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The Teichmüller space from a view point of group representations

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Abstract: The Teichmüller space $\mathcal{T}_g$ is the branched universal covering of the moduli $\mathcal{M}_g$ of compact Riemann surfaces of genus g. The present note gives an exposition on a short cut construction of the Teichmüller space together with its complex structure and Kähler structure from the view point of representation of Fuchsian group.

§0. Introduction

The study of moduli of Riemann surface of genus g has long history. Already Riemann stated that $\mathcal{M}_g$ depends on 3g-3 complex parameters (for $g>1$) through an analysis of branched coverings. In the study of automorphic functions and Fuchsian groups by F. Klein and H. Poincaré (1880's) we see a starting of the study of moduli, which later on was developped by R.Fricke, W.Fenchel and J.Nielsen. Teichmüller (1940'~) described $\mathcal{T}_g$

in terms of quasiconformal maps,recognizing the importance of quadratic differentials. The theory of Teichmüller was rigolously based by L.V. Ahlforse and L. Bers by a heavy analysis on Beltrami equations (50's).

Owing to a series of works of A. Weil[2-3], one is able to give a rigolous foundation of Teichmüller space from the view point of Fricke [F]: to understand the moduli of Riemann surfaces as that of Fuchsian group Γ. Since then, there are increasing number of works in the direction [Mc][Mc-S][Ha][He][C-S][Wo][L-M][Mo-S][G][Br][Se-So] and others. The present paper is one trial in the same direction following [Sa1-4]. This has a motivation to give a foundation for the defining domain of certain infinite series ω_Γ introduced in [Sa2]. Some readers may be suggested to skip the §1 to §2, from where we start the study of representations of a finitely generated group Γ into SL_2.

As is originaly due to Helling [He], we regard $\mathcal{T}_g$ as a component of a real affine algebraic variety. We give *an explicit system of infinite number of defining polynomial equations in terms of infinite number of variables* associated to the group Γ_g (§2-3). Thanks to that expression, one is able to describe the tangent space of $\mathcal{T}_g$ explicitly and obtains its comparison with the cohomology of Γ_g in §4 (this is one of what Weil intended [W2-3]). So far is the real algebraic descriptions of $\mathcal{T}_g$. To add the complex and Kähler structure on it, we use Eichler integral of quadratic differentials. Namely their periods are cocycles of the group Γ_g (§5) so that they give arise of a Hodge decomposition of the tangent space of $\mathcal{T}_g$. In this way, one recovers the well known almost complex structure and a Kähler metric on $\mathcal{T}_g$ in §6. The integrability was shown readily in [Sa1, §9]. On the Kähler metric, one is refered to [Wo1-7].

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§1. Moduli and Teichmüller space for compact Riemann surfaces

We give a quick review on the moduli and Teichmüller spaces, which are more or less standard. Nevertheless, we take special care on the orientations of the surfaces (see the *Remark* 1. at the end of this §).

By a Riemann surface X, we mean an 1-dimensional complex manifold. We denote by $|X|$ the underlying real 2-dimensional manifold. As is well known, $|X|$ is orientable and the complex structure on X automatically induces an orientation. So if X is compact, there is a cannonical isomorphism: $H^2(X,\mathbb{Z})\simeq\mathbb{Z}$. In the case, its diffeomorphism class is determined by the first Betti number 2g, where g is called the genus of $|X|$.

Let S be a compact oriented real 2-dimensional surface of genus g. We ask a question: how much Riemann surface structures on S such that the induced orientation agree with the fixed one on S does there exist? Precisely, we construct the following spaces and ask for their study.

(1.1) $\mathcal{M}_g$:=the set of all complex structures on a surface of genus g /~,

where X~Y, if and only if there is a biholomophic map from X to Y.

(1.2) $\mathcal{T}_g(S)$:=the set of all complex structures on a surface S, whose induced orientations agree with the fixed one on S /~,

where X~Y, if and only if there is a biholomorphic map from X to Y which is isotopic to the identity map of S. In fact, $\mathcal{T}_g(S)$ depends only on g but not on the chosen surface S, since a diffeomorphism $S \simeq S'$ induces a bijection $\mathcal{T}_g(S) \simeq \mathcal{T}_g(S')$. Therefore, we shall denote $\mathcal{T}_g(S)$ by $\mathcal{T}_g$.

Let $Diff^+(S)$ be the group of all orientation preserving diffeomorphisms of S and let $Diff_0^+(S)$ be its subgroup consisting of all diffeomorphisms which are isotopic to the identity. The quotient group $\mathrm{Diff}^+(S)/\mathrm{Diff}_0^+(S)$ is called the mapping class group. It is known to be finitely presented and residualy finite([Bi][B-L]). By definition, the mapping class group acts naturally on the space $\mathcal{T}_g$ so that its quotient space is identified with $\mathcal{M}_g$ (use that there is an orientation reversing diffeomorphism of a surface of genus g to itself). In another word, forgetting the isotopy condition on the elements of $\mathcal{T}_g$, one obtains a natural quotient map:

(1.3) $$\pi : \mathcal{T}_g \longrightarrow \mathcal{M}_g$$

by the mapping class group action on $\mathcal{T}_g$. The $\mathcal{M}_g$ and $\mathcal{T}_g$ are called the moduli space or the Teichmüller space of compact Riemann surfaces genus g respectively. We shall see that $\mathcal{T}_g$ and $\mathcal{M}_g$ are complex manifolds and the map π is the quotient map by a properly discontinous holomorphic action of the mapping class group on $\mathcal{T}_g$. More strongly, it is known that $\mathcal{M}_g$ has a structure of an algebraic variety defined over $\mathbb{Z}$ [ACGH].

To describe the mapping class group algebraically, we recall some basic facts on compact surface topology. Let S be a compact orientable surface of genus g with a *base point* $*$ and a *fixed orientation*. There is a system of simple closeded pathes a_i, b_i $(i=1,\ldots,g)$ on S such that i)they mutually intersect only at $\{*\}$ with the multiplicity $\langle a_i,b_j\rangle=\delta_{ij}$, $\langle a_i,a_j\rangle=\langle b_i,b_j\rangle=0$ for $1\le i,j\le g$ (note that the sign of the intersection depends on the orientation of the surface), and ii) $S\setminus(\{*\}\amalg a_i\amalg b_i)$ is a 4g-gon. The system is called a *canonical dissection of* S with the base point $*$ and the orientation. Denote by a_i and b_i the homotopy class in $\pi_1(S,*)$ represented by the same pathes. Then we obtain an expression:

(1.4) $$\Gamma_g \simeq \pi_1(S,*)$$

for the abstract group:

(1.5) $$\Gamma_g := \langle a_1,\dots,a_g,b_1,\dots,b_g \mid \prod_{i=1}^{g} (a_i b_i a_i^{-1} b_i^{-1}) \rangle .$$

An automorphism $\alpha \in \mathrm{Aut}(\Gamma_g)$ is called orientation preserving (resp. reversing), if its induced action on $H^2(\Gamma_g,\mathbb{Z})\simeq\mathbb{Z}$ is an identity (resp. minus identity). This is equivalent to the following: Let J be the symplectic form on $\Gamma_g/[\Gamma_g,\Gamma_g]$ $(\simeq H^1(\Gamma_g,\mathbb{Z})$ given by $\langle a_i,b_j\rangle=\delta_{ij}$ and $\langle a_i,a_j\rangle=\langle b_i,b_j\rangle=0$. An $\alpha\in \mathrm{Aut}(\Gamma_g)$ is orientation preserving (rep. reversing), if the induced element $\tilde{\alpha}\in GL(\Gamma_g/[\Gamma_g,\Gamma_g])$ acts on J as $\tilde{\alpha}(J)=J$ (resp. $\tilde{\alpha}(J)=-J$) Let $\mathrm{Aut}^+(\Gamma_g)$ and $\mathrm{Out}^+(\Gamma_g):=\mathrm{Aut}^+(\Gamma_g)/\mathrm{Inn}(\Gamma)$ be the subgroups consisting of all orientation preserving (outer) automorphisms.

Theorem (Nielsen [N]) *The natural map induces an isomorphism:*

(1.6) $$\mathrm{Diff}^+(S)/\mathrm{Diff}_0^+(S) \simeq \mathrm{Out}^+(\Gamma_g) .$$

Let us return to the study of the moduli and the Teichmüller spaces. As explained in the introduction, we aproach them from a view point of deformation of discrete subgroups of $SL_2(\mathbb{R})$, where the following theorem on the uniformization of a Riemann surface is fundamental.

Theorem (Poincare [Po], Koebe [Ko]) *Any simply connected Riemann surface is biholomorphic to either the Riemann sphere* $\hat{\mathbb{C}}$, *the whole complex plane* $\mathbb{C}$ *or the complex upper half plane* $\mathbb{H}$.

Apply the theorem to the universal cover $\tilde{X}$ of a compact Riemann surface X of genus $g\geq 2$. We see that $\tilde{X}$ is biholomorphic to $\mathbb{H}$. Since the group of biholomorphic automorphisms of $\mathbb{H}$ is isomorphic to $PSL_2(\mathbb{R})$ (through its fractional linear action on $\mathbb{H}$), the covering transformation group of the cover $\tilde{X}\longrightarrow X$ is embedded into a discrete subgroup of $PSL_2(\mathbb{R})$. This means that the moduli of Riemann surfaces of genus $g\geq 2$ may be described in terms of the moduli of discrete subgroups of $PSL_2(\mathbb{R})$. To achieve this idea precisely, we attach to X the following three aditional structures and study the moduli of such "structured Riemann surfaces" X.

i) a point $*$ in X and $\tilde{*}$ in $\tilde{X}$ over $*$,

ii) a biholomorhism $\varphi:\tilde{X} \longrightarrow \mathbb{H}$ which brings $\tilde{*}$ to the unit $i\in\mathbb{H}$,

iii) a canonical dissection of $|X|$ with the base point $*$ and with the orientation induced from the complex structure on X.

Let a surface X with i)-iii) be given. Owing to the datum i), the fundamental group $\pi_1(X,*)$ acts left on $\tilde{X}$ as the covering transformation group. Using ii),the action of an element $\gamma\in\pi_1(X,*)$ is represented by an element $\varphi\circ\gamma\circ\varphi^{-1}$ of $Aut(\mathbb{H})\simeq PSL(2,\mathbb{R})$. Combining this with the isomorphism (1.4) induced from the iii), we obtain a homomorphims (representation):

$$\rho : \Gamma_g \longrightarrow PSL_2(\mathbb{R}) \tag{1.7}$$

Clear from the construction, the map $\tilde{X} \xrightarrow{\sim} \mathbb{H}$ induces an expression:

$$X \simeq \rho(\Gamma_g)\backslash\mathbb{H}$$

of the Riemann surface X. So ρ needs to be injective with discrete and co-compact image $\rho(\Gamma_g)$ in $PSL_2(\mathbb{R})$.

