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On the b-Function of Nonisolated Hypersurface Singularities

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Let f be a germ of holomorphic function of n variables, and $b_f(s)$ the b-function (i.e. Bernstein polynomial) of f. It is the monic generator of the ideal consisting of polynomials $b(s)$ which satisfy the relation

(0.1) $$b(s)f^s = Pf^{s+1} \quad \text{in} \quad \underline{O}_X[f^{-1}][s]f^s$$

for $P \in \underline{D}_X[s]$, where $\underline{D}_X$ denotes the germs of holomorphic differential operators on $X := (\mathbb{C}^n, 0)$, and $\underline{D}_X[s] = \underline{D}_X \otimes_{\mathbb{C}} \mathbb{C}[s]$. Substituting $s = -1$, we can check easily that $b_f(s)$ is divisible by $s+1$. Let $\tilde{b}_f(s) = b_f(s)/(s+1)$, R_f the roots of $\tilde{b}_f(-s)$, $\alpha_f = \min R_f$, and $m_\alpha(f)$ the multiplicity of a root α of $\tilde{b}_f(-s)$. By Kashiwara [7], we have

(0.2) **Theorem.** $\alpha_f > 0$, and $R_f \subset \mathbb{Q}$.

Assume f has isolated singularity and $n > 1$. Let $H''_f = \Omega_X^n / df \wedge d\Omega_X^{n-2}$, following Brieskorn [2]. Then H''_f is a free $\mathbb{C}\{\{t\}\}$-module of rank μ (the Milnor number of f), and has a regular singular meromorphic connection. Let $\tilde{H}''_f = \sum_{i \geq 0} (t\partial_t)^i H''_f \subset H''_f[t^{-1}]$ (the saturation of H''_f). By Malgrange [13], we have

(0.3) **Theorem.** $\tilde{b}_f(s)$ is the minimal polynomial of the action of $-\partial_t t$ on $\tilde{H}''_f / t\tilde{H}''_f$.

Combined with a result of Varchenko [29] (and [26]), this implies (see also [17]) :

(0.4) **Theorem.** $R_f \subset [\alpha_f, n - \alpha_f]$.

(0.5) **Theorem.** $m_\alpha(f) \leq n - \alpha_f - \alpha + 1$ $(\leq n - 2\alpha_f + 1)$.

In the isolated singularity case, we proved also (see [16]) :

(0.6) **Proposition.** $Y = f^{-1}(0)$ has rational singularity if and only if $\alpha_f > 1$.

Using the theory of mixed Hodge Modules [18] [19] [20], we extend these to the nonisolated singularity case (see [23] [24]), i.e.

(0.7) **Theorem.** (0.4–6) are valid also in the nonisolated singularity case, where we assume Y reduced in (0.6).

Note that (0.5) is an improvement of $m_\alpha(f) \leq n - \delta_{\alpha,1}$ (where $\delta_{\alpha,1}$ is Kronecker's delta) which is shown in [14] as a corollary of the relation with Deligne's vanishing cycle sheaf

$\varphi_f\mathbb{C}_X$ [3]. See also [8]. This relation implies for example that $\exp(2\pi i\alpha)$ for $\alpha \in R_f$ are the eigenvalues of the monodromy on $\varphi_f\mathbb{C}_X$. But $\varphi_f\mathbb{C}_X$ cannot be replaced with the reduced cohomology of a Milnor fiber at the origin as in the isolated singularity case, because we have to take the Milnor fibration at several points of $\mathrm{Sing}\, f^{-1}(0)$ even when we consider the b-function of f at the origin.

For the proof of the generalization of (0.4–5), we introduce the notion of microlocal b-function (1.1), and show an assertion (1.2) which may be viewed as a generalization of (0.3). Using this, we can prove the Thom-Sebastiani type theorem for b-function in some case (2.8). In the nondegenerate Newton boundary case [12], we get an estimate of α_f by the Newton polyhedron (2.7). Note that the b-function is also related with the spectrum [27] of f, and with a result of Deligne-Dimca [5]. See [23].

§1. Microlocal b-Function

(1.1) Let $\delta(t-f)$ denote the delta function on $X' := X \times (\mathbb{C}, 0)$ with support $\{f = t\}$, where t is the coordinate of $\mathbb{C}$. Then, setting $s = -\partial_t t$, f^s and $\delta(t-f)$ satisfy the same relation (see for example [13]). So f^s in (0.1) can be replaced by $\delta(t-f)$, and f^{s+1} by $t\delta(t-f)$. We define the *microlocal* b-*function* $\tilde{b}_f(s)$ by the monic generator of the ideal consisting of polynomials b(s) which satisfy the relation

(1.1.1) $$b(s)\delta(t-f) = P\partial_t^{-1}\delta(t-f) \quad \text{in} \quad \underline{O}_X[\partial_t, \partial_t^{-1}]\delta(t-f)$$

for $P \in \underline{D}_X[\partial_t^{-1}, s]$. Here we can also allow for P a microdifferential operator [7] [9] [10] [25] satisfying a condition on the degree of t and ∂_t (see [24, (1.4)]).

