数理解析研究所講究録
第821巻 1992年 82-97

Computational algebraic analysis and connection formula

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(*January 31, 1992*)

Abstract. We introduce "fundamental problems" in "computational algebraic analysis" and explain that mechanical derivation and proving of connection formulas, which are those of the most important special function identities, are reduced to solving the fundamental problems. We give a new efficient algorithm of solving the fundamental problems. The algorithm decomposes a problem in n-variables to problems in $n-1$ variables.

§1. Fundamental problems in computational algebraic analysis

Put

$$A_n = \mathbf{C}\langle x_1, \cdots, x_n, \partial_1, \cdots, \partial_n \rangle,$$

$$\partial_i x_j - x_j \partial_i = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$$

and

$$B_n = \mathbf{C}\langle x_1, \cdots, x_n, E_1, \cdots, E_n \rangle,$$

$$E_i x_j - x_j E_i = \begin{cases} E_i, & i = j, \\ 0, & i \neq j. \end{cases}$$

The ring A_n is the ring of differential operators and B_n is the ring of difference operators. Let $\mathfrak{A}$ and $\mathfrak{P}$ be left ideals of the ring A_n and B_n respectively. We will call the following problems "fundamental problems in computational algebraic analysis";

Find Gröbner basis of the following modules;

$$A_{n-1} \text{ module } A_n/(\mathfrak{A} + \partial_n A_n), \tag{1.1}$$

$$A_{n-1} \text{ module } A_n/(\mathfrak{A} + x_n A_n), \tag{1.2}$$

$$B_{n-1} \text{ module } B_n/(\mathfrak{P} + (E_n - 1)B_n). \tag{1.3}$$

Notice that, for example, $\mathfrak{A} + \partial_n A_n$ is the sum of the left ideal $\mathfrak{A}$ and the right ideal $\partial_n A_n$ of A_n.

We can solve the following problems by solving problems (1.1), (1.2) and (1.3) respectively.

$$\text{Find} \quad A_{n-1} \cap (\mathfrak{A} + \partial_n A_n). \tag{1.4}$$

$$\text{Find} \quad A_{n-1} \cap (\mathfrak{A} + x_n A_n). \tag{1.5}$$

$$\text{Find} \quad B_{n-1} \cap (\mathfrak{P} + (E_n - 1)B_n). \tag{1.6}$$

We will also call these problems "fundamental problems in computational algebraic analysis".
These fundamental problems are related to problems in the classical analysis as follows;

$$
\begin{array}{ll}
A_n/\mathfrak{A} \mapsto A_n/(\mathfrak{A} + \partial_n A_n) & f(x_1, \cdots, x_n) \mapsto \int_{-\infty}^{\infty} f(x_1, \ldots, x_n) dx_n \\
 & \text{definite integral with parameters} \\
A_n/\mathfrak{A} \mapsto A_n/(\mathfrak{A} + x_n A_n) & f(x_1, \cdots, x_n) \mapsto f(x_1, \cdots, x_{n-1}, 0) \\
 & \text{restriction} \\
B_n/\mathfrak{P} \mapsto B_n/(\mathfrak{P} + (E_n - 1)B_n) & f(x_1, \cdots, x_n) \mapsto \sum_{x_n=-\infty}^{\infty} f(x_1, \cdots, x_n) \\
 & \text{summation with parameters}
\end{array}
$$

Let us see the correspondence of the above by using an example. Put $f(x_1, x_2) = \binom{x_1}{x_2}$ (the binomial coefficient). Here, we put $\binom{x_1}{x_2} = 0$ when $x_2 < 0$ or $x_2 > x_1$. We define the action of E_1 and E_2 to the function as follows;

$$E_1 f(x_1, x_2) = f(x_1 + 1, x_2), \quad E_2 f(x_1, x_2) = f(x_1, x_2 + 1).$$

Put

$$
\begin{aligned}
\ell_1 &= (x_1 + 1 - x_2)E_1 - (x_1 + 1), \\
\ell_2 &= (x_2 + 1)E_2 - (x_1 - x_2).
\end{aligned}
$$

Then we have $\ell_1 f = \ell_2 f = 0$. Let $\mathfrak{P}$ be the left ideal of B_2 generated by ℓ_1 and ℓ_2. Notice that we have $\mathfrak{P} f = 0$. Let us find an element of $(\mathfrak{P} + (E_2 - 1)B_2) \cap B_1$. It is convinient to write the operators ℓ_1 and ℓ_2 as

$$
\begin{aligned}
\ell_1 &= E_1(x_1 - x_2) - (x_1 + 1), \\
\ell_2 &= (E_2 - 1)x_2 - x_1 + 2x_2.
\end{aligned}
$$

Since we have

$$
\begin{aligned}
2\ell_1 + E_1\ell_2 &= 2E_1(x_1 - x_2) - 2(x_1 + 1) + (E_2 - 1)E_1 x_2 - E_1 x_1 + 2E_1 x_2 \\
&= 2E_1 x_1 - 2(x_1 + 1) - E_1 x_1 + (E_2 - 1)E_1 x_2 \\
&= E_1 x_1 - 2(x_1 + 1) + (E_2 - 1)E_1 x_2,
\end{aligned}
$$

we can see that

$$E_1 x_1 - 2(x_1 + 1) \in (\mathfrak{P} + (E_2 - 1)B_2) \cap B_1.$$

For any element $\ell \in B_2$, we have $\sum_{x_2=-\infty}^{\infty} (E_2 - 1)\ell \binom{x_1}{x_2} = 0$. Therefore we have

$$[E_1 x_1 - 2(x_1 + 1)] \sum_{x_2=-\infty}^{\infty} \binom{x_1}{x_2} = 0.$$

We have the following proposition in a similar way.