Conversely, let an injective representation ρ (1.7) with discrete and cocompact image be given. Then, by putting $X:=\rho(\Gamma_g)\backslash\mathbb{H}$ and $\tilde{*}:=i\in\mathbb{H}$, one recovers a Riemann surface with the data i) and ii), but not yet iii). The obstruction to get iii) lies in the fact that the pathes in $\rho(\Gamma_g)\backslash\mathbb{H}$ corresponding to the homotopy classes a_i and b_j in $\Gamma_g\simeq\pi_1(X,*)$ may not give desired sign of the intersection numbers. Let us call ρ (1.7) to be *positive* (resp. *negative*), if the intersection number of a_i and b_j in $\rho(\Gamma_g)\backslash\mathbb{H}$ is equal to δ_{ij} (resp. $-\delta_{ij}$). According to this definition, a representation ρ (1.7) induced from a surface with iii) is positive.

According to [W2], for a Lie group G and an abstract group Γ, put

(1.8) $R(\Gamma, G):=$ the set of all group homomorphisms from Γ to G,

and

(1.9) $R_0(\Gamma,G):=\{\rho\in R(\Gamma,G) : \rho$ is injective, discrete and cocompact$\}$.

Since G caries a structure of a real analytic variety with the classical topology, $R(\Gamma,G)$ is a real analytic variety by the weak toplogy. It is a basic theorem by Weil [W2,I] that $R_0(\Gamma,G)$ is an open subset of $R(\Gamma,G)$.

In case $G=PSL_2(\mathbb{R})$ or $SL_2(\mathbb{R})$ and $\Gamma=\Gamma_g$, we define further for $\varepsilon \in \{\pm\}$

(1.10) $R_0^{\varepsilon}(\Gamma_g,G):=\{\rho\in R_0(\Gamma_g,G)$: ρ is positive (negative) for $\varepsilon=+\ (-)\}$,

where, by definition, one has the disjoint union decomposition:

$$R_0(\Gamma_g,G) = R_0^+(\Gamma_g,G) \amalg R_0^-(\Gamma_g,G) \ .$$

By the use of these notations, one has established a bijection:

(1.11) {Riemann surfaces of genus g with i)-iii)}/$\sim$ $\simeq R_0^+(\Gamma_g,PSL_2(\mathbb{R}))$,

where $\sim$ is the natural equivalence (biholomorphic map preserving the structures i)-iii)). On the right hand of (1.11), the groups $Aut(\Gamma_g)$ and $G=PGL_2(\mathbb{R})$act: for $\alpha,g\in Aut(\Gamma_g)\times G$ and $\rho\in R(\Gamma_g,G)$ put $\alpha\cdot\rho\cdot Ad(g)\in R(\Gamma_g,G)$ by $\alpha\cdot\rho\cdot Ad(g)(\gamma):=g^{-1}\rho(\alpha(\gamma))g$. Note that the actions of $Aut^+(\Gamma_g)$ and of G preserve $R_0^{\varepsilon}(\Gamma_g,G)$, but the action of $Aut(\Gamma_g)\setminus Aut^+(\Gamma_g)$ interchange $R_0^+(\Gamma_g,G)$ and $R_0^-(\Gamma_g,G)$. Note also that the left action of $Inn(\Gamma_g)$ is "absorbed" in the right action of $PSL_2(\mathbb{R})$, for $Ad(\gamma)\cdot\rho=\rho\cdot Ad(\rho(\gamma))$.

Following equivalence in (1.11) are easy to check.

To forget the structure iii) is equivalent to divide by $Aut^+(\Gamma_g)$.

To forget the structure ii) is equivalent to divide by PSO(2).

To forget the point $\tilde{*}$ is equivalent to divide by $Inn(\Gamma_g)$.

To forget the structure i) and ii) is equivalent to divide by $PSL_2(\mathbb{R})$

In summery, we obtain an expression:

$$(1.11)\qquad \mathcal{M}_g \simeq \mathrm{Out}^+(\Gamma_g)\backslash R_0^+(\Gamma_g,PSL_2(\mathbb{R}))/\mathrm{Ad}(PSL_2(\mathbb{R}))$$

The $\mathrm{Aut}(PSL_2(\mathbb{R}))=PGL_2(\mathbb{R})$ (=an index 2 extension of $PSL_2(\mathbb{R})$) acts adjointly on $R_0(\Gamma_g,PSL_2(\mathbb{R}))$. Since the fractional linear transformation of an element of $PGL_2(\mathbb{R})\backslash PSL_2(\mathbb{R})$ interchanges the upper and the lower half planes, we see that its action on $R_0(\Gamma_g,G)$ interchanges $R_0^+(\Gamma_g,G)$ and $R_0^-(\Gamma_g,G)$. In view of this fact, let us consider the space:

$$(1.12)\qquad \mathrm{Out}^+(\Gamma_g)\backslash R_0(\Gamma_g,PSL(2,\mathbb{R}))/\mathrm{Ad}(PSL_2(\mathbb{R}))$$

which decomposes into two parts:

$$\mathrm{Out}^+(\Gamma_g)\backslash R_0^+(\Gamma_g,PSL(2,\mathbb{R}))/\mathrm{Ad}(PSL_2(\mathbb{R}))$$
$$\amalg\ \mathrm{Out}^+(\Gamma_g)\backslash R_0^-(\Gamma_g,PSL(2,\mathbb{R}))/\mathrm{Ad}(PSL_2(\mathbb{R})).$$

There are *two mutualy commutative involutions* acting on the space (1.12) interchanging two parts: the generator of $PGL_2(\mathbb{R})/PSL_2(\mathbb{R})$ and that of $\mathrm{Out}(\Gamma_g)/\mathrm{Out}^+(\Gamma_g)$. By any of them, the space (1.12) may be regarded as a double covering of $\mathcal{M}_g$. As a *convention*, we employ $PGL_2(\mathbb{R})/PSL_2(\mathbb{R})$ as for the covering transformation group. In fact, this choice seems to have a naturality from a view point of invariant theory (cf §3 and its Remark). According to this convention, we rewrite (1.11) as:

$$(1.14)\qquad \mathcal{M}_g \simeq \mathrm{Out}^+(\Gamma_g)\backslash R_0(\Gamma_g,PSL_2(\mathbb{R}))/\mathrm{Ad}(PGL_2(\mathbb{R})),$$

on which $\mathrm{Out}(\Gamma_g)/\mathrm{Out}^+(\Gamma_g)$ acts non trivially. In view of (1.3) and the theorem of Nielsen, we obtain a description of the Teichmüller space:

$$(1.15)\qquad \mathcal{T}_g \simeq R_0^+(\Gamma_g,PSL_2(\mathbb{R}))/\mathrm{Ad}(PSL_2(\mathbb{R}))$$
$$\simeq R_0(\Gamma_g,PSL_2(\mathbb{R}))/\mathrm{Ad}(PGL_2(\mathbb{R}))$$

$$\simeq \quad \mathrm{Inn}(\Gamma_g)\backslash R_0(\Gamma_g, PSL_2(\mathbb{R}))/Ad(PGL_2(\mathbb{R}))$$

The expressions (1.15) is not only set theoretic but has real analytic meaning. By solving implicit function theorem, it is easy to see that $R_0(\Gamma_g, G)$ for $G=PSL_2(\mathbb{R})$ or $SL_2(\mathbb{R})$ and are real analytic manifolds. The adjoint actions of $PSL_2(\mathbb{R})$ are proper anlytic and fixed point free. The action has also local analytic sections. Therefore the quotient spaces $R_0^{\varepsilon}(\Gamma_g, G)/Ad(PSL_2(\mathbb{R}))$ are naturally endowed with Hausdorf topology with natural real analytic manifold structure such that quotient map is a principal $PSL_2(\mathbb{R})$-bundle. These facts are standard. As for a proof fit to our situation, see for instance [Sa1, §4-6] and its references.

Remark. 1. Even historically notations for Teichmüller spaces are fixed already, it would have been better to call the space (1.2) by $\mathcal{T}_g^+(S)$ and the samely defined space with reversed orientation by $\mathcal{T}_g^-(S)$. Then one can introduce $\mathcal{T}_g$ as the quotient of $\mathcal{T}_g^+ \amalg \mathcal{T}_g^-$ by the involution $PGL_2(\mathbb{R})/SL_2(\mathbb{R})$ and so that one can have natural expressions:

$$\mathcal{T}_g^{\varepsilon} \simeq \mathrm{Inn}(\Gamma_g)\backslash R_0^{\varepsilon}(\Gamma_g, PSL_2(\mathbb{R}))/Ad(PSL_2(\mathbb{R})) ,$$

$$\mathcal{T}_g^+ \amalg \mathcal{T}_g^- \simeq \mathrm{Inn}(\Gamma_g)\backslash R_0(\Gamma_g, PSL_2(\mathbb{R}))/Ad(PSL_2(\mathbb{R})) ,$$

$$\mathcal{T}_g \simeq \mathrm{Inn}(\Gamma_g)\backslash R_0(\Gamma_g, PSL_2(\mathbb{R}))/Ad(PGL_2(\mathbb{R})) .$$

The quotient of $\mathcal{T}_g$ by $\mathrm{Out}(\Gamma_g)$ is the space of all conformal structures.

2. Introduce some more spaces by putting

$$S^1\mathfrak{X}_g := \mathrm{Inn}(\Gamma_g)\backslash R_0^+(\Gamma_g, PSL_2(\mathbb{R})),$$

$$\mathfrak{X}_g := \mathrm{Inn}(\Gamma_g)\backslash R_0^+(\Gamma_g, PSL_2(\mathbb{R}))/SO(2).$$

It is shown in [Sa1] that $\mathcal{T}_g$ and $\mathfrak{X}_g$ carry natural integrable complex structures such that the natural projection $\mathfrak{X}_g \longrightarrow \mathcal{T}_g$ is the universal family of Riemann surfaces of genus g. The natural projection $S^1\mathfrak{X}_g \longrightarrow \mathfrak{X}_g$ is a S^1-bundle embedded in the relative complex tangent bundle $T(\mathfrak{X}_g/\mathcal{T}_g)$ as the unit circles with respect to the Poincaré metric. (To be exact, in the literature, we used the space $R_0(\Gamma,G)$ instead of $R_0^+(\Gamma,G)$, which gives a double covering space of what we consider in the present note).