We can show (see [24, (1.5)]):

(1.2) **Proposition.** $b_f(s) = (s+1)\tilde{b}_f(s)$.

(1.3) Let $R_X = \underline{D}_X[t, \partial_t]$, $\tilde{R}_X = \underline{D}_X[t, \partial_t, \partial_t^{-1}]$, and

(1.3.1) $$\underline{B}_f = \underline{O}_X[\partial_t]\delta(t-f), \quad \tilde{\underline{B}}_f = \underline{O}_X[\partial_t, \partial_t^{-1}]\delta(t-f),$$

where $\underline{O}_X[\partial_t]\delta(t-f)$ is a free module of rank one over $\underline{O}_X[\partial_t]$ with a basis $\delta(t-f)$ (similarly for $\tilde{\underline{B}}_f$). Then $\underline{B}_f$, $\tilde{\underline{B}}_f$ have naturally a structure of R_X-module and $\tilde{R}_X$-module respectively.

Let V be the filtration on R_X, $\tilde{R}_X$ by the differences of the degrees of t and ∂_t, i.e.,

(1.3.2) $$V^p R_X = \textstyle\sum_{i-j\ge p} \underline{D}_X t^i \partial_t^j \quad (\text{same for } \tilde{R}_X).$$

We define a decreasing filtration G on $\underline{B}_f$, $\tilde{\underline{B}}_f$ by

(1.3.3) $$G^p\underline{B}_f = V^p R_X \delta(t-f), \quad G^p\tilde{\underline{B}}_f = V^p\tilde{R}_X\delta(t-f),$$

and an increasing filtration F by

(1.3.4) $$F_p\underline{B}_f = \oplus_{0\le i\le p}\, \underline{O}_X\partial_t^i\delta(t-f),\quad F_p\underline{\tilde{B}}_f = \oplus_{i\le p}\, \underline{O}_X\partial_t^i\delta(t-f).$$

Then we have

(1.3.5) $$\partial_t^i : G^p\underline{\tilde{B}}_f \xrightarrow{\sim} G^{p-i}\underline{\tilde{B}}_f,\quad \partial_t^i : F_p\underline{\tilde{B}}_f \xrightarrow{\sim} F_{p+i}\underline{\tilde{B}}_f$$

(1.3.6) $$\underline{D}_X[s](F_p\underline{\tilde{B}}_f) \subset G^{-p}\underline{\tilde{B}}_f.$$

(1.4) *Remark.* $b_f(s)$ and $\tilde{b}_f(s)$ are the minimal polynomial of the action of $s := -\partial_t t$ on $Gr_G^0\underline{B}_f$ and $Gr_G^0\underline{\tilde{B}}_f$ respectively, because s belongs to the center of $Gr_G^0 R_X = Gr_G^0\tilde{R}_X = \underline{D}_X[s]$.

§2. Filtration V

(2.1) Let V denote the filtration of Kashiwara [8] and Malgrange [14] on $\underline{B}_f$ indexed by $\mathbb{Q}$. Here we index V decreasingly so that the action of $\partial_t t - \alpha$ on $Gr_V^\alpha\underline{B}_f$ is nilpotent, where $Gr_V^\alpha = V^\alpha/V^{>\alpha}$ with $V^{>\alpha} = \cup_{\beta>\alpha} V^\beta$. By [7] (see also (0.2) above), we have

(2.2.1) $$F_0\underline{B}_f \subset V^{>0}\underline{B}_f.$$

We can show (see [24, (2.2) and (2.4)]):

(2.2) **Lemma.** We have a decreasing filtration V on $\underline{\tilde{B}}_f$ such that

(2.2.1) $$V^\alpha\underline{\tilde{B}}_f = V^\alpha\underline{B}_f + \underline{O}_X[\partial_t^{-1}]\partial_t^{-1}\delta(t-f)\quad \text{for}\quad \alpha \le 1,$$

(2.2.2) $$\partial_t^j : V^\alpha\underline{\tilde{B}}_f \xrightarrow{\sim} V^{\alpha-j}\underline{\tilde{B}}_f\quad \text{for any } j,\ \alpha.$$

