math. Phys. 143 (1991), 371-403.

[3] Douglas, M. Phys. Lett. 238B (1990) 176-180.

[4] David, F., Mod. Phys. Lett. A 5 (1990), 1019-1029.
Dijkgraaf, R., Verlinde, E., and Verlinde, H., Nucl. Phys. B348 (1991), 435-456.
Fukuma, M., Kawai, H., and Nakayama, R., Int. J. Mod. Phys. A6 (1991), 1385-1406.

[5] Boyer, C.P., and Plebanski, J.F., J. Math. Phys. 26 (1985), 229-234.
Takasaki, K., J. Math. Phys. 31 (1990), 1877-1888.
Park, Q-Han, Phys. Lett. 238B (1990), 287-290.

[6] Zumino, B., Phys. Lett. 87B (1979), 206-209.
Alvarez-Gaumé, L., Freedman, D.Z., Phys. Lett. 94B (1980), 171.
Hitchin, N.J., Kahlhede, A., Lindström, U., and Roček, M., Commun. Math. Phys. 108 (1987), 535-589.

[7] Floratos, E.G, and Leontaris, G.K., Phys. Lett. 223B (1989), 153-156.

[8] Bakas, I., Commun. Math. Phys. 134 (1990), 487-508.
Park, Q-Han, Phys. Lett. 236B (1990), 429-432.

[9] Ooguri, H., and Vafa, C., Mod. Phys. Lett. A5 (1990), 1389-1398; Nucl. Phys. B361 (1991), 469-518; Nucl. Phys. B367 (1991), 83-104.

[10] For a review of W algebras and symmetries in field theory, see: Sezgin, E., Aspects of W_∞ symmetry, Texas A& M preprint CTP-TAMU-9/91; Area-preserving diffeomorphisms, w_∞ algebras and w_∞ gravity, Texas A& M preprint CTP-TAMU-13/92.

[11] Penrose, R., Gen. Rel. Grav. 7 (1976), 31-52.

[12] Plebanski, J.F., J. Math. Phys. 16 (1975), 2395-2402.

[13] Gindikin, S.G., in: H.D. Doebner and T. Weber (eds.), *Twistor Geometry and Nonlinear Systems*, Lecture Notes in Mathematics vol. 970 (Springer-Verlag, 1982).

[14] Bergshoeff, E., Pope, C.N., Romans, L.J., Sezgin, E., Shen, X., and Stelle, K.S., Phys. Lett. 243B (1990), 350-357.

Bergshoeff, E., and Pope, C.N., Phys. Lett. 249B (1990), 208-215.

[15] Park, Q-Han, in Ref. 8.

Yamagishi, K., and Chapline, F., Class. Quantum Grav. 8 (1991), 427-446.

Hull, C.M., Phys. Lett. 269B (1991), 257-263.

[16] Date, E., Kashiwara, M., Jimbo, M., and Miwa, T., in: M. Jimbo and T. Miwa (eds.), *Nonlinear Integrable Systems – Classical Theory and Quantum Theory* (World Scientific, Singapore, 1983).

Sato, M., and Sato, Y., in: P.D. Lax, H. Fujita, and G. Strang (eds.), *Nonlinear Partial Differential Equations in Applied Sciences* (North-Holland, Amsterdam, and Kinokuniya, Tokyo, 1982).

[17] Orlov, A.Yu., in: V.G. Bar'yakhtar et al. (eds.), *Plasma Theory and Nonlinear and Turbulent Processes in Physics* (World Scientific, Singapore, 1987).

Grinevich, P.G., and Orlov, A.Yu., in: A. Belavin et al. (eds.), *Problems of modern quantum field theory* (Springer-Verlag, 1989).

[18] Awada, M., and Sin, S.J., Twisted W_∞ symmetry of the KP hierarchy and the string equation of $d = 1$ matrix models, Florida preprint UFITFT-HEP-90-33.

Yoneya, T., Toward a canonical formalism of non-perturbative two-dimensional gravity, UT-Komaba 91-8 (February, 1991).

[19] Kodama, Y., Phys. Lett. 129A (1988), 223-226.

Kodama, Y., and Gibbons, J., Phys. Lett. 135A (1989), 167-170.

Krichever, I.M., Commun. Math. Phys. 143 (1991), 415-426.

[20] Takasaki, K., and Takebe, T., SDiff(2) KP hierarchy, Kyoto preprint RIMS-814 (October, 1991; revised version December, 1991).