We remark that the space $S^1\mathfrak{X}_g$ can be regarded as the universal family of uniformization maps from $\mathbb{H}$ to Riemann surfaces of genus g: let us consider the natural $SL_2(\mathbb{R})$ action $S^1\mathfrak{X}_g \times SL_2(\mathbb{R}) \longrightarrow S^1\mathfrak{X}_g$ from the right. Then take the quotient by $SO(2)$ of the initial and the target spaces:

$$S^1\mathfrak{X}_g \times \mathbb{H} \longrightarrow \mathfrak{X}_g$$

For each point of $S^1\mathfrak{X}_g$, the map gives a uniformization of a Riemann surface of genus g and so $S^1\mathfrak{X}_g$ parametrize whole such uniformzations. This interpretation is the starting point of the present work (cf[Sa1])

§2. Representation space of a finitely generated group Γ into SL_2

We start with an algebraic study of representaion space of finitely generated group Γ into $SL_2(\mathbb{R})$ instead into $PSL_2(\mathbb{R})=SL_2(\mathbb{R})/\pm 1$, since $SL_2(\mathbb{R})$ is more convenient than $PSL_2(\mathbb{R})$ from a view point of invariants. To treat the spaces $R_0(\Gamma,SL_2(\mathbb{R}))$ and $R_0(\Gamma,SL_2(\mathbb{R}))/Ad(PGL_2(\mathbb{R}))$ functorially, we represent them as the subsets of real valued points of certain affine algebraic schemes defined over $\mathbb{Z}$ (cf. [He], [C-S], [Mo-S], [L-M]) For proof and details of some theorems on invariants in §2-4, see [Sa4].

Let Γ be a finitely generated group. Let R be any commutative ring with the unit 1. We denote by $Hom(\Gamma,SL_2(R))$ the set of all homomorphisms of Γ into $SL_2(R)$ (if $R=\mathbb{R}$, this is the same as the $R(\Gamma,SL_2(\mathbb{R}))$ in Weil's notation). This set of representations can be regarded as the set of R-valued point of certain affine algebraic scheme Spec($A_2(\Gamma)$) as below.

Assertion. There exists a finitely generated algebra $A_2(\Gamma)$ *over* $\mathbb{Z}$ *and a representaion* $\sigma:\Gamma \longrightarrow SL_2(A_2(\Gamma))$ *such that the correspondence* $\varphi\in$ Hom($A_2(\Gamma)$,R) *(=set of ring homomorphisms) to* $\varphi\circ\sigma\in$Hom(Γ,SL_2(R)) *(=set of group homomorphisms) is a bijection*:

(2.1) $$\mathrm{Hom}(A_2(\Gamma),R) \simeq \mathrm{Hom}(\Gamma,SL_2(R)).$$

In fact the pair ($A_2(\Gamma)$, σ) is unique up to an isomorphism. $A_2(\Gamma)$ is obtained explicily as a quotient of the infintely generated polynomial ring $\mathbb{Z}[a_{ij}(\gamma),\ 1\le i,j\le 2,\ \gamma\in\Gamma]$ of the indeterminates $a_{ij}(\gamma)$ for $1\le i,j\le 2$ and for $\gamma\in\Gamma$ divided by the ideal generated by the entries of the matrix relations $\sigma(e)-I_2$, $\sigma(\gamma\delta)-\sigma(\gamma)\sigma(\delta)$ for $\gamma,\delta\in\Gamma$ and $\det(\sigma(\gamma))-1$ for $\gamma\in\Gamma$, where $\sigma(\gamma):=\left(a_{ij}(\gamma)\right)_{ij}$. The map $\sigma:\Gamma\longrightarrow SL_2(A_2(\Gamma))$, which by definition is a representation,will be refered as the universal representation for Γ.

The $X\in PGL_2$ acts on $A_2(\Gamma)$ by sending $\sigma(\gamma)$ to $X^{-1}\sigma(\gamma)X$, inducing the ring homomorphism $A_2(\Gamma)\longrightarrow A_2(\Gamma)\otimes A(PGL_2)$, where $A(PGL_2)$:=degree 0 part of the graded localization of the polynomial ring generated by entries of X by det(X). Taking R-valuded points of the rings, the homomorphism yiels the adjoint action of PGL_2 on the representation space:

(2.2) $$\mathrm{Ad} : \mathrm{Hom}(\Gamma,SL_2(R)) \times PGL_2(R) \longrightarrow \mathrm{Hom}(\Gamma,SL_2(R))$$

The categoriacal quotient space $\mathrm{Hom}(\Gamma,SL_2(R))//PGL_2(R)$ of the action is realized as $\mathrm{Hom}(A_2(\Gamma)^{PGL_2}, R)$:= the set of R-valued points of the affine scheme $\mathrm{Spec}(A_2(\Gamma)^{PGL_2})$, where $A_2(\Gamma)^{PGL_2}$:={polynomial$\in A_2(\Gamma)$ which is invariant by the adjoint action of $X\in PGL_2$} (cf [Mu, The.1.1]), and we have the universal categorical quotient map:

$$\text{(2.3)} \qquad \mathrm{Spec}(A_2(\Gamma)) \longrightarrow \mathrm{Spec}(A_2(\Gamma)^{PGL_2})$$

by the adjoint action of PGL_2. The quotient map (2.3) admits some group actions: i) $\alpha\in \mathrm{Aut}(\Gamma)$ (resp. $\mathrm{Out}(\Gamma)$) acts on $A_2(\Gamma)$ (resp. $A_2(\Gamma)^{PGL_2}$) equivariantly by sending the entries of $\sigma(\gamma)$ to that of $\sigma(\alpha(\gamma))$ for $\gamma\in\Gamma$, ii) the spin: $\chi\in\mathrm{Hom}(\Gamma,\mathbb{Z}/2)$ acts on $A_2(\Gamma)$ and $A_2(\Gamma)^{PGL_2}$ equivariantly by sending the entries of $\sigma(\gamma)$ to that of $\chi(\gamma)\sigma(\gamma)$ for $\gamma\in\Gamma$.

A structural study on the invariant ring and the quotient map (2.3) will be done in a Theorem in the next §3. In the remaining of this §, we devote our attention for an analysis of $\mathbb{R}$-valued points of the functor in connection with semi-algebraic geometry of the Teichmüller space.

The Weil's theorem[W2] says that $R_0(\Gamma,SL_2(\mathbb{R}))$ is an open subset of the real affine algebraic variety $\mathrm{Hom}(A_2(\Gamma),\mathbb{R})$ $(=R(\Gamma,SL_2(\mathbb{R})))$. Further, owing to several authors ([He],[Jo],[C-S]), it is shown to be a closed subset. For instance, a result of Jørgensen [Jø] states that a representation $\rho:\Gamma\longrightarrow SL_2(\mathbb{R})$ is discrete and faithfull if and only if, for every pair of γ, δ of Γ which are not commutative, the following inequality holds.

$$|\mathrm{tr}([\rho(\gamma),\rho(\delta)])-2| + |\mathrm{tr}(\rho(\gamma))^2-4| \geq 1.$$

Note that if Γ_g is discrete and faithfully represented in $SL_2(\mathbb{R})$, then it is automatically co-compact, since Γ_g is torsion free and $\rho(\Gamma_g)$ can not have an elliptic fixed point and hence $H^2(\rho(\Gamma_g)\backslash\mathbb{H},\mathbb{Z})\simeq H^2(\Gamma_g,\mathbb{Z})\simeq\mathbb{Z}$. Therefore, $R_0(\Gamma_g,SL_2(\mathbb{R}))$ *is a finite union of connected component of* $\mathrm{Hom}(A_2(\Gamma_g),\mathbb{R})$ *with respect to the classical topology* ([He],[C-S],[M-S]). (For informations on the other components of $\mathrm{Hom}(\Gamma_g,SL_2(\mathbb{R}))$, see [Go]).

We now compare two representation spaces by the natural projection:

$$R_0(\Gamma_g,SL_2(\mathbb{R})) \longrightarrow R_0(\Gamma_g,PGL_2(\mathbb{R}))$$

induced from the map $SL_2(\mathbb{R}) \longrightarrow PSL_2(\mathbb{R})\simeq SL_2(\mathbb{R})/\{\pm1\}$. Let us verify that

Assertion. The map is a 2^{2g}-fold covering.

Surjectivity: if $\bar{A}_i$ and $\bar{B}_i \in PSL_2(\mathbb{R})$ $(1 \le i \le g)$ generate a discrete subgroup of $PSL_2(\mathbb{R})$ isomorphic to Γ_g satisfying the relation $\prod_{i=1}^{g}(\bar{A}_i\bar{B}_i\bar{A}_i^{-1}\bar{B}_i^{-1})=1$, then do the representative matrices $A_i, B_i \in SL_2(\mathbb{R})$ of $\bar{A}_i$ and $\bar{B}_i$ satisfy the relation $\prod_{i=1}^{g}(A_iB_iA_i^{-1}B_i^{-1})=1$ in $SL_2(\mathbb{R})$? This question ([Si1]) was answered affirmatively by several authors $(Z(SL_2(\mathbb{R})) \mid c_1(K_{X_\rho})$, see [Pa]).