(2.3) **Proposition.** We have

(2.3.1) $$Gr_V^\alpha\underline{\tilde{B}}_f = \underline{D}_X(F_p Gr_V^\alpha\underline{\tilde{B}}_f)\quad \text{if}\quad F_{-p-1}Gr_V^{n-\alpha}\underline{\tilde{B}}_f = 0.$$

(2.4) *Proof of* (0.4) *in the general case.* We have $G^1Gr_V^\alpha\underline{\tilde{B}}_f \supset \underline{D}_X(F_{-1}Gr_V^\alpha\underline{\tilde{B}}_f)$ by (1.3.6). So it is enough to show $Gr_V^\alpha\underline{\tilde{B}}_f = \underline{D}_X(F_{-1}Gr_V^\alpha\underline{\tilde{B}}_f)$ for $\alpha > n - \alpha_f$ by (1.4), because it implies $Gr_G^0Gr_V^\alpha\underline{\tilde{B}}_f = Gr_V^\alpha Gr_G^0\underline{\tilde{B}}_f = 0$. By definition of α_f, we have

(2.4.1) $$F_0Gr_V^\alpha\underline{\tilde{B}}_f = G^0Gr_V^\alpha\underline{\tilde{B}}_f = 0\quad \text{for}\quad \alpha < \alpha_f$$

using (1.3.6). So the assertion follows from (2.3) applied to $p = -1$.

By a similar argument, we prove (0.5) using also the monodromy filtration W. Here W is uniquely characterized by the properties (see [4]):

(2.4.1) $$NW_i \subset W_{i-2},\quad N^j : Gr_j^W \xrightarrow{\sim} Gr_{-j}^W\ (j > 0),$$

where $N = s + \alpha$ on $Gr_V^\alpha\underline{\tilde{B}}_f$. See [24, (2.8)] for the details. We can show also the following:

(2.5) *Remark.* Let $\varphi_f\mathbb{C}_X$ be Deligne's vanishing cycle sheaf [3], and T_u, T_s denote respectively the unipotent and semisimple part of the monodromy T on $\varphi_f\mathbb{C}_X$. Let $\varphi_f^\alpha\mathbb{C}_X$ = Ker(T_s − exp(−2πiα)) (as a shifted perverse sheaf [1]), and $N = \log T_u / 2\pi i$. Then we have $N^{r+1} = 0$ on $\varphi_f^\alpha\mathbb{C}_X$ for $\alpha \in [\alpha_f, \alpha_f + 1)$ and $r = [n - \alpha_f - \alpha]$. In particular, $N^{r+1} = 0$ on $\varphi_f\mathbb{C}_X$ for $r = [n - 2\alpha_f]$. See [24, (0.6)].

(2.6) *Remark.* If Sing $f^{-1}(0)$ is isolated and f is a quasi-homogeneous polynomial of weight $(w_1, \cdots, w_n)$ (i.e. f is a linear combination of monomials $x_1^{m_1}\cdots x_n^{m_n}$ such that $m_1w_1 + \cdots + m_nw_n = 1$), then it is well-known that $m_\alpha(f) = 1$ for $\alpha \in R_f$, and α belongs to R_f if and only if the coefficient of t^α in

$$\prod_i (t^{w_i} - t)/(1 - t^{w_i}) \tag{2.6.1}$$

is nonzero. This follows for example from Steenbrink [28] (using [13] [29]) and also from Brieskorn or Kashiwara (unpublished). In particular, we have $\max R_f = n - \alpha_f$ in this case.

(2.7) *Remark.* If f has nondegenerate Newton boundary, we can show $\alpha_f \geq 1/t$ for $(t, \cdots, t) \in \partial\Gamma_+(f)$ (see [24, (3.3)]), where $\Gamma_+(f)$ is the Newton polygon of f. In the isolated singularity case, it is known that the equality holds. (See also [22].)

(2.8) *Remark.* Let g be a holomorphic function on a germ of complex manifold Y. Let $Z = X \times Y$, and $h = f + g \in \underline{O}_Z$. We define R_g, R_h as in the introduction. Then $R_fR_g \subset R_h + \mathbb{Z}_{\geq 0}$, $R_h \subset R_fR_g + \mathbb{Z}_{\leq 0}$. Furthermore, if there is a holomorphic vector field ξ such that $\xi g = g$, then $R_fR_g = R_h$, and $m_\gamma(h) = \max_{\alpha+\beta=\gamma}\{m_\alpha(f) + m_\beta(g) - 1\}$. See [24, (4.3–4)]. The last assertion is proved in [30] if f and g have isolated singularities.