Proposition 1.1 *Let $\mathfrak{P}$ be a left ideal of B_n and f be a function of $x_1 \cdots x_n$ such that $f(x_1, \cdots, x_n) = 0 \quad (|x_n| >> 0)$. If $\mathfrak{P} f(x_1, \cdots, x_n) = 0$, then*

$$(\mathfrak{P} + (E_n - 1)B_n) \cap B_{n-1} \sum_{x_n=-\infty}^{\infty} f(x_1, \cdots, x_n) = 0.$$

Thus, if the B_n module structure of the function f is $B_n/\mathfrak{P}$, then the B_{n-1} module structure of the function $\sum_{x_n=-\infty}^{\infty} f(x_1, \cdots, x_n)$ is

$$B_{n-1}/(\mathfrak{P} + (E_n - 1)B_n) \cap B_{n-1}.$$

In the theory of holonomic systems, we consider the B_{n-1} module $B_n/(\mathfrak{P} + (E_n - 1)B_n)$ rather than the above module, because it is more natural concept and easy to treat. It is important to consider the fundamental problems for $\mathcal{D}_n$-module and q-Weyl algebra. These studies have just started. As for $\mathcal{D}_n$-module, see [SO1].

§2. An application to the mechanical theorem proving of binomial identities and special function identities

Studies on mechanical theorem proving of mathematical theorems have been done since the beginning of computer science and computers have proved many theorems which are hard to prove. For example, [Wu1] and [Chou1] found algorithms and implemented systems to prove theorems on elementary geometry by using the Wu-Ritt characteristic sets and the Gröbner basis. The quantifier elimination method and the decision procedure over the real numbers have been widely studied. As for special function identities, [Tak1] found an algorithm to find contiguity relations of special function by using the Gröbner basis.

D.Zeilberger [Zei1] found a general method to prove binomial coefficients identities and special functions identities; he explained that proving these identities reduces to solving the fundamental problems (1.4) and (1.6).

Let us explain his ideal by using an example.

Our goal is to prove the identity

$$\sum_{x_2=-\infty}^{\infty} \binom{x_1}{x_2} = 2^{x_1}.$$

We have shown in §2 that

$$[E_1 x_1 - 2(x_1+1)] \sum_{x_2=-\infty}^{\infty} \binom{x_1}{x_2} = 0$$

by computing the ideal intersection

$$(\mathfrak{P} + (E_2 - 1)B_2) \cap B_2.$$

Since $E_1 x_1 - 2(x_1+1) = (x_1+1)(E_1 - 2)$, we have

$$[E_1 x_1 - 2(x_1+1)]2^{x_1} = 0,$$

which means that the both sides of the identity satisfy the same difference equation. Hence, it is enough for the proof to show that the both sides satisfy a same initial condition. In fact, when $x_1 = 0$, we have $1 = 1$. It completes the proof.

The most important part of the above proof is to solve the fundamental problems (1.3) or (1.6). D. Zeilberger gave an efficient algorithm, which is a modification of Gosper's decision procedure ([Go1]), to solve the problems (1.4) and (1.6) in case of annihilating ideals of binomial coefficients and hyperexponential functions ([Zei2], [AZ1]). He has also a similar algorithm for the q-case. On the other hand, the author found an algorithm, which is a modification of the Buchberger algorithm ([Buch1]), to solve the problems (1.1) $\sim$ (1.6) ([Tak2], [Tak3]). By using these algorithms, we can easily give proofs, for example, to the following binomial identities;

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}, \qquad \sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k} = \frac{2^{n+1}-1}{n+1},$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \frac{(2n)!}{(n!)^2}, \qquad \sum_{k=0}^{p} \binom{n}{k}\binom{m}{p-k} = \binom{n+m}{p}, \text{ (Chu-Vandermonde)},$$

$$\sum_{k} (-1)^k \binom{n+b}{n+k}\binom{n+c}{c+k}\binom{b+c}{b+k} = \frac{(n+b+c)!}{n!b!c!}, \text{ (Dixson)}.$$

§3. A new "fundamental problem" and its application

In the preceding sections, we considered mechanical theorems proving of binomial and special functions identities. In this section, we will consider algorithms of deriving and proving connection formulas of special functions and introduce a new fundamental problem in computational algebraic analysis.

Put $\theta_i = x_i \partial_i$. Let $\mathfrak{A}$ be an left ideal of the Weyl algebra A_n. We will call also the following problem a fundamental problem;

Find

$$(\mathfrak{A} + x_n A_{n-1}\langle x_n, \theta_n\rangle) \cap \mathbf{C}\langle x_1, \cdots, x_{n-1}, \theta_n\rangle; \tag{3.1}$$

in particular, find the set of roots

$$\{\theta \in \mathbf{C} \mid {}^{\forall} f(x_1, \cdots, x_{n-1}, \theta) = 0, f \in (3.1)\}.$$

We call the above set of roots *exponents* of the ideal $\mathfrak{A}$ along the line $x_n = 0$.

We will give an efficient algorithm which give a sufficient answer for applications to the problem (3.1). Computations of exponents along regular singularities, derivations and making proofs of connection formulas and finding a sufficient answer to the fundamental problem (1.5) are applications of the problem (3.1) and Algorithm 4.1 given in Section 4.

Before going to giving a general algorithm, we will explain the method of solving (3.1) and (1.5) and its applications by using an example. The general procedure will be given in the next section. Our method is a kind of the pruning.

It is known that the following identity holds.

If $c - b - 1 \notin \mathbf{Z}$, then

$$z_1 = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} z_7 + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} z_9 \tag{3.2}$$

where

$$z_1 = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}(1)_m(1)_n} x^m y^n,$$

$$z_7 = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(1+a+b-c)_m(c-b)_n(1)_m(1)_n} (1-x)^m y^n,$$

$$z_9 = (1-x)^{c-a-b} \sum_{m,n=0}^{\infty} \frac{(c-b)_{m+n}(c-a)_m(b')_n}{(1+c-a-b)_m(c-b)_n(1)_m(1)_n} (1-x)^m y^n,$$

$$(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1) = \Gamma(\alpha+k)/\Gamma(\alpha)$$

and $\Gamma(\alpha)$ is the Gamma function.