Covering transformation: since $PSL_2(\mathbb{R}) \simeq SL_2(\mathbb{R})/(\mathbb{Z}/2\mathbb{Z})$ where $\mathbb{Z}/2\mathbb{Z}$ is the center $Z(SL_2(\mathbb{R}))$, a spin $\chi \in Hom(\Gamma_g, \mathbb{Z}/2\mathbb{Z}) \simeq (\mathbb{Z}/2\mathbb{Z})^{2g}$ acts on $\rho \in R_0(\Gamma_g, SL_2(\mathbb{R}))$ by letting $\chi\rho(\gamma) := \chi(\gamma)\rho(\gamma)$. One has a set theoretic identification: $Hom(\Gamma_g, \mathbb{Z}/2\mathbb{Z}) \backslash R_0(\Gamma_g, SL_2(\mathbb{R})) \simeq R_0(\Gamma_g, PSL_2(\mathbb{R}))$. On the other hand, since $|tr(\rho(\gamma))| > 2$ for $\gamma \ne 1$, the action of the spins induces a simple action on the set of signs of $tr(\rho(a_i))$ and $tr(\rho(b_i))$ and therefore on the set of connected components. Hence it is fixed point free and covering map.□

As can be reduced to the case of §1, the action of $PGL_2(\mathbb{R})$ on $R_0(\Gamma_g, SL_2(\mathbb{R}))$ is fixed point free and proper with local transversal sections. So the quotient $R_0(\Gamma_g, SL_2(\mathbb{R}))/Ad(PGL_2(\mathbb{R}))$ is naturaly a real analytic manifold and $R_0(\Gamma_g, SL_2(\mathbb{R}))$ is a total space of a principal $PGL_2(\mathbb{R})$ bundle over the quotient space. On the other hand, it will be shown in a theorem in §3 that the categorical quotient and geometric quotient of $Spec(A_2(\Gamma))$ by the action of PGL_2 coincide in a domain in $Hom(A_2(\Gamma_g), \mathbb{R})$ containg $R_0(\Gamma_g, SL_2(\mathbb{R}))$, which is an inverse image of an open subset of $Hom(A_2(\Gamma_g)^{PGL_2}, \mathbb{R})$. This implies that the quotient real analytic variety

(2.4) $$\tilde{\mathcal{T}}_g := R_0(\Gamma_g, SL_2(\mathbb{R}))/PGL_2(\mathbb{R})$$

is embedded into the real affine algebraic variety $Hom\left(A_2(\Gamma_g)^{PGL_2}, \mathbb{R}\right)$ as an open and closed subset. In view of (1.15), we see that

(2.5) $$\mathcal{T}_g \simeq \mathrm{Hom}(\Gamma_g, \mathbb{Z}/2\mathbb{Z}) \setminus \tilde{\mathcal{T}}_g .$$

Here the action of the spin group on $\tilde{\mathcal{T}}_g$ is simple on the set of components of $\tilde{\mathcal{T}}_g$. That is: $\tilde{\mathcal{T}}_g$ decomposes into 2^{2g} parts, each of which is isomorphic to $\mathcal{T}_g$. We may call $\tilde{\mathcal{T}}_g$ *the Teichmüller space for spin Riemann surfaces*, or *the Teichmüller space* for an abuse of terminology.

So far as complex or Kähler structure are concerned, $\mathcal{T}_g$ and $\tilde{\mathcal{T}}_g$ does not make difference. We shall devote our attention on $\tilde{\mathcal{T}}_g$ here after in the present note. It is therefore quite necessary to determine the invariant subring $A_2(\Gamma_g)^{PGL_2}$, at least the place where the map (2.3) is a geometric quotient map. This will be achieved in §3. For the purpose, let us recall traces (=characters) of the universal representation σ.

For any $\gamma\in\Gamma$, put

(2.6) $$\Sigma(\gamma) := \mathrm{tr}(\sigma(\gamma))$$

Of course, the relation $\mathrm{tr}(X^{-1}\sigma(\gamma)X)=\mathrm{tr}(\sigma(\gamma))$ implies that $\Sigma(\gamma) \in A_2(\Gamma)^{PGL_2}$. Fricke [F-K] has found several algebraic relations among the $\Sigma(\gamma)$ for $\gamma\in\Gamma$, intending to show that the ring genrated by all such characters $\Sigma(\gamma)$ is in fact finitely generated and 6g-6 are algebraically independent in case $\Gamma=\Gamma_g$. This was rigoloulsy proved by Helling [He] and Horowitz [Ho] independently (cf [Mg]). Let us recall some simplest relations among them. Surprisingly, as we shall see in the §3, these relations are (essentially) sufficient to recover the invariant ring.

The first trivial relation is

(2.7) $$\Sigma(e) = 2$$

for the unit element e of Γ. The next relation well known as the trace relation ([F-K, formula (2), p.338]), which is essentially due to the Caylay-Hamilton relation, is the following:

(2.8) $$\Sigma(\gamma)\Sigma(\delta) = \Sigma(\gamma\delta) + \Sigma(\gamma^{-1}\delta).$$

Proof. By a use of the fact $\det(\sigma(\gamma))=1$, it is imediate to see the fact $\sigma(\gamma)+\sigma^{-1}(\gamma)=\mathrm{tr}(\sigma(\gamma))\cdot I_2$. Multiply $\sigma(\delta)$ and take the trace. □

§3. Invariant ring by the adjoint action of PGL_2

We "aproximate" the invariant ring $A_2(\Gamma)^{PGL_2}$ by a ring R_Γ, whose generators and relations can be explicitly written down. By the use of the ring R_Γ, we determine the Zariski open subset of $\mathrm{Spec}(A_2(\Gamma))$, where the adoint action of PGL_2 induces geometric quotient (Theorem). Then $\tilde{\mathcal{T}}_g$ is regarded as an open-closed subset of the affine variety $\mathrm{Hom}(R_\Gamma,\mathbb{R})$.

As in §2, let Γ be a finitely generated group. We introduce a ring:

(3.1) $$R_\Gamma := \mathbb{Z}[s(\gamma)\ \ (\gamma\in\Gamma)]/(s(e)-2,\ s(\gamma)s(\delta)-s(\gamma\delta)-s(\gamma^{-1}\delta)\ \ (\gamma,\delta\in\Gamma))$$

generated by indeterminates: $s(\gamma)$ for $\gamma\in\Gamma$,
which stand for the traces $\Sigma(\gamma)$, and divided by the ideal generated by:

$$s(e)-2 \quad\text{and}\quad s(\gamma)s(\delta)-s(\gamma\delta)-s(\gamma^{-1}\delta) \text{ for } \gamma,\delta\in\Gamma$$

which stand for the trace relations in (2.7) and (2.8). As an abstract algebra, one can show the finitely generatedness of the ring R_Γ over $\mathbb{Z}$ likewise the proof for the character ring in §2. The correspondence $s(\gamma) \longmapsto \Sigma(\gamma)$ induces a ring homomorphism:

(3.2) $$R_\Gamma \longrightarrow A_2(\Gamma)^{PGL_2} \subset A_2(\Gamma).$$

We are interested in a comparison of R_Γ and $A_2(\Gamma)^{PGL_2}$. By definition, the image generates the ring of characters so that $R_\Gamma\otimes_{\mathbb{Z}}\mathbb{Q}$ surjects to $A_2(\Gamma)^{PGL_2}\otimes_{\mathbb{Z}}\mathbb{Q}$ (cf [Pr]).

On the other hand, we show below that they coicides outside of some Zariski closed set D (=intersection of discriminant loci defined below)

First, we introduce a discriminant for two elements α and β in Γ by

$$\Delta(\alpha,\beta) := s(\alpha\beta\alpha^{-1}\beta^{-1})-2 \ . \tag{3.3}$$

$$= s(\alpha)^2+s(\beta)^2+s(\alpha\beta)^2-s(\alpha)s(\beta)s(\alpha\beta)-4.$$

It is defined as an element in R_Γ, but its image in $A_2(\Gamma)$ will be denoted by the same notation $\Delta(\alpha,\beta)$. This element first appeared in [F] and has been studied extensively by many authors.

The following theorem, inspired by the work of Helling [He], is one key result in whole of the present paper (cf [Sa4]).

Theorem *Let* α *and* β *be any fixed pair of elemements of* Γ. *Consider the localizations of* R_Γ *and* $A_2(\Gamma)$ *by* $\Delta:=\Delta(\alpha,\beta)$. *Then*

1. $A_2(\Gamma)_\Delta$ *is faithfully flat over* $A_2(\Gamma)_\Delta^{PGL_2}$.

2. *The morphism* $\mathrm{Spec}(A_2(\Gamma)_\Delta) \longrightarrow \mathrm{Spec}(A_2(\Gamma)_\Delta^{PGL_2})$ *is the geometric quotient map by* PGL_2.

3. *The homomorphism* (3.2) *induces an isomorphism:*

$$R_{\Gamma,\Delta} \simeq A_2(\Gamma)_\Delta^{PGL_2} \tag{3.4}$$

This implies that i)outside of the zero loci of $\Delta(\alpha,\beta)$, the categorical quotient map (2.3) is in fact the geometric quotient map by the action of PGL_2, and ii) the structure ring of the quotient variety (outside of zero loci of $\Delta=\Delta(\alpha,\beta)$) is given by (3.1). These two facts are exibited by the following two comutative diagrams symbolically.

(3.5)

$$\begin{array}{ccc}
\text{Spec } A_2(\Gamma) & \underset{\text{open}}{\supset} & \text{Spec } A_2(\Gamma)_\Delta \\
\Big\downarrow \begin{smallmatrix}\text{categorical}\\ \text{quotient by } PGL_2\end{smallmatrix} & & \Big\downarrow \begin{smallmatrix}\text{geometric}\\ \text{quotient by } PGL_2\end{smallmatrix} \\
\text{Spec } A_2(\Gamma)^{PGL_2} & \underset{\text{open}}{\supset} & \text{Spec } A_2(\Gamma)_\Delta^{PGL_2} \\
\Big\downarrow & & \Big\downarrow \wr \text{ isomorphism} \\
\text{Spec } R_\Gamma & \underset{\text{open}}{\supset} & \text{Spec } R_{\Gamma,\Delta}
\end{array}$$

The union of Zariski open set $\mathfrak{U}:=\cup_{\alpha,\beta}\text{Spec } A_2(\Gamma)_{\Delta(\alpha,\beta)}=\cup_{\alpha,\beta}\{\Delta(\alpha,\beta)\neq 0\}$ in Spec $A_2(\Gamma)$ covers the places of our interest. Then, the following is a slite modification of the fact known by several authors ([He], [Si], [C-S]).

Assertion. Let $\mathfrak{p}$ *be a prime ideal of* $A_2(\Gamma)$. *Then* $\mathfrak{p}$ *belongs to* $\text{Spec}(A_2(\Gamma))\setminus\mathfrak{U}$, *if and only if either the image* $\sigma_{\mathfrak{p}}(\Gamma)$ *is abelian or* $\sigma_{\mathfrak{p}}$ *is a reducible representation. Here* F *is the fractional field of* $A_2(\Gamma)/\mathfrak{p}$ *and* $\sigma_{\mathfrak{p}}$ *is the representation* $\Gamma \longrightarrow SL_2(F)$ *induced from the universal* σ.