§3. Rational Singularity

(3.1) Let Y be a reduced complex analytic space. We say that Y has *rational singularity*, if the natural morphism

$$\underline{O}_Y \rightarrow R\pi_*\underline{O}_{Y'} \tag{3.1.1}$$

is an isomorphism for a resolution of singularity $\pi : Y' \rightarrow Y$. If Y is Cohen-Macaulay and pure dimensional, it is equivalent to the bijectivity of the trace morphism

$$\pi_*\omega_{Y'} \rightarrow \omega_Y \tag{3.1.2}$$

by duality [15], because $R^i\pi_*\omega_{Y'} = 0$ for $i > 0$ by [6] (this follows also from [11] [21]) where π is assumed projective. Here ω_Y denotes the dualizing sheaf (i.e., the dualizing complex [15] shifted by the dimension to the right). The trace morphism (3.1.2) is injective, and its image is independent of the choice of resolution, because (3.1.2) is an isomorphism if

Y is smooth. We will denote by $\tilde{\omega}_Y$ the image of (3.1.2).

(3.2) Assume Y is a reduced divisor D on the germ of complex manifold X in the introduction. Let f be a reduced defining equation of D.

Using the coordinate system $(x_1, \cdots, x_n)$ of X, we have the involution of $\underline{D}_X$ such that $(PQ)^* = Q^*P^*$, $(x_i)^* = x_i$, $(\partial/\partial x_i)^* = -\partial/\partial x_i$. So the right $\underline{D}$-module ω_X is identified with the left $\underline{D}$-module $\underline{O}_X$ using the basis $dx = dx_1 \wedge \cdots \wedge dx_n$ of ω_X, and we get isomorphisms

(3.2.2) $$\underline{B}_f = \omega_X[\partial_t]\delta(t-f), \quad \tilde{\underline{B}}_f = \omega_X[\partial_t, \partial_t^{-1}]\delta(t-f).$$

We can show (see [23]) :

(3.3) **Theorem.** We have a commutative diagram

(3.3.1) $$\begin{array}{ccccccccc} 0 & \to & \tilde{\omega}_D & \to & \omega_D & \to & \omega_D/\tilde{\omega}_D & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & F_0W_0\mathrm{Gr}_V^1\underline{B}_f & \to & F_0(\underline{B}_f/V^{>1}\underline{B}_f) & \to & F_0(\tilde{\underline{B}}_f/V^{>1}\tilde{\underline{B}}_f) & \to & 0, \end{array}$$

such that the vertical morphisms are isomorphisms.

(3.4) *Remark.* The horizontal short exact sequences correspond to the short exact sequence of mixed Hodge modules [19] :

(3.4.1) $$0 \to \mathbb{Q}_D^H[n-1] \to \psi_f\mathbb{Q}_X^H[n] \to \varphi_f\mathbb{Q}_X^H[n] \to 0.$$

In fact, taking Gr_V of (3.3.1), we get F_{1-n} of the underlying filtered $\underline{D}$-module of (3.4.1) (using (2.2.1)), because the underlying filtered $\underline{D}$-modules $\psi_f\omega_X$, $\varphi_f\omega_X$ of $\psi_f\,\mathbb{Q}_X^H[n]$, $\varphi_f\mathbb{Q}_X^H[n]$ are defined by

(3.4.2) $$\psi_f\omega_X = \oplus_{0<\alpha\le 1}\mathrm{Gr}_V^\alpha\underline{B}_f, \quad \varphi_f\omega_X = \oplus_{0<\alpha\le 1}\mathrm{Gr}_V^\alpha\tilde{\underline{B}}_f.$$

Here we have a shift of the filtration F coming from the transformation of left and right filtered $\underline{D}$-modules (see [23]). Furthermore, $\tilde{\omega}_D$ is F_{1-n} of the underlying filtered $\underline{D}$-module of the intersection complex $IC_D\mathbb{Q}^H$ which is a quotient of $\mathbb{Q}_D^H[n-1]$.

As a corollary of (3.3), we get (0.6) and the following

(3.5) **Corollary.** We have a canonical isomorphism

(3.5.1) $$F_{1-n}(\varphi_f\omega_X) = \oplus_{0<\alpha\le 1}\ \mathrm{Gr}_V^\alpha(\omega_D/\tilde{\omega}_D),$$

such that $\mathrm{Gr}_V^\alpha(\omega_D/\tilde{\omega}_D)$ corresponds to the $\exp(-2\pi i\alpha)$-eigenspace of $\varphi_f\omega_X$ with respect to the action of monodromy.

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