EXAMPLE 3.1 Prove the identity (3.2) by using the connection formula of the Gauss hypergeometric function and the fact that the functions z_1, z_7 and z_9 are solutions of the system of differential equations

$$\ell_0 u = \ell_1 u = \ell_2 u = 0$$

where

$$\ell_0 = \theta_x(\theta_x + \theta_y + c - 1) - x(\theta_x + \theta_y + a)(\theta_x + b)$$
$$\ell_1 = \theta_y(\theta_x + \theta_y + c - 1) - y(\theta_x + \theta_y + a)(\theta_y + b')$$
$$\ell_2 = (x - y)\frac{\partial^2}{\partial x \partial y} - b'\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$$
$$\theta_x = x\partial_x = x\frac{\partial}{\partial x}, \quad \theta_y = y\partial_y = y\frac{\partial}{\partial y}.$$

Solution. In order to prove the identity (3.2), we need the following lemma.

LEMMA 3.1 Let P be an operator of the form

$$P=\sum_{k=0}^{s} y^k p_k(\theta_y, x, \theta_x)$$

and f be a holomorphic function at $(x,y)=(0,0)$ such that $f(x,0)\equiv 0$ and $Pf=0$. Suppose that $p_0=p_0(\theta_y,x)$ does not depend on θ_x and $p_0(k,x)\neq 0$ for all non-negative integers k. Then we have $f(x,y)\equiv 0$.

Proof. Since the function f can be written as

$$f=\sum_{k=0}^{\infty} y^k f_k(x),$$

we have

$$Pf=\sum_{k=0}^{\infty}\sum_{i=0}^{s} y^{i+k} p_k(k,x,\theta_x) f_k(x)=0.$$

Hence, we have the following difference equation

$$p_0(k,x)f_k=-p_1(k-1,x,\theta_x)f_{k-1}-\cdots-p_s(k-s,x,\theta_x)f_{k-s}.$$

Since $f_0\equiv 0$, then we have $f_k\equiv 0$. ▯

Putting

$$f=z_1-\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}z_7-\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}z_9,$$

we will use Lemma 3.1. In order to use it, we need to find an operator of the form

$$p_0(\theta_y,x)+y[\cdots\cdots]$$

in the left ideal $R\ell_0+R\ell_1+R\ell_2$ where $R=\mathbf{C}\langle x,y,\theta_x,\theta_y\rangle$. This problem is the problem (3.1).

In order to find an operator of the form $p_0(\theta_y,x)+y[\cdots\cdots]$, we need not to consider the higher order terms with respect to the variable y, because we have

$$\theta_y y = y(\theta_y+1) \text{ in } R.$$

In fact, the operators ℓ_0,ℓ_2 can be written as

$$\begin{aligned} xy\ell_2 &= x\theta_y(\theta_x+b)-y\theta_x(\theta_x+b') \\ &= x\theta_y(\theta_x+b)+O(y) \\ \ell_0 &= \theta_y(\theta_x+\theta_y+c-1)+O(y) \end{aligned}$$

and we need to consider only

$$\begin{aligned} \ell_3 &= x\theta_y(\theta_x+b), \\ \ell_4 &= \theta_y(\theta_x+\theta_y+c-1). \end{aligned}$$

We compute

$$\mathrm{ideal}(\ell_3,\ell_4)\cap\mathbf{C}(b,c)\langle x,\theta_y\rangle.$$

This elimination can be done by the Buchberger algorithm for the ring of differential operators. In this case, we can easily eliminate θ_x by taking the S-pair

$$x\ell_3-\ell_4=x\theta_y(\theta_y+c-1-b)\in\mathbf{C}(b,c)\langle x,\theta_y\rangle.$$

Finally, we have

$$\begin{aligned}
&x \cdot xy\ell_2 - \ell_0 \\
=&x(\ell_3 + O(y)) - (\ell_4 + O(y)) \\
=&x\theta_y(\theta_y + c - 1 - b) + y[-x(\theta_x + \theta_y + a)(\theta_y + b') + \theta_x(\theta_y + b')].
\end{aligned}$$

Here, $O(y)$ is a term expressed as $y[\cdots\cdots]$. Hence, it follows from Lemma 3.1 that if we have $f(x,0) \equiv 0$, then we complete the proof of (3.1).

The restricted function $f(x,0)$ can be written as

$$\begin{aligned}
f(x,0) = &\sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{(c)_m(1)_m} x^m \\
&- \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{(1+a+b-c)_m(1)_m} x^m \\
&- \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-x)^{c-a-b} \sum_{m=0}^{\infty} \frac{(c-b)_m(c-a)_m}{(1+c-a-b)_m(1)_m} x^m,
\end{aligned}$$

which is equal to zero by reason of the connection formula of the Gauss hypergeometric function ([WW 14.51]). It completes the proof of (3.2).

EXAMPLE 3.2 Put $b_i(x) = z_i|_{y=0}$. Show that the function b_i satisfies the Gauss hypergeometric differential equation

$$[\theta_x(\theta_x + c - 1) - x(\theta_x + a)(\theta_x + b)]v = 0$$

by using only the fact that the function z_i is a solution of the system of differential equation

$$\ell_0 u = \ell_1 u = \ell_2 u = 0.$$

Solution. Notice that if we find an operator of the form

$$p_0(x, \partial_x) - yp_1(x, \partial_x, y, \partial_y) \in \mathfrak{A} = R\ell_0 + R\ell_1 + R\ell_2,$$

then we have $p_0(x, \partial_x)b_i = 0$. Since

$$\mathfrak{A} \ni \ell_0 = \theta_x(\theta_x + c - 1) - x(\theta_x + a)(\theta_x + b) + y[\partial_y\theta_x - x\partial_y(\partial_x + b)],$$

we have

$$[\theta_x(\theta_x + c - 1) - x(\theta_x + a)(\theta_x + b)]b_i = 0.$$

REMARK. 3.2 Here, we have proved (3.2) by solving the fundamental problem (3.1). The next question is how to derive connection formulas like (3.2). In fact, it is possible to derive connection formulas, with a limitation, by solving the fundamental problems (3.1) and (1.5). The limitation is that we cannot obtain higher order terms of connection formulas by the method. For example, we can find the following formula

$$z_1 = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}(1 + O(x-1, y)) + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-x)^{c-a-b}(1 + O(x-1, y))$$

by the derivation method. We need new notions and theorems, which are explained in [Tak3], for the derivation. Here, we only notice that we can derive connection formulas by solving the fundamental problems (3.1) and (1.5) and by using the notions and theorems in [Tak3].