In case of surface group Γ_g for $g\geq 2$, the (open) subset $R_0(\Gamma_g, SL_2(\mathbb{R}))$ of $\text{Hom}(A_2(\Gamma_g),\mathbb{R})$ is contained in any Zariski open set $\Delta(\alpha,\beta)\neq 0$ for non commutative pair α,β of Γ_g, since $\rho(\alpha\beta\alpha^{-1}\beta^{-1})\in SL_2(\mathbb{R})$ is a nontrivial hyperbolic matrix so that $\Delta=\text{tr}\rho(\alpha\beta\alpha^{-1}\beta^{-1})-2\neq 0$. Therefore applying the theorem, the quotient variety $\tilde{\mathfrak{T}}_g := R_0(\Gamma_g, SL_2(\mathbb{R}))/\text{Ad}(PGL_2(\mathbb{R}))$ is openly embedded in the real affine algebraic variety $\text{Hom}(A_2(\Gamma)^{PGL_2}, \mathbb{R})$. The image is closed in view of the Jørgensen's result. So $\tilde{\mathfrak{T}}_g$ is a union of some connected components of the affine variety $\text{Hom}(A_2(\Gamma)^{PGL_2}, \mathbb{R})$, readily known ([He], [C-S], [M-S]).

$$
(3.6)\qquad
\begin{array}{ccc}
\mathrm{Hom}(A_2(\Gamma),\ \mathbb{R}) & \supset & R_0(\Gamma,\ SL_2(\mathbb{R})) \\
\big\downarrow {\scriptstyle //PGL_2(\mathbb{R})} & & \big\downarrow {\scriptstyle /PGL_2(\mathbb{R})} \\
\mathrm{Hom}(A_2(\Gamma)^{PGL_2},\mathbb{R}) & \supset & \tilde{\mathfrak{T}}_g := R_0(\Gamma,SL_2(\mathbb{R}))/Ad(PGL_2(\mathbb{R})) \\
\big\downarrow {\scriptstyle \text{closed immersion}} & & \\
\mathrm{Hom}(R_\Gamma,\ \mathbb{R}) & &
\end{array}
$$

Since $R_\Gamma \otimes_{\mathbb{Z}} \mathbb{R} \longrightarrow A_2(\Gamma)^{PGL_2} \otimes_{\mathbb{Z}} \mathbb{R}$ is surjective, $\mathrm{Hom}(A_2(\Gamma)^{PGL_2},\mathbb{R})$ is a closed subvariety of $\mathrm{Hom}(R_\Gamma,\mathbb{R})$. Let us regard now $\tilde{\mathfrak{T}}_g$ as a subset of the affine variety $\mathrm{Hom}(R_\Gamma,\ \mathbb{R})$. We show that:

$\tilde{\mathfrak{T}}_g$ is an open and closed subset of the real affine variety $\mathrm{Hom}(R_\Gamma,\mathbb{R})$. *The real anlytic variety structure on $\tilde{\mathfrak{T}}_g$ as the geometric quotient variety coincides with that induced from real affine variety* $\mathrm{Hom}(R_\Gamma,\mathbb{R})$.

Proof. $\tilde{\mathfrak{T}}_g$ is open since it is an open subset of an open subset $\mathrm{Hom}(A_2(\Gamma)_\Delta^{PGL_2},\ \mathbb{R}) \simeq \mathrm{Hom}(R_{\Gamma,\Delta},\ \mathbb{R})$ of $\mathrm{Hom}(R_\Gamma,\ \mathbb{R})$. $\tilde{\mathfrak{T}}_g$ is closed since it is a closed subset of a closed subset $\mathrm{Hom}(A_2(\Gamma)^{PGL_2},\mathbb{R})$ of $\mathrm{Hom}(R_\Gamma,\mathbb{R})$.□

Therefore, here after we regard R_Γ as *the structure ring for the Teichmüller space $\tilde{\mathfrak{T}}_g$ and $\tilde{\mathfrak{T}}_g$ is regarded as an open closed semialgebraic subset of* $\mathrm{Hom}(R_\Gamma,\mathbb{R})$.

Remark. As we see in §1 and also is pointed in Helling [He,], the $R_0(\Gamma_g,PSL_2(\mathbb{R}))$ consisits of two connected components due to two orientation classes. As stated in §1, we choose the adjoint action of $PGL_2(\mathbb{R})/PSL_2(\mathbb{R})$ to identify the two components (cf (1.14), (1.15)). This enable us to develop a PGL_2-invariant theory as in this §3.

In [M-S, p451] and [Br, p65-66], they consider the adjoint action of SL_2 to apply the geometric invariant theory, and claim the second vertical map of (3.6) is a $SL_2(\mathbb{R})$ principal bundle. But, since the character variety $X(\Gamma)$ they consider as the target space of the map, is the quotient variety of $R_0(\Gamma_g, SL(2,\mathbb{R}))$ by the $PGL_2(\mathbb{R})$, it seems necessary to correct the action from SL_2 to PGL_2.

§4. Group cohomology of Γ and the tangent space of $\tilde{\mathcal{T}}_g$

We compare the tangent space of $\tilde{\mathcal{T}}_g$ with the cohomology group of Γ_g as originated in [W2-3]. The tangent space of $\tilde{\mathcal{T}}_g$ is described by the derivations of the algebra R_Γ in view of the last statement in §3.

In general, let R be a commutative algebra with the unit 1. Consider a point $t \in Hom(R_\Gamma, R)$, represented as a ring homomorphism:

$$t: R_\Gamma \longrightarrow R \tag{4.1}$$

(which preserves the unit). A t-derivation w of R_Γ is, by definition, an additive map $w: R_\Gamma \longrightarrow R$ satisfying the Leibniz rule:

$$w(fg) = w(f)t(g) + t(f)w(g) \tag{4.2}$$

for any f and $g \in R_\Gamma$. The set of all t-derivations of R_Γ is denoted by

$$Der_t(R_\Gamma, R) := \{t\text{-derivations of } R_\Gamma\}, \tag{4.3}$$

which is naturally a right R-module.

Particularly, if $R=\mathbb{R}$ and $t \in \tilde{\mathcal{T}}_g$ then the tangent space $T_t\tilde{\mathcal{T}}_g$ (recall that $\tilde{\mathcal{T}}_g$ (or equivalently $\mathcal{T}_g$) is a smooth real analytic variety (§1)), is mapped naturally to the space of derivations $Der_t(R_\Gamma, \mathbb{R})$. The map is bijective, if and only if $\tilde{\mathcal{T}}_g$ is regular at p as the affine scheme of R_Γ. In fact, this is true seen by (3.6) and its following statements.

Since the generators of the algebra R_Γ are given by $s(\gamma)$ $\gamma \in \Gamma$ (2.1.1), a t-derivation w is uniquely determined if we know the values $w(s(\gamma))$ $\in R$ for $\gamma \in \Gamma$. For a sake of simplicity, let us denote $w(s(\gamma))$ by $w(\gamma)$ and $t(s(\gamma))$ by $t(\gamma)$. In view of the description (3.1) of the algebra R_Γ (or recall the relations (2.7) and (2.8)), a map

$$w: \Gamma \longrightarrow R$$

gives a t-derivation if and only if it satisfies:

$$w(e) = 0, \tag{4.4}$$

$$w(\gamma\delta) + w(\gamma^{-1}\delta) = w(\gamma)t(\delta) + t(\gamma)w(\delta) \tag{4.5}$$

for $\gamma, \delta \in \Gamma$. Therefore we have an identification:

$$\mathrm{Der}_t(R_\Gamma, R) \simeq \{w:\Gamma \longrightarrow R \mid (4.4) \text{ and } (4.5)\} \tag{4.6}$$

We compare the space of derivations with the cohomolgy group of Γ. First, recall and fix notations of the cohomology with coefficient in a right Γ-module M. The module of cocycles is defined by

$$Z^1(\Gamma,M) := \{z:\Gamma \longrightarrow M \mid \text{a map satifying the cocycle condition: } z(\gamma\delta) = z(\gamma)\cdot\delta + z(\delta) \text{ for } {}^\forall\gamma,\delta \in \Gamma\}.$$

The coboundary map δ is a homomorphism: $M \longrightarrow Z^1(\Gamma,M)$ given by

$$\delta(m)(\gamma) := m\cdot\gamma - m$$

for $m \in M$ and $\gamma \in \Gamma$. The cohomology group is the module defined by

$$H^1(\Gamma,M) := Z^1(\Gamma,M)/\delta M .$$

If M is a left R-module, then so are $Z^1(\Gamma,M)$ and $H^1(\Gamma,M)$ and δ is a left R-homomorphism. If a representaion $\rho:\Gamma \longrightarrow SL_2(R)$ is given, then by a composition with ρ, any $SL_2(R)$-module M becomes a Γ-module. In such case, to stress the ρ dependence, we put subscript ρ in the notations of the module of cocycles, coboundary maps and cohomology groups.

Let $t_\rho:R_\Gamma \longrightarrow R$ be the ring homomorphism defined as a composition of (3.2) with a homomorphism $A_2(\Gamma) \longrightarrow R$ associated to a representation ρ

(§2 Assertion). That is: $t_\rho(\gamma)=t_\rho(s(\gamma)):= tr(\rho(\gamma))$. In such situation, we define a comparison R-homomorphism

$$(4.7) \qquad H^1_\rho(\Gamma, sl_2(R)) \longrightarrow Der_{t_\rho}(R_\Gamma, R) \ ,$$

by the correspondence:

$$z \in Z^1_\rho(\Gamma, sl_2(R)) \longmapsto tr(\rho z) \in Der_{t_p}(R_\Gamma, R)$$

$$tr(\rho z)(\gamma) := tr(\rho(\gamma)\cdot z(\gamma)) \ .$$

To show that this is well defined, we need to verify two statements:

i) *For* $z\in Z^1_\rho(\Gamma, sl_2(R))$, *the map* $tr(\rho z):\Gamma \longrightarrow R$ *satisfies the relation* (4.4-5) *for the* t_ρ*-derivations.*

ii) *The* t_ρ*-derivation* $tr(\rho\delta_\rho(X))$ *associated to the coboundary* $\delta_\rho(X)$ *for any* $X\in sl_2(R)$ *is equal to zero in* $Der_{t_\rho}(R_\Gamma, R)$.