[21] Dijkgraaf, R., Verlinde, H., and Verlinde, E., Topological strings in $d < 1$, PUPT-1204, IASSNS-HEP-9o/71 (October, 1990).

Blok, B., and Varchenko, A., Topological conformal field theories and flat coordinates, IASSNS-HEP-91/05 (January, 1990).

[22] Kodama, Y., Phys. Lett. 147A (1990), 477-482.

[23] Golenisheva-Kutuzova, M.I., and Reiman, A.G., Zap. Nauch. Semin. LOMI 169 (1988), 44. (in Russian).

Saveliev, M.V., and Vershik, A.M., Commun. Math. Phys. 126 (1989), 367-378.

Kashaev, R.M., Saveliev, M.V., Savelieva, S.A., and Vershik, A.M., in: *Ideas and Methods in Mathematics and Physics* (Cambridge University Press, Cambridge, 1991).

[24] Takasaki, K., and Takebe, T., Lett. Math. Phys. 23 (1991), 205-214.

[25] Boyer, C., and Finley, J.D., J. Math. Phys. 23 (1982), 1126-1128.

Gegenberg, J.D., and Das, A., Gen. Rel. Grav. 16 (1984), 817-829.

[26] Dhar, A., Jayaraman, T., Narain, K.S., and Wadia, S.R., Mod. Phys. Lett. A5 (1990), 863.

[27] Hitchin, N.J., in: H.D. Doebner and T. Weber (eds.), *Twistor Geometry and Non-linear Systems*, Lecture Notes in Mathematics vol. 970 (Springer-Verlag, 1982).

Jones, P.E., and Tod, K.P., Class. Quantum Grav. 2 (1985), 565-577.

Ward, R.S., Class. Quantum Grav. 7 (1990). L95-L98.

[28] Flaherty, E.J., Gen. Rel. Grav. 9 (11) (1978), 961-978.

[29] Park, Q-Han, Integrable deformations of self-dual gravity, Cambridge preprint DAMTP R-91/5 (July, 1991).

[30] Takasaki, K., Volume-preserving diffeomorphisms in integrable deformations of self-dual gravity, Kyoto preprint KUCP-0046/92 (March, 1992).

[31] Polchinski, J., Nucl. Phys. B362 (1991), 125-140.

Avan, J., and Jevicki, A., Phys. Lett. 266B (1991), 35-41.

Das, S.R., Dhar, A., Mandal, G., and Wadia, S.R., Gauge theory formulation of the $c = 1$ matrix model: symmetries and discrete states, ETH-TH-91/30, IASSNS-HEP-91/52, TIFR-TH-91/44 (September, 1991); W-infinity Ward identities and correlation functions in the $c = 1$ matrix model, IASSNS-HEP-91/79, TIFR-TH-91/57 (February, 1992).

Witten, E., Ground ring of two dimensional string theory, IASSNS-HEP-91/51 (August, 1991).

[32] Hoppe, J., Olshanetsky, M., and Theisen, S., Dynamical systems on quantum tori algebras, Kahlsruhe preprint KA-THEP-10/91 (October, 1991).

Hoppe, J., Dynamical systems on W_∞ and universal enveloping algebras, Kahlsruhe preprint KA-THEP-11/91 (December, 1991).

[33] Fairlie, D.B., and Zachos, C.K., Phys. Lett. 224B (1989), 101-107.

[34] Hoppe, J., Phys. Lett. 250B (1990), 44-48.

[35] Degasperis, A. Lebedev, D., Olshanetskii, M., Pakuliak, S., Perelomov, A., Santini, P., Recent development for integrable integrodifferential equations, BONN-HE-90-13 (Nov 1990); Nonlocal integrable partners to generalized mKdV and two-dimensional Toda lattice equations in the formalism of a dressing method with quantized spectral parameter, BONN-HE-90-14 (Nov 1990).

[36] Lebedev, D., Orlov, A., Pakuliak, S., Zabrodin, A., Nonlocal integrable equations as reduction of the Toda hierarchy, BONN-HE-91-05 (Feb 1991).

[37] Bakas, I., and Kiritsis, E., Grassmannian coset models and unitary representation of W_∞, UBC-PTH-90/16, UMD-PP90-215 (April, 1990).

[38] Witten, E., Surprises with topological field theories, IASSNS-HEP-90/37 (April, 1990).