REMARK. 3.3 We can obtain the operator of the form

$$p_0(x,\theta_y)+yp_1(x,\partial_x,y,\theta_y)$$

by computing the intersection

$$\mathfrak{A}\cap\mathbf{C}\langle x,y,\theta_y\rangle.$$

The computation of the intersection above is more expensive than the method explained in Example 3.1, because the elimination is done in the ring of 4 variables in this case and in the method explained in Example 3.1 the elimination is done in the ring of 3 variables. Notice that we can see that the operator

$$x\theta_y(\theta_y-1)(b-c-\theta_y+1)+y\theta_y(\theta_y+b')(ax-bx+c+x\theta_y+\theta_y-1)-y^2(\theta_y+b')(\theta_y+b'+1)(\theta_y+a),$$

is in the left ideal

$$R\ell_0+R\ell_1+R\ell_2.$$

§4. New algorithms

We will explain new efficient algorithms to solve the fundamental problems (3.1) and (1.5) in sufficient forms for the application to proof and derivation of connection formulas. We assume the variables are x, y, ∂_x and ∂_y and the number of generators of a given left ideal $\mathfrak{A}$ is 2 for the simplicity.

Put

$$A_2=\mathbf{C}\langle x,y,\partial_x,\partial_y\rangle,\quad A_2'=\mathbf{C}\langle x,y,\theta_x,\partial_y\rangle,\quad \tilde{A}=\mathbf{C}\langle\theta_x,y,\partial_y\rangle.$$

For given operators $P,Q\in A_2'$, we have unique expansions

$$P=\sum_{k=0}^{K_1}x^kp_k,$$
$$Q=\sum_{k=0}^{K_2}x^kq_k,\quad p_k,q_k\in\tilde{A}.$$

Put $\mathfrak{A}=A_2P+A_2Q$. Then the set $\mathfrak{A}$ is the left ideal of the ring A_2. We will show an algorithm of solving the fundamental problem (3.1) for the left ideal $\mathfrak{A}$.

We inductively define operators $R_j^{(i)}$ and $S_j^{(i)}$ $(j=0,1,2,\dots)$ satisfying the condition

$$R_j^{(i)}P+S_j^{(i)}Q=O(x^{j+1})$$

where $O(x^{j+1})$ denotes a term expressed as $x^{j+1}w$, $w\in A_2'$. The space

$$\{(R,S)\mid Rp_0+Sq_0=0,\ (R,S)\in\tilde{A}^2\}$$

is the left $\tilde{A}$ module. For $j=0$, let $(R_0^{(i)},S_0^{(i)})$ $(i=1,\cdots,g_0)$ be generators of the left $\tilde{A}$ module. For j, we put

$$R_j^{(i)}=\sum_{k=1}^{g_{j-1}}d_{jk}^{(i)}R_{j-1}^{(k)}+x^jr_j^{(i)},\quad i=1,\cdots,g_j,$$
$$S_j^{(i)}=\sum_{k=1}^{g_{j-1}}d_{jk}^{(i)}S_{j-1}^{(k)}+x^js_j^{(i)},\quad i=1,\cdots,g_j.$$

Here, $(d_{j1}^{(i)},\cdots,d_{jg_{j-1}}^{(i)},r_j^{(i)},s_j^{(i)})$, ($i=1,\cdots,g_j$) are generators of the left $\tilde{A}$ module

$$\{(d_1,\cdots,d_{g_{j-1}},r_j,s_j)\in\tilde{A}^{g_{j-1}+2}\mid\sum_{k=1}^{g_{j-1}}d_kT_{j-1}^{(k)}+(r_jp_0+s_jq_0)=0\}$$

where we define the operator $T^{(k)}_{j-1} \in \tilde{A}$ as

$$R^{(k)}_{j-1}P + S^{(k)}_{j-1}Q = x^j T^{(k)}_{j-1} + O(x^{j+1}).$$

Notice that we have

$$R^{(i)}_{j}P + S^{(i)}_{j}Q = O(x^{j+1})$$

from the definition. Eliminating the operator ∂_y from $\{T^k_j \mid k = 1, \cdots, g_j\}$, we can solve the fundamental problem (3.1). The correctness will be proved in Theorem 4.1. Summarizing the above procedure, we have the following algorithm.

ALGORITHM 4.1 findBoundaryData(P, Q, J)
(1) exponents := $\emptyset$
(2) boundaryValues := $\emptyset$
(3) $T^{(1)}_{-1} := p_0$
(4) $T^{(2)}_{-1} := q_0$
(5) $g_{-1} := 2$
(6) exponents := findExponents($\{T^{(k)}_{-1} \mid k = 1, 2\}$)
(7) boundaryValues := findBoundaryValues(exponents,$\{T^{(k)}_{-1} \mid k = 1, 2\}$)
(8) Let $(R^{(i)}_0, S^{(i)}_0)$ $(i = 1, \cdots, g_0)$ be generators of the left $\tilde{A}$ module:

$$\{(R, S) \mid Rp_0 + Sq_0 = 0,\ (R, S) \in \tilde{A}^2\}$$

(9) for $j := 1$ to J do
(10) $T^{(k)}_{j-1}$:= the coefficient of x^j of $R^{(k)}_{j-1}P + S^{(k)}_{j-1}Q$, $(k = 1, \cdots, g_{j-1})$
(* $R^{(k)}_{j-1}P + S^{(k)}_{j-1} = x^j T^{(k)}_{j-1} \bmod x^{j+1}A'_2$ *)
(11) exponents := findExponents($\{T^{(k)}_j \mid j = -1, \cdots, j-1, k = 1, \cdots, g_{j-1}\}$)
(12) boundaryValues := findBoundaryValues($\{T^{(k)}_j \mid j = -1, \cdots, j-1, k = 1, \cdots, g_{j-1}\}$)
(13) Let $(d^{(i)}_{j1}, \cdots, d^{(i)}_{jg_{j-1}}, r^{(i)}_j, s^{(i)}_j)$, ($i = 1, \cdots, g_j$) be generators of the left $\tilde{A}$ module

$$\{(d_1, \cdots, d_{g_{j-1}}, r_j, s_j) \in \tilde{A}^{g_{j-1}+2} \mid \sum_{k=1}^{g_{j-1}} d_k T^{(k)}_{j-1} + (r_j p_0 + s_j q_0) = 0\}$$