Proof of i). Let $z\in Z^1_\rho(\Gamma, sl_2(R))$ as above and put $w:=tr(\rho z)$. Clearly $w(e)=tr(\rho(e)z(e))=tr(0)=0$. For $\gamma,\delta \in\Gamma$, consider

$$w(\gamma\delta) + w(\gamma^{-1}\delta) := tr\Big(\rho(\gamma\delta)z(\gamma\delta) + \rho(\gamma^{-1}\delta)z(\gamma^{-1}\delta)\Big).$$

By a use of the cocycle condition on z, this is rewritten as:

$$tr\Big(\rho(\gamma)\rho(\delta)\Big(\rho(\delta^{-1})z(\gamma)\rho(\delta)+z(\delta)\Big)+\rho(\gamma^{-1})\rho(\delta)\Big(\rho(\delta^{-1})z(\gamma^{-1})\rho(\delta)+z(\delta)\Big)\Big).$$

Using the fact $z(\gamma^{-1})= -\rho(\gamma)z(\gamma)\rho(\gamma^{-1})$ (easily seen from the cocycle condition), this is rewritten as

$$*) \qquad tr\Big(\Big(\rho(\gamma)z(\gamma)-z(\gamma)\rho(\gamma^{-1})\Big)\rho(\delta)\Big) + tr\Big(\Big(\rho(\gamma)+\rho(\gamma^{-1})\Big)\rho(\delta)z(\delta)\Big).$$

Let $\rho(\gamma)=\begin{bmatrix} a & b \\ c & d\end{bmatrix}$ and $z(\alpha)=\begin{bmatrix} p & q \\ r & s\end{bmatrix}$. Due to the facts $\rho(\gamma)\in SL_2(R)$ and $z(\gamma)\in sl_2(R)$, one has $\rho(\gamma^{-1})=\begin{bmatrix} d & -b \\ -c & a\end{bmatrix}$ and $s=-p$. Therefore, onw obtains:

$$**) \qquad \rho(\gamma) + \rho(\gamma^{-1}) = \begin{bmatrix} a & b \\ c & d\end{bmatrix} + \begin{bmatrix} d & -b \\ -c & a\end{bmatrix} = tr(\sigma(\gamma))\ E,$$

$$***) \qquad \rho(\gamma)z(\gamma)-z(\gamma)\rho(\gamma^{-1}) = \begin{bmatrix} a & b \\ c & d\end{bmatrix}\begin{bmatrix} p & q \\ r & -p\end{bmatrix} - \begin{bmatrix} p & q \\ r & -p\end{bmatrix}\begin{bmatrix} d & -b \\ -c & a\end{bmatrix}$$

$$= (ap+br+cq-dp)\ E = tr(\sigma(\gamma)z(\gamma))\ E.$$

Substitute **) and ***) in *), and one obtains the formula,

$$w(\gamma\delta) + w(\gamma^{-1}\delta) = tr(\rho(\gamma)z(\gamma))tr(\rho(\delta)) + tr(\rho(\gamma))tr(\rho(\delta)z(\delta))$$
$$= w(\gamma)\ t_\rho(\delta) + t_\rho(\gamma)\ w(\delta)\ . \qquad \square$$

Proof of ii). For $X \in sl_2(R)$ and $\gamma \in \Gamma$,

$$tr\Big(\rho(\gamma)(\delta_\rho(X)(\gamma))\Big) = tr\Big(\rho(\gamma)(X-\rho(\gamma^{-1})X\rho(\gamma))\Big) = tr\Big(\rho(\gamma)X-X\rho(\gamma)\Big) = 0. \qquad \square$$

We give a criterium for the map (4.7) to be isomorphic. The theorem in §3 applied to the $R[\varepsilon]/(\varepsilon^2)$-valued points implies the following.

Theorem. Suppose there exist α *and* $\beta \in \Gamma$ *such that* $t_\rho(\Delta(\alpha,\beta))$ *is invertible in* R. *Then the map* (4.7) *is an isomorphism of* R-*modules.*

$$H^1_\rho(\Gamma, sl_2(R)) \simeq Der_{t_\rho}(R_\Gamma, R)\ .$$

The advantage of the isomorphism lies in the fact that the right hand side module $Der_t(R_\Gamma,R)$ (4.6) is described only in terms of the point t (4.1) in $Hom(R_\Gamma, R)$ and is not necessary to refer to the equivalence class of representations ρ over t, whereas the left hand side module $H^1_\rho(\Gamma,sl_2(R))$ depends on the representative ρ over t. But exactly for that reason, the left hand side carries various structures.

§5. Eichler integrals of the quadratic differentials

We recall a notion of an Eichler integral. Their original form are given in [Bo],[E1]. For detailed account on Eichler-Shimura isomorphism one is refered to Shimura [Sh1-2]. For relations with Kleinean groups, see [K1]. In connection with the complex structure on the Teichmüller space and a generalization of abelian integrals, see [Sa3].

Let us fix a cocompact discrete faithfull representation $\rho: \Gamma_g \longrightarrow SL_2(\mathbb{R})$. Let $\mathbb{H}$ be the complex upper half plane with the coordinate z and consider the associated Riemann surface $X_\rho := \rho(\Gamma)\backslash\mathbb{H}$. The fact that the representation ρ is not into $PSL_2(\mathbb{R})$ but into $SL_2(\mathbb{R})$ implies that the Riemann surface X_ρ caries not only the structure of the uniformization (as discussed in i),ii) and iii) in §1) but also some more: called the spin structure as described below.

For any integer $n\in\mathbb{Z}$, let $\mathcal{O}_{\mathbb{H}}(K_{\mathbb{H}}^{-n/2})$ be the sheaf of $\mathcal{O}_{\mathbb{H}}$-free module of rank 1 generated by a symbol $\left(\frac{d}{dz}\right)^{n/2}$. In case of $n<0$, we denote the symbol $\left(\frac{d}{dz}\right)^{n/2}$ by $dz^{-n/2}$. We define a right action of $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R})$ on the module $\mathcal{O}_{\mathbb{H}}(K_{\mathbb{H}}^{-n/2})$ by:

$$(5.1) \qquad Ad^{n/2}(A): \quad \varphi(z)\left(\frac{d}{dz}\right)^{n/2} \mapsto \varphi\left(\frac{az+b}{cz+d}\right)(cz+d)^n\left(\frac{d}{dz}\right)^{n/2}.$$

We call this *the adjoint action of* A and denote it by $ad^{n/2}(A)$. Composing $Ad^{n/2}$ with ρ, Γ_g acts on the module $\mathcal{O}_{\mathbb{H}}(K_{\mathbb{H}}^{-n/2})$ so that the quotient is a sheaf of $\mathcal{O}_{X_p}$-module denoted by $\mathcal{O}_{X_\rho}(K_{X_\rho}^{-n/2})$ over X_ρ.

Let $\Gamma(\mathbb{H},\mathcal{O}_{\mathbb{H}}(K_{\mathbb{H}}^{-n/2}))$ be the module of global sections over $\mathbb{H}$. Then the invarant subspace $\Gamma(\mathbb{H},\mathcal{O}_{\mathbb{H}}(K_{\mathbb{H}}^{-n/2}))^{\rho(\Gamma_g)}$ is canonically isomorphic to the finite dimensional complex vector space $\Gamma(X_\rho,\mathcal{O}_{X_\rho}(K_{X_\rho}^{-n/2}))$, whose rank is equal to 0, 1, g or $-(n+1)(g-1)$ according as $n>0$, $n=0$, $n=-2$ or $n\leq-3$ respectively, easily calculated from Riemann-Roch theorem. More in general,one has the natural identification: $H_\rho^i(\Gamma_g,\Gamma(\mathbb{H},\mathcal{F}_{\mathbb{H}})) \simeq H^i(X_\rho,\mathcal{F}_{X_\rho})$ for a sheaf $\mathcal{F}_{\mathbb{H}}$ on $\mathbb{H}$ admitting $SL_2(\mathbb{R})$ action with $\mathcal{F}_{X_\rho} := \mathcal{F}_{\mathbb{H}}/\rho(\Gamma_g)$.

For each $n\geq 0$, consider a map defined by a differentiation:

$$(5.2) \qquad \partial^{n+1}: \Gamma(\mathbb{H}, \mathcal{O}_{\mathbb{H}}(K_{\mathbb{H}}^{-n/2})) \longrightarrow \Gamma(\mathbb{H}, \mathcal{O}_{\mathbb{H}}(K_{\mathbb{H}}^{n/2+1}))$$

$$\varphi(z)\left(\frac{d}{dz}\right)^{n/2} \longmapsto \varphi^{(n+1)}(z)dz^{n/2+1}$$

Here $\varphi^{(n+1)}$ denotes the n+1-th derivative of φ with respect to the coordinate z. It was a remarkable observation due to G. Bol [Bo] that the map ∂^{n+1} is equivariant with the $SL_2(\mathbb{R})$ actions on the bothsides. That is: one has the relation:

$$\left(\frac{d}{dz}\right)^{n+1}\left(\varphi\left(\frac{az+b}{cz+d}\right)(cz+d)^n\right) = \varphi^{(n+1)}\left(\frac{az+b}{cz+d}\right)(cz+d)^{-n-2}$$

for any function φ in one variable z. This fact can be elementarily checked by induction on n, whose verification is left to the reader.

The map ∂^{n+1} is surjective. The kernel of ∂^{n+1} is the n+1 dimensional vector space over $\mathbb{C}$ of forms with coefficients in polynomials of degree less or equal than n in the variable z. Let us observe the kernel more closely. Let $\mathbb{R}^2$ be the space of 2-column vector $\begin{bmatrix} u \\ v \end{bmatrix}$, on which $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\in SL_2(\mathbb{R})$ acts naturally by $\begin{bmatrix} au+bv \\ cu+dv \end{bmatrix}$. Let $Sym^n(\mathbb{R}^2)$ be the n-th symmetric tensor product of $\mathbb{R}^2$, which is naturally identified with the space of binary n-forms u^n, $u^{n-1}v, \ldots, v^n$ spanned over $\mathbb{R}$. Using an inhomogeneous coordinate: $z=u/v$, the correspondence: $P(u/v)v^n \longmapsto P(z)\left(\frac{d}{dz}\right)^{n/2}$ gives a natural isomorphims:

$$(5.3) \qquad Sym^n(\mathbb{R}^2)\otimes_{\mathbb{R}}\mathbb{C} \simeq Ker(\partial^{n+1})$$

as $SL_2(\mathbb{R})$-module (cf (5.1)). This implies a quite important consequence the kernel of ∂^{n+1} obtained a *real structure* so that it admits an operation of complex conjugate.

Let us denote by $Sym^n(\mathbb{R}^2)_{\mathbb{H}}$ the local system with constant fiber $Sym^n(\mathbb{R}^2)$ defined over $\mathbb{H}$. The $SL_2(\mathbb{R})$ acts naturaly on the local system equivariant with its action on the sheaf $\mathcal{O}_{\mathbb{H}}(K_{\mathbb{H}}^{-n/2})$ w.r.t. the embedding (5.3). So we obtain an $SL_2(\mathbb{R})$-equivariant short exact sequence:

$$(5.4) \qquad 0 \longrightarrow Sym^n(\mathbb{R}^2)_{\mathbb{H}}\otimes_{\mathbb{R}}\mathbb{C} \longrightarrow \mathcal{O}_{\mathbb{H}}(K_{\mathbb{H}}^{-n/2}) \xrightarrow{\partial^{n+1}} \mathcal{O}_{\mathbb{H}}(K_{\mathbb{H}}^{n/2+1}) \longrightarrow 0.$$

Let us return to the map (5.2).