(14) Return(exponents, boundaryValues)

ALGORITHM 4.2 findExponents($\{T_i \mid i = 1, \cdots \ell\}$)
(1) G := the Gröbner basis of $\{T_i\}$ in $\tilde{A}$ by the order $\{y, \partial_y\} > \theta_x$
(2) $G := G \cap \mathbf{C}\langle y, \theta_x\rangle$
(3) Return($\{\theta \in C \mid f(y, \theta) = 0, \forall f \in G\}$)

ALGORITHM 4.3 findBoundaryValues(exponents, $\{T_i \mid i = 1, \cdots, \ell\}$)
(1) $B := \emptyset$
(2) for $e \in$ exponents do
(3) $G := \{T_i|_{\theta_x \to e} \mid i = 1, \cdots, \ell\}$
(4) $B := B \cup \{$ Gröbner basis of $G\}$
(5) Return(B)

Suppose that there exist $R', S' \in A_2$ such that

$$R'P + S'Q = T(\theta_x, y) + xU(x, y, \theta_x, \partial_y).$$

In this case, there exists N satisfying

$$x^N R', x^N S' \in A_2' = \mathbf{C}\langle x, y, \theta_x, \partial_y\rangle.$$

The following theorem shows that if the number J is sufficiently large, then Algorithm 4.1 (findBoundaryData(P, Q, J)) returns the exponents and solves the fundamental problem (3.1). Notice that the upperbound of the number J is not known.

THEOREM 4.1 *If there exist a number N, an operator $T \in \mathbf{C}\langle\theta_x, y\rangle$ and operators $P, Q, R, S \in \mathbf{C}\langle x, \theta_x, y, \partial_y\rangle$ such that*

$$RP + SQ = x^{N+1}T(\theta_x, y) + O(x^{N+2}),$$

then the operator T is the member of the left ideal of $\tilde{A} = \mathbf{C}\langle\theta_x, y, \partial_y\rangle$ generated by

$$T_N^{(k)},\ k = 1, \cdots, g_N,\ T_{-1}^{(1)} = p_0,\ T_{-1}^{(2)} = q_0.$$

We need a lemma to prove the theorem.

LEMMA 4.1 *Let (R, S) be a solution of*

$$RP + SQ = O(x^{N+1}).$$

Then the solution (R, S) can be expressed as

$$(R, S) = \sum_{k=1}^{g_N} c_k^N (R_N^{(k)}, S_N^{(k)}) + O(x^{N+1}), \quad c_k^N \in \tilde{A}.$$

Proof. We can suppose that the degrees of R and S are N with respect to the variable x. We use the induction on N. We have the conclusion in case of $N = 0$ from the definition of $R_0^{(k)}$ and $S_0^{(k)}$. We express the operators R and$S$ as

$$R = R' + x^N r, \quad S = S' + x^N s$$

where the degrees of R' and S' with respect to x are less than $N - 1$. Since

$$RP + SQ = R'P + S'Q + x^N(rp_0 + sq_0) + O(x^{N+1}),$$

we have $R'P + S'Q = 0(x^N)$. It follows from the assumption of the induction that

$$(R', S') = \sum_{i=1}^{g_{N-1}} c_i^{N-1}(R_{N-1}^{(i)}, S_{N-1}^{(i)}), \quad c_i^{N-1} \in \tilde{A}.$$

Therefore $R'P + S'Q$ can be written as

$$R'P + S'Q = x^N \sum_{i=1}^{g_{N-1}} c_i^{N-1} T_{N-1}^{(i)} + O(x^{N+1}).$$

Since $R'P + S'Q + x^N(rp_0 + sq_0) = O(x^{N+1})$, we have

$$\sum_{i=1}^{g_{N-1}} c_i^{N-1} T_{N-1}^{(i)} + rp_0 + sq_0 = 0.$$

Any solution

$$(c_1^{N-1}, \cdots, c_{g_{N-1}}^{N-1}, r, s)$$

of the above linear indefinite equation can be expressed as

$$\sum_{i=1}^{g_i} d_i(d_{N1}^{(i)}, \cdots, d_{Ng_{N-1}}^{(i)}, r_N^{(i)}, s_N^{(i)}), \quad d_i \in \tilde{A}.$$

We have the conclusion from the above and the definition of $(R_N^{(k)}, S_N^{(k)})$. $\Box$

Proof of Theorem 4.1. The operators R and S can be expressed as

$$R = R' + x^{N+1}r + O(x^{N+2}),$$
$$S = S' + x^{N+1}s + O(x^{N+2}),$$

where the degree of R' and S' with respect to x is less than N. It follows from Lemma 4.1 that there exist $c_k^N \in \tilde{A}$ such that

$$R'P + S'Q = \sum_{k=1}^{g_N} c_k^N T_N^{(k)} + O(x^{N+2}).$$

Since we have

$$RP + SQ = x^{N+1}T(\theta_x, y) + O(x^{N+2}),$$

then

$$\sum_{k=1}^{g_N} c_k^N T_N^{(k)} + rp_0 + sq_0 = T(\theta_x, y),$$

which means that

$$T(\theta_x, y) \in \sum \tilde{A}T_N^{(k)} + \tilde{A}p_0 + \tilde{A}q_0. \Box$$

Let α be one of the exponents of the ideal $\mathfrak{A}$. In order to derive connection formulas, we need to find the left ideal for the boundary values with respect to α;

$$(x^{-\alpha}\mathfrak{A}x^\alpha + xA_2) \cap \mathbf{C}\langle y, \partial_y \rangle.$$

The above problem is the fundamental problem (1.5). We gave algorithms for solving (1.5) in [Tak2] and [Tak3]. Algorithm 4.1 does not always give correct answer, but it is faster than the algorithms given in [Tak2] and [Tak3]. Let T' be the generator of

$$(x^{-\alpha}\mathfrak{A}x^\alpha + xA_2) \cap \mathbf{C}\langle y, \partial_y \rangle.$$

The following theorem gives a sufficient condition for the correctness of Algorithm 4.1 — findBoundaryData(P, Q, J).