Definition. For a $\omega\in\Gamma(X_\rho,\ \mathcal{O}_{X_\rho}(K_{X_\rho}^{n/2+1}))$, any of its inverse image by the map ∂^{n+1} in $\Gamma(\mathbb{H},\mathcal{O}_{\mathbb{H}}(K_{\mathbb{H}}^{-n/2}))$ will be called an *Eichler integral* of ω and will be denoted by $\int\omega$. By definition, the Eichler integral exists for any ω and is unique up to an addition of an element of $\mathrm{Sym}^n(\mathbb{C}^2)$. The ambiguity of the integral will be called the *integral constant.*

Let $\int\omega$ be an Eichler integral. For any $\gamma\in\Gamma$, the image by the adjoint action $(\int\omega)\cdot\mathrm{Ad}^{n/2}(\rho(\gamma))$ is again an Eichler integral for ω, due to the equivariance of ∂^{n+1} and γ invariance of ω. Namely

$$\partial^{n+1}\Big((\int\omega)\cdot\mathrm{Ad}^{n/2}(\rho(\gamma))\Big)=\Big(\partial^{n+1}\int\omega\Big)\cdot\mathrm{Ad}^{n/2}(\rho(\gamma))=\omega\cdot\mathrm{Ad}^{n/2}(\rho(\gamma))=\omega.$$

So the difference $(\int\omega)\cdot\mathrm{Ad}^{n/2}(\rho(\gamma))-\int\omega$ is an integral constant in $\mathrm{Sym}^n(\mathbb{R})\otimes_{\mathbb{R}}\mathbb{C}$, which we shall denote by $\int_\gamma\omega$ and call the *period* of the integral $\int\omega$ for $\gamma\in\Gamma_g$. So:

$$(5.5)\qquad \int_\gamma\omega = (\int\omega)\cdot\mathrm{Ad}^{n/2}(\rho(\gamma)) - \int\omega$$

By definition, the period of the integral satisfies the following addition relation for γ and $\delta\in\Gamma_g$:

$$(5.6)\qquad \int_{\gamma\delta}\omega = (\int_\gamma\omega)\cdot\mathrm{Ad}^{n/2}(\rho(\delta)) + \int_\delta\omega$$

For instance, if $n=0$, $\omega\in\Gamma(X_\rho,\mathcal{O}_X(K_X))$ is the abelian differential of the first kind. Then $\int\omega$ is the indefinite integral of ω viewed as a function on the universal covering $\mathbb{H}$ of X. The period $\int_\gamma\omega$ is nothing but the period in the classical sense, so the (5.6) is the standard addition relation of the periods.

In general, for a fixed $\omega\in\Gamma(X_\rho,\mathcal{O}_{X_\rho}(K_{X_\rho}^{n/2+1}))=\Gamma(\mathbb{H},\mathcal{O}_{\mathbb{H}}(K_{\mathbb{H}}^{n/2+1}))^{\rho(\Gamma_g)}$, the relation (5.6) is the cocycle condition for the map $\gamma\in\Gamma\mapsto\int_\gamma\omega\in$

$Sym^n(\mathbb{R}^2)\otimes_{\mathbb{R}}\mathbb{C}$, so the map is a cocycle $Z^1_\rho(\Gamma, Sym^n(\mathbb{R})\otimes_{\mathbb{R}}\mathbb{C})$. The cocycle $\int_\gamma \omega$ is determined from ω up to an ambiguity of adding a coboundary of an integral constant from $Sym^n(\mathbb{R})\otimes_{\mathbb{R}}\mathbb{C}$. So we obtain a map, which associates to a form ω the cohomology class of its periods of Eichler integral.

$$(5.7)\quad P_n: \Gamma(X_\rho, \mathcal{O}_{X_\rho}(K_{X_\rho}^{n/2+1})) \longrightarrow Z^1_\rho(\Gamma_g, Sym^n(\mathbb{R}^2)\otimes_{\mathbb{R}}\mathbb{C})/\delta_\rho(Sym^n(\mathbb{R}^2)\otimes_{\mathbb{R}}\mathbb{C}) \simeq H^1_\rho(\Gamma_g, Sym^n(\mathbb{R}^2))\otimes_{\mathbb{R}}\mathbb{C}$$

The source space is a complex vector space of rank g for $n=0$ and rank $(n+1)(g-1)$ for $n>0$ as well known by a use of Riemann Roch thorem. On the other hand, one has

$$\mathrm{rank}_{\mathbb{R}}\ H^1(\Gamma_g, Sym^n(\mathbb{R}^2)) = \begin{cases} 2g & n=0 \\ 2(n+1)(g-1) & n>0 \end{cases}.$$

This is calculated roughly as follows. Since Γ_g has $2g$ generators with a single relation, $Z^1(\Gamma_g, Sym^n(\mathbb{R}^2)) = \{(z_i)\in\left(Sym^n(\mathbb{R}^2)\right)^{2g} |$ a single cocycle condition$\}$. Then, rank of $H^1(\Gamma_g, Sym^n(\mathbb{R}^2)) = Z^1(*)/\delta\left(Sym^n(\mathbb{R}^2)\right)$ for $n>0$ is equal to $2g\cdot\mathrm{rank}(Sym^n(\mathbb{R}^2)) - \mathrm{rank}(Sym^n(\mathbb{R}^2)) - \mathrm{rank}(Sym^n(\mathbb{R}^2))$. Here one needs to check that the cocyle condition and the coboundary map are non-degenrate. For $n=0$, the Γ_g action on $Sym^0(\mathbb{R}^2)=\mathbb{R}$ becomes trivial. Then the cocycle condition and the coboundary map degenerate so that the rank is $2g\cdot\mathrm{rank}_{\mathbb{R}}(\mathbb{R})$. □

Thus the source space of the map P_n (5.7) is half dimensional of the target space. We shall show that the image $\mathrm{Im}(P_n)$ does not intersect with the real subspace $H^1(\Gamma, Sym^n(\mathbb{R}^2))$. This can be achieved through the following Eichler Shimura isomorphism:

Theorem ([E], [Sh]) *The inclusion map* $Sym^n(\mathbb{R}^2)_{\mathbb{H}} \longrightarrow \mathcal{O}_{\mathbb{H}}(K_{\mathbb{H}}^{-n/2})$ (5.3) *induces an isomorphism of* $\mathbb{R}$*-vector spaces:*

$$(5.8)\qquad H^1_\rho(\Gamma_g, Sym^n(\mathbb{R}^2)) \simeq H^1_\rho(\Gamma_g, \Gamma(\mathbb{H}, \mathcal{O}_{\mathbb{H}}(K_{\mathbb{H}}^{-n/2})))$$

$$\simeq H^1(X_\rho, \mathcal{O}_{X_\rho}(K_{X_\rho}^{-n/2}))$$

The isomorphism of the first and the second line is a general fact in a comparison theory of group cohomology and sheaf cohomology.

We draw some consequences of the theorem. The exact sequence (5.4) yiels an exact sequence: $0 \longrightarrow \mathrm{Sym}^n(\mathbb{C}^2)_X \longrightarrow \mathcal{O}_X(K_X^{-n/2}) \longrightarrow \mathcal{O}_X(K_X^{n/2+1}) \longrightarrow 0$ on X_ρ by dividing by $\rho(\Gamma_g)$. Take the long exact sequence of the cohomology associated to that. Applying the vanishings $\Gamma(X, \mathcal{O}_X(K_X^{-n/2})) = \{0\}$ and $H^1(X, \mathcal{O}_X(K_X^{n/2+1})) = 0$ $(n > 0)$ to the long exact sequence, we obtain:
(5.9)

$$0 \longrightarrow \Gamma(X_\rho, \mathcal{O}_{X_p}(K_{X_\rho}^{n/2+1})) \longrightarrow H^1(\Gamma_g, \mathrm{Sym}^n(\mathbb{R}^2)) \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow H^1(X_\rho, \mathcal{O}_{X_\rho}(K_{X_\rho}^{-n/2})) \longrightarrow 0$$

where the first morphism is the map P_n (5.7). The isomorphism (5.8) says that real subspace $H^1(\Gamma_g, \mathrm{Sym}^n(\mathbb{R}^2))$ of the middle module is isomorphic to the third module. Particularly the real subspace should not intersect with the kernel of second morphism = $\mathrm{Im}(P_n)$:

$$H^1(\Gamma_g, \mathrm{Sym}^n(\mathbb{R}^2)) \cap \mathrm{Im}(P_n) = \{0\}.$$

This implies that the complexification $H^1(\Gamma_g, \mathrm{Sym}^n(\mathbb{R}^2)) \otimes_{\mathbb{R}} \mathbb{C}$ decomposes into a direct sum $\mathrm{Im}(P_n) \oplus \overline{\mathrm{Im}(P_n)}$ of conjugate subspaces. In view of the exact sequence (5.9), the $\mathrm{Im}(P_n)$ (resp. $\overline{\mathrm{Im}(P_n)}$) is cannonically isomorphic to $\Gamma(X_\rho, \mathcal{O}_{X_p}(K_{X_\rho}^{n/2+1}))$ (resp. $H^1(X, \mathcal{O}_X(K_X^{-n/2}))$). Thus one obtains a decomposition:

$$(5.10) \qquad H^1(\Gamma_g, \mathrm{Sym}^n(\mathbb{R}^2)) \otimes_{\mathbb{R}} \mathbb{C} \simeq \Gamma(X_\rho, \mathcal{O}_{X_\rho}(K_{X_\rho}^{n/2+1})) \oplus H^1(X_\rho, \mathcal{O}_{X_\rho}(K_{X_\rho}^{-n/2}))$$

which is the goal of this paragraph. We remark that the two factor spaces are $\mathbb{C}$-dual to each other due to the Serre duality.

§6. Complex structure on the tangent space of $\tilde{\mathcal{T}}_g$

Let $t\in\tilde{\mathcal{T}}_g\subset \mathrm{Hom}(R_\Gamma,\mathbb{R})$ be a point of the Teichmüller space. By denoting $t(\gamma):=t(s(\gamma))$ for $\gamma\in\Gamma$, to give the point t is equivalent to give a map $t\colon \Gamma_g \longrightarrow \mathbb{R}$ satisfying i) the algebraic relations:

$$(2.7\text{-}8) \qquad t(e)=2, \quad t(\gamma)t(\delta)=t(\gamma\delta)+t(\gamma^{-1}\delta)$$

for $\gamma,\delta\in\Gamma_g$, and ii) the semi-algebraic conditions (ie. for t to be the trace of a faithful and discrete representation, for instance the Jørgensen's inequalities): $|t([\gamma,\delta])-2|+|t(\gamma)^2-4|\geq 1$ for $\gamma,\delta\in\Gamma$.