THEOREM 4.2. *If there exist operators ℓ and U satisfying*

$$\ell = T' + xU, \ \ell \in x^{-\alpha}\mathfrak{A}x^\alpha, \ U \in A_2' = \mathbf{C}\langle x, \theta_x, y, \partial_y \rangle,$$

then for sufficiently large number J, Algorithm 4.1 returns a generator of the left ideal for the boundary value with respect to the exponent α.

Proof. Without loss of generality, we assume that $\alpha = 0$. It is enough to prove that if there exists a number N such that

$$RP + SQ = x^{N+1}[T(y, \partial_y) + \theta_x U(\theta_x, y, \partial_y)] + O(x^{N+2}),$$
$$T, U, R, S \in \mathbf{C}\langle x, \theta_x, y, \partial_y\rangle,$$

then the operator T is the member of the left ideal of $C\langle y, \partial_y\rangle$ generated by

$$T_N^{(k)}{}_{|\theta_x=0}, \quad (k = 1, \cdots, g_k), T_{-1}^{(k)}{}_{|\theta_x=0}, \quad (k = 1, 2).$$

We can see, in a similar way to the proof of Theorem 4.1, that

$$T(y, \partial_y) + \theta_x U(\theta_x, y, \partial_y) \in \sum \tilde{A} T_N^{(k)} + \sum \tilde{A} T_{-1}^{(k)}.$$

Putting $\theta_x = 0$ in the above, we obtain the conclusion. $\square$

REMARK. 4.1. (Complexity) The complexity of Algorithm 4.1 is a research problem. We only notice the following observation. Let $\ell_1, \cdots, \ell_r$ be elements of $\mathbf{C}\langle\theta_x, y, \partial_y\rangle$. Put $d = \max_{i=1,\cdots,r} \deg \ell_i$. Suppose that $T(d)$ and $D(d)$ are upper bounds of the time of the construction of the Gröbner basis of $\ell_1, \cdots, \ell_r$ by the order

$$\partial_y > \{\theta_x, y\}$$

and the maximum of the degrees of elements in the Gröbner basis. Put $d_0 = \max\{\deg(p_0), \deg(q_0)\}$. Then an upper bound of the execution time of Algorithm 4.1 is $\sum_{k=0}^{J} T(D_k)$ where

$$D_k = \max\{\underbrace{D \circ \cdots \circ D}_{k}(d_0), D_{k-1}\}.$$

EXAMPLE 4.1. (The Appell function F_1.) Put

$$P = \theta_x(\theta_x + y\partial_y + c - 1) - x(\theta_x + y\partial_y + a)(\theta_x + b),$$
$$Q = y\partial_y(\theta_x + y\partial_y + c - 1) - y(\theta_x + y\partial_y + a)(y\partial_y + b').$$

We call findBoundaryData(P,Q,-1). In this case, we have

$$p_0 = \theta_x(\theta_x + y\partial_y + c - 1),$$
$$q_0 = y\partial_y(\theta_x + y\partial_y + c - 1) - y(\theta_x + y\partial_y + a)(y\partial_y + b').$$

Let G be the Gröbner basis of the left ideal generated by p_0 and q_0 by the order $\{y, \partial_y\} > \theta_x$. We have

$$G \cap \mathbf{C}\langle y, \theta_x\rangle \ni (a - c + 1)y\theta_x(\theta_x + c - b' - 1).$$

In fact, we have

$$\begin{aligned} t_1 &:= y \cdot y\partial_y p_0 + \theta_x q_0, \\ t_2 &:= t_1 - y\partial_y p_0, \\ t_3 &:= t_2 - y(c - 1 - b' - a)p_0, \\ &= (a - c + 1)y\theta_x(\theta_x + c - b' - 1). \end{aligned}$$

Hence we have exponents $= \{0, 1+b'-c\}$. Since we have

$$\begin{aligned} p_0|_{\theta_x=0} &= 0, \\ q_0|_{\theta_x=0} &= y\partial_y(y\partial_y + c - 1) - y(y\partial_y + a)(y\partial_y + b') \end{aligned}$$

and

$$\begin{aligned} p_0|_{\theta_x=1+b'-c} &= (-c+b'+1)(y\partial_y + b'), \\ q_0|_{\theta_x=1+b'-c} &= [y\partial_y - y(y\partial_y + a - c + b' + 1)](y\partial_y + b'), \end{aligned}$$

then

$$\text{boundaryValues} = \{q_0|_{\theta_x=0}, p_0|_{\theta_x=1+b'-c}\}.$$

We can prove that the above "boundaryValues" are the maximal annihilating left ideal of the boundary values by using [Tak3; Proposition 2.1].

EXAMPLE 4.2 Put

$$\begin{aligned} P &= \theta_x(\theta_x + y\partial_y + c - 1) - x(\theta_x + y\partial_y + a)(\theta_x + b), \\ Q &= -y\theta_x(y\partial_y + b') + xy\partial_y(\theta_x + b). \end{aligned}$$

We call findBoundaryData$(P, Q, 0)$. Notice that the operators P and Q above generates the same left ideal with the ideal generated by P and Q in Example 4.1 in the ring $\mathbf{C}(a, b, b', c, x, y)\langle\partial_x, \partial_y\rangle$.

Put

$$\begin{aligned} p_0 &= \theta_x(\theta_x + y\partial_y + c - 1), \\ q_0 &= -y\theta_x(y\partial_y + b'). \end{aligned}$$

The Gröbner basis G of the ideal generated by p_0 and q_0 by the order $\{y, \partial_y\} > \theta_x$ is

$$\{\theta_x(\theta_x + y\partial_y + c - 1), y\theta_x(\theta_x + c - 1 - b'), -\theta_x(\theta_x - 2 + c)(-\theta_x + c + b' - c)\}.$$

Hence, the step (6) returns $\{0, 1+b'-c\}$ and the step (7) returns $\{0, (1+b'-c)(y\partial_y + b')\}$. In this case, the ideal for the boundary value with respect to the exponent 0 is not correct. The step (8) computes generators of the module, which can be easily obtained from the Gröbner basis of p_0 and q_0. In fact, we have

$$\begin{aligned} (R_0^{(1)}, S_0^{(1)}) &= (y(y\partial_y + b'), y\partial_y + c - 2 + \theta_x), \\ (R_0^{(2)}, S_0^{(2)}) &= \cdots, \\ &\cdots. \end{aligned}$$

In the step (10), we have

$$\begin{aligned} T_{-1}^{(1)} &= -y(y\partial_y + b')(\theta_x + y\partial_y + a)(\theta_x + b) + (y\partial_y + c - 1 + \theta_x)y\partial_y(\theta_x + b), \\ T_{-1}^{(2)} &= \cdots. \end{aligned}$$

The step (12) returns the correct ideal for the boundary value with the exponent 0;

$$b[y\partial_y(y\partial_y + c - 1) - y(y\partial_y + a)(y\partial_y + b')].$$

EXAMPLE 4.3. (The zonal spherical system on $SL(3, \mathbf{R})/SO(3)$.)

Consider the system of differential equation ([Sek1; 14p])

$$(\Delta_2 - L_2)u = (\Delta_3 + L_3)u = 0 \tag{4.1}$$

where $L_2, L_3 \in \mathbf{C}$ and

$$
\begin{aligned}
\Delta_2 = & -4(\theta_1^2 - \theta_1\theta_2 + \theta_2^2) \\
& + (2\frac{1+x_1}{1-x_1} - \frac{1+x_2}{1-x_2} + \frac{1+x_1x_2}{1-x_1x_2})\theta_1 \\
& + (2\frac{1+x_2}{1-x_2} - \frac{1+x_1}{1-x_1} + \frac{1+x_1x_2}{1-x_1x_2})\theta_2 - 1, \\
\Delta_3 = & -8\theta_1\theta_2(\theta_1 - \theta_2) \\
& + 2(\frac{1+x_2}{1-x_2} + \frac{1+x_1x_2}{1-x_1x_2})\theta_1^2 + 4(\frac{1+x_1}{1-x_1} - \frac{1+x_2}{1-x_2})\theta_1\theta_2 \\
& - 2(\frac{1+x_1}{1-x_1} + \frac{1+x_1x_2}{1-x_1x_2})\theta_2^2 \\
& - \frac{2(1+x_1)(1-x_1x_2^2)}{(1-x_1)(1-x_2)(1-x_1x_2)}\theta_1 + \frac{2(1+x_2)(1-x_1^2x_2)}{(1-x_1)(1-x_2)(1-x_1x_2)}\theta_2.
\end{aligned} \tag{4.2}
$$

Here, we use the notation $\theta_i = x_i\partial_i$ and $\partial_i = \frac{\partial}{\partial x_i}$. Changing the independent variables (x_1, x_2) into $(x'', y'') = T_1 \circ T_2(x_1, x_2)$ where $T_1(x', y') = (x' - 1, y' - 1)$ and $T_2(x_1, x_2) = (1/x_1, x_2)$, we obtain a system of differential equations $\mathcal{P}_L$ of which locus of singularities is

$$\{(x'', y'') \mid x''y''(x''-1)(y''-1)(x''-y'') = 0\}.$$

Notice $(0,0) = T_1 \circ T_2(1,1)$. Blowing-up (x'', y'')-plane at the origin, we naturally obtain the following system of differential equations from the system $\mathcal{P}_L$:

$$Pu = Qu = 0 \tag{4.3}$$

where

$$
\begin{aligned}
P = & \, p_0 + xp_1 + x^2p_2 \\
Q = & \, q_0 + xq_1 + x^2q_2 + x^3q_3 \\
p_0 = & \, 2(2y^3\theta_x^2 - 4y^3\theta_x\theta_y + 2y^3\theta_y^2 - 2y^2\theta_x^2 + 6y^2\theta_x\theta_y - 4y^2\theta_y^2 + \\
& y^2\theta_y - 2y\theta_x\theta_y - y\theta_x + 4y\theta_y^2 + y\theta_y - 2\theta_y^2) \\
p_1 = & \, 2y(4y^2\theta_x^2 - 6y^2\theta_x\theta_y + y^2\theta_x + 2y^2\theta_y^2 - 4y\theta_x^2 + 8y\theta_x\theta_y - 2y \\
& \theta_x + 4y\theta_y - 2\theta_x\theta_y - \theta_x - 2\theta_y^2) \\
p_2 = & \, y^2(L_2y - L_2 + 4y\theta_x^2 - 4y\theta_x\theta_y + 2y\theta_x + 4y\theta_y^2 + 2y\theta_y + y - 4\theta_x^2 \\
& + 4\theta_x\theta_y - 4\theta_x - 4\theta_y^2 + 2\theta_y - 1) \\
q_0 = & \, 4(2y^2\theta_x^2\theta_y + 2y^2\theta_x^2 - 4y^2\theta_x\theta_y^2 - 4y^2\theta_x\theta_y + 2y^2\theta_y^3 \\
& + 2y^2\theta_y^2 - 2y\theta_x^2\theta_y - y\theta_x^2 + 6y\theta_x\theta_y^2 + 2y\theta_x\theta_y - 4y\theta_y^3 - 2\theta_x\theta_y^2 + 2\theta_y^3 - 2\theta_y^2) \\
q_1 = & \, 2(4y^3\theta_x^2\theta_y + 2y^3\theta_x^2 - 8y^3\theta_x\theta_y^2 - 4y^3\theta_x\theta_y + 4y^3\theta_y^3 \\
& + 2y^3\theta_y^2 + 4y^2\theta_x^2\theta_y + 8y^2\theta_x^2 - 12y^2\theta_x\theta_y + 2y^2\theta_x - 4y^2\theta_y^3 \\
& + 8y^2\theta_y^2 - y^2\theta_y - 8y\theta_x^2\theta_y - 4y\theta_x^2 + 12y\theta_x\theta_y^2 + 4y\theta_x\theta_y - y\theta_x \\
& - 4y\theta_y^3 - 8y\theta_y^2 - y\theta_y - 4\theta_x\theta_y^2 + 4\theta_y^3 - 2\theta_y^2) \\
q_2 = & \, 2y(8y^2\theta_x^2\theta_y + 4y^2\theta_x^2 - 12y^2\theta_x\theta_y^2 - 4y^2\theta_x\theta_y + y^2\theta_x \\
& + 4y^2\theta_y^3 + 2y^2\theta_y^2 - 4y\theta_x^2\theta_y + 4y\theta_x^2 + 12y\theta_x\theta_y^2 - 8y\theta_x\theta_y \\
& + 2y\theta_x - 8y\theta_y^3 - 4y\theta_y - 4\theta_x^2\theta_y - 2\theta_x^2 - \theta_x + 4\theta_y^3 - 2\theta_y^2) \\
q_3 = & -y^2(L_3y - L_3 - 8y\theta_x^2\theta_y - 4y\theta_x^2 + 8y\theta_x\theta_y^2 - 2y\theta_x + 4y\theta_y^2 \\
& + 2y\theta_y + 8\theta_x^2\theta_y - 8\theta_x\theta_y^2 + 8\theta_x\theta_y - 4\theta_y^2 + 2\theta_y)
\end{aligned}
$$