The real tangent space of $\mathrm{Hom}(R_\Gamma,\mathbb{R})$ at the point t is given by the set of t-derivations $\mathrm{Der}_t(R_\Gamma,\mathbb{R})$. Recall that to give a t-derivation w is equivalent to give a map $w\colon \Gamma_g \longrightarrow \mathbb{R}$ satisfying

$$(4.4\text{-}5) \qquad w(e)=0$$

and

$$w(\gamma)t(\delta) + w(\delta)t(\gamma) = w(\gamma\delta) + w(\gamma^{-1}\delta) .$$

On the other hand, recall that t-derivations can be obtained also from cocycles of Γ_g with coefficients in $sl_2(\mathbb{R})$ as follows. Namely, let $\rho\colon \Gamma_g\longrightarrow SL_2(\mathbb{R})$ be any representaion over t. That is: ρ and t are related by $t(\gamma)=tr(\rho(\gamma))$ for $\gamma\in\Gamma$. The Γ_g acts on $sl_2(\mathbb{R})$ by the compositon of the adjoint action of $SL_2(\mathbb{R})$ with the map ρ. Then for any cocycle $z\in Z^1_\rho(\Gamma_g, sl_2(\mathbb{R}))$, the map $tr(\rho z)\colon\gamma\in\Gamma \mapsto tr(\rho(\gamma)z(\gamma))\in \mathbb{R}$ satisfies the derivation condition (4.4-5) (cf(4.7)). This means that one obtains a $\mathbb{R}$-linear map

$$(6.1) \qquad H^1_\rho(\Gamma_g,\ sl_2(\mathbb{R})) \xrightarrow{\ \sim\ } \mathrm{Der}_t(R_{\Gamma_g},\ \mathbb{R}),$$

which was shown to be isomorphic (§4 Theorem). By complexifying the same procedure, one obtains a $\mathbb{C}$-linear isomorphism:

$$(6.1)' \qquad H^1_\rho(\Gamma_g,\ sl_2(\mathbb{R}))\otimes_{\mathbb{R}}\mathbb{C} \ \simeq\ \mathrm{Der}_t(R_{\Gamma_g},\ \mathbb{R})\otimes_{\mathbb{R}}\mathbb{C}$$

preserving the real structure.

On the other hand, let us give a natural $SL_2(\mathbb{R})$ isomorphism

$$\mathrm{Sym}^2(\mathbb{R}^2) \simeq sl_2(\mathbb{R}) . \tag{6.2}$$

This is obtained as follows. The infinitesimalization of the action of $SL_2(\mathbb{R})$ on $\mathbb{H}$ induces a Lie algebra homomorphism from $sl_2(\mathbb{R})$ to the Lie algebra of the global holomorphic vector fields on $\mathbb{H}$. Explicitly, it is given by the map

$$X \in sl_2(\mathbb{R}) \mapsto (-1,z)X\begin{bmatrix} z \\ 1 \end{bmatrix} \frac{d}{dz} \in \Gamma(\mathbb{H},\Theta) = \Gamma(\mathbb{H}, \mathcal{O}_{\mathbb{H}}(K_{\mathbb{H}}^{-1}))$$

Clealy by the map, $sl_2(\mathbb{R})$ is mapped to the space of vector fields of real polynomial coefficeints of degree less or equal than 2. This together with (5.3) means the isomorphism (6.2).

Recall also the map (5.7)

$$P_2 \ : \ \Gamma(X_\rho,\mathcal{O}_{X_\rho}(K_{X_\rho}^2)) \longrightarrow H^1(\Gamma_g,\mathrm{Sym}^2(\mathbb{R}^2))\otimes_{\mathbb{R}}\mathbb{C}$$

by associating to a quadratic differential $\omega \in \Gamma(\mathbb{H},\mathcal{O}_{\mathbb{H}}(K_{\mathbb{H}}^2))^{\rho(\Gamma_g)} \simeq \Gamma(X,\mathcal{O}_X(K_X^2))$, the period $\int_\gamma \omega$ of its Eichler integral. Combining P_2 with the isomorphisms (6.2) and (6.1),one obtains an injective $\mathbb{C}$-linear map:

$$P_2\colon \Gamma(X_\rho,\mathcal{O}_{X_\rho}(K_{X_\rho}^2)) \longrightarrow \mathrm{Der}_t(R_{\Gamma_g},\mathbb{R})\otimes_{\mathbb{R}}\mathbb{C} \tag{6.3}$$

whose source is half (=3g-3) dimesional complex vector subspace of the target space. As was shown in §5, P_2 is injective and its image does not intersect with the real subspace $\mathrm{Der}_t(R_\Gamma,\mathbb{R})$ in the target. We show:

The image of P_2 *does not depend on the chosen* ρ.

Proof. Let $\rho' = \rho\cdot\mathrm{Ad}(A) = A^{-1}\rho A$ for an $A\in PSL_2(\mathbb{R})$ be another representaion over t. Then the map $z\mapsto A(z)$ induces the isomorphism $X_{\rho'}\simeq X_\rho$ and hence $\mathrm{Ad}^{-4}(A)\colon \mathcal{O}_{\mathbb{H}}(K_{\mathbb{H}}^2) \longrightarrow \mathcal{O}_{\mathbb{H}}(K_{\mathbb{H}}^2)$ induces the isomorphism $\Gamma(X_\rho,\mathcal{O}_{X_\rho}(K_{X_\rho}^2)) \simeq \Gamma(X_{\rho'},\mathcal{O}_{X_{\rho'}}(K_{X_{\rho'}}^2))$. Let us show that the t-derivation

$w := tr(\rho\cdot\int\omega)$ associated to the period of $\omega\in\Gamma(X_\rho,\mathcal{O}_{X_\rho}(K^2_{X_\rho}))$ coincides with the t-derivation $w' := tr((\rho\cdot Ad(A)\cdot(\int\omega\cdot Ad^{-4}(A)))$ for the period of $\omega\cdot Ad^{-4}(A)$. Since $\int(\omega\cdot Ad^{-4}(A))=(\int\omega)\cdot Ad^2(A)$ (equivariance of ∂^3 with the $SL_2(\mathbb{R})$ action), one has a relation among the peiods:

$$\int_\gamma(\omega\cdot Ad^{-4}(A)) = (\int\omega)\cdot Ad^2(A)Ad^2(A^{-1}\rho(\gamma)A) - (\int\omega)\cdot Ad^2(A)$$
$$= (\int_\gamma\omega)\cdot Ad^2(A).$$

So the t-derivation w' associated to the period of $\omega\cdot Ad^{-4}(A)$ is given b

$$w'(\gamma) = tr(A^{-1}\rho(\gamma)A\cdot((\int_\gamma\omega)\cdot Ad(A))) = tr(A^{-1}\rho(\gamma)A\cdot A^{-1}\cdot(\int_\gamma\omega)\cdot A)$$
$$= tr(A^{-1}\rho(\gamma)(\int_\gamma\omega)A) = tr(\rho(\gamma)\int_\gamma\omega)$$
$$= w(\gamma).$$

This implies the coincidence $w=w'$ of the t-derivations. □

Applying the decomposition (5.10) to this situation. In summery,

Assertion. At any point $t\in\tilde{\mathcal{T}}_g\subset Hom(R_{\Gamma_g},\mathbb{R})$, *the complexification of its real tangent space has a canonical decomposition:*

$$(6.4)\qquad Der_t(R_{\Gamma_g},\mathbb{R})\otimes_{\mathbb{R}}\mathbb{C} \simeq \Gamma(X_\rho,\mathcal{O}_{X_\rho}(K^2_{X_\rho}))\oplus H^1(X_\rho,\mathcal{O}_{X_\rho}(K^{-1}_{X_\rho}))$$

where i) *the decomposition does not depend on the choice of* ρ *over* t

ii) *the two factor spaces are complex conjugate of each other,*

iii) *the two factor spaces are* $\mathbb{C}$*-dual space of each other.*

By this description, the real tangent space $Der_t(R_\Gamma,\mathbb{R})$ obtains a complex structure and a Hermitian structure as follows:

i) as the complex structur for $Der_t(R_\Gamma,\mathbb{R})$, we employ the second factor of the decomposition (6.4). That is: the almost complex structure J (an endomorphism on $\mathrm{Der}_t(R_\Gamma,\mathbb{R})$ with $J^2=-1$) is defined by a multiplication of $\sqrt{-1}$ (resp. $-\sqrt{-1}$) on the second (resp. the first factor) of (6.4).

More directly, the isomorphism

$$(6.5)\qquad \mathrm{Der}_t(R_\Gamma,\mathbb{R}) \underset{(4.7)}{\simeq} H^1_\rho(\Gamma,\mathrm{sl}_2(\mathbb{R})) \underset{\substack{(5.8)\\(6.2)}}{\simeq} H^1(X_\rho,\mathcal{O}_{X_\rho}(K_{X_\rho}{}^{-1}))$$

defines the complex structure on the tangent space $T_t\tilde{\mathcal{T}}_g \simeq \mathrm{Der}(R_\Gamma,\mathbb{R})$ at t.

ii) a hermitian metric on the complexified tangent space $T_t\tilde{\mathcal{T}}_g$ is given as follows. Let ξ, η be tangent vectors in $\mathrm{Der}_t(R_\Gamma,\mathbb{R})$ at $t \in \tilde{\mathcal{T}}_g$. By the isomorphism (6.5), consider the vector ξ and η to belong to $H^1(X_\rho,\mathcal{O}_X(K_X{}^{-1}))$. Use the complex conjugate (Assertion ii)) so that $\bar{\eta}$ belongs to $\Gamma(X_\rho,\mathcal{O}_X(K_X{}^2))$. Then the duality (Assertion iii)) yiels the hermitian form g:

$$(6.6)\qquad g(\xi,\eta) := \langle \xi,\bar{\eta}\rangle .$$

It is rather a formal task to identify the above complex structure with the standard complex structure on the Teichmüller space by a use of Beltrami differentials (cf [Be]), and the above hermitian metric with the well known Weil Petersson metric [Wl][Ah]. So if we assume these identification, the almost complex structure is integrable and the metric is Kählerian. But from our view point, it is desirable to show theses facts directly in terms of representation spaces, without use such identifications.

The integrability of the above almost complex structure on the space $R_0(\Gamma_g,SL_2(\mathbb{R}))/Ad(PSL_2(\mathbb{R}))$ (and hence on $\tilde{\mathcal{T}}_g$) is readily shown in 9.5 Theorem [Sa1]. In a similar contex, it may be nice to give such description for the Kählerity of the metric. For details of this metric, one is refered to a series of works of S. Wolpert [Wo1-7].

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