where $x'' = x$ and $y'' = xy$. Notice that $x = 0$ is the exceptional curve of the blowing-up.

In this example, we obtain the exponents and boundary values of the system (4.3) along the line $x = 0$ which were not known ([Sek1]).

PROPOSITION 4.1

(1) *If there exists a solution of* (4.3) *around the origin of the form*

$$x^{\mu}\sum_{k=0}^{\infty}a_k(y)x^k, \quad a_0(y)\not\equiv 0,$$

then $\mu = 0$ *or* $\pm\frac{1}{2}$.

(2) *If* $\mu = 0$, *then the function* $a_0(y)$ *satisfies the differential equation*

$$\partial_y a_0(y) = 0.$$

If $\mu = \frac{1}{2}$, *then the function* $a_0(y)$ *satisfies the differential equation*

$$p_{01}a_0(y) = 0.$$

If $\mu = -\frac{1}{2}$, *then the function* $a_0(y)$ *satisfies the differential equation*

$$p_{02}a_0(y) = 0.$$

Here,

$$\begin{aligned}
p_{01} &= p_0|_{\theta_x=1/2}, \\
&= 4y^3\theta_y^2 - 4y^3\theta_y + y^3 - 8y^2\theta_y^2 + 8y^2\theta_y - y^2 + 8y\theta_y^2 - y - 4\theta_y^2 \\
&= y^3(2\theta_y - 1)^2 - y^2(8\theta_y^2 - 8\theta_y + 1) + y(8\theta_y^2 - 1) - 4\theta_y^2 \\
p_{02} &= p_0|_{\theta_x=-1/2}, \\
&= 4y^3\theta_y^2 + 4y^3\theta_y + y^3 - 8y^2\theta_y^2 - 4y^2\theta_y - y^2 + 8y\theta_y^2 + 4y\theta_y + y - 4\theta_y^2 \\
&= y^3(2\theta_y + 1)^2 - y^2(8\theta_y^2 + 4\theta_y + 1) + y(8\theta_y^2 + 4\theta_y + 1) - 4\theta_y^2
\end{aligned}$$

Proof of (1). Put

$$\begin{aligned}
R = &-(2y^4\partial_y - 4y^3\partial_y - 4y^3 + 4y^2\partial_y + 7y^2 - 2y\partial_y - 4y\theta_x - 5y + 2\theta_x + 2)P \\
&+ (y^4 - 2y^3 + 3y^2 - 2y + 1)Q
\end{aligned}$$

and

$$\begin{aligned}
S = &(6y^4\partial_y - 12y^3\partial_y - 8y^3\theta_x - 24y^3 + 12y^2\partial_y + 12y^2\theta_x + 39y^2 - 6y\partial_y - 33y - 2\theta_x + 12)R \\
&+ (y^2(4y\theta_x^2 - y - 4\theta_x^2 + 1))9(y-1)P.
\end{aligned}$$

Then the operator S has the following form:

$$S = s_0 + xs_1 + x^2s_2 + x^3s_3$$

where

$$s_i \in \mathbf{C}\langle\theta_x, y, \partial_y\rangle,$$

$$s_0(\theta_x, y) = 4y^2(y^2 - y + 1)^2(y-1)\theta_x^2(2\theta_x - 1)(2\theta_x + 1)$$

and

$$s_i(k, y, \partial_y) \not\equiv 0, \ s_i(k + 1/2, y, \partial_y) \not\equiv 0, \ s_i(k - 1/2, y, \partial_y) \not\equiv 0$$

for $i = 1, 2, 3$ and $k \in \{0, 1, 2, \cdots\}$. Since

$$s_0(\mu + k, y)a_k(y) = -\sum_{i=1}^{3} s_k(\mu + k - i, y, \theta_y)a_{k-i},$$

the parameter μ must be 0 or $\pm 1/2$.

Proof of (2). Omitted.

REMARK.

$$p_{01} = 4y(y-1)(y^2-y+1)\partial_y^2 + 4(2y-1)\partial_y + (y^2-y-1)$$

and

$$p_{02} = 4y(y-1)\partial_y^2 + 4(2y-1)\partial_y + 1.$$

The singularity $y^2 - y + 1$ must be apparent.

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