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# SYSTEMS OF NONLINEAR VARIATIONAL INEQUALITIES ARISING FROM PHASE TRANSITION PHENOMENA

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## 1. Introduction

We consider an evolution system, consisting of a nonlinear second-order parabolic PDE and a nonlinear fourth-order parabolic PDE with constraint, which is described as follows:

$$\rho(u)_t + \lambda(w)_t - \Delta u = h(t,x) \qquad \text{in } Q := (0,T)\times\Omega, \tag{1.1-1}$$

$$\frac{\partial u}{\partial n} + n_o u = h_o(t,x) \qquad \text{on } \Sigma := (0,T)\times\Gamma, \tag{1.1-2}$$

$$u(0,\cdot) = u_o \qquad \text{in } \Omega, \tag{1.1-3}$$

$$w_t - \Delta(-\nu\Delta w + \xi + g(w) - \lambda_o(w)u) = 0 \qquad \text{in } Q, \tag{1.2-1}$$

$$\frac{\partial w}{\partial n} = 0, \quad \frac{\partial}{\partial n}(-\nu\Delta w + \xi + g(w) - \lambda_o(w)u) = 0 \qquad \text{on } \Sigma, \tag{1.2-2}$$

$$\xi \in \beta(w) \qquad \text{on } Q, \tag{1.2-3}$$

$$w(0,\cdot) = w_o \qquad \text{in } \Omega. \tag{1.2-4}$$

Here $\Omega$ is a bounded domain in $\mathbf{R}^N$ $(1 \le N \le 3)$ with smooth boundary $\Gamma = \partial\Omega$; $\rho : \mathbf{R} \to \mathbf{R}$, $g : \mathbf{R} \to \mathbf{R}$ and $\lambda : \mathbf{R} \to \mathbf{R}$ are given functions and $\lambda_o(r) = \lambda'(r)$ (= the derivative of $\lambda$) for $r \in \mathbf{R}$; $\nu > 0$ and $n_o \ge 0$ are given constants, and $h$ and $h_o$ are given functions on $Q$ and $\Sigma$, respectively; $u_o$ and $w_o$ are initial data; $\beta$ is a given maximal monotone graph in $\mathbf{R}\times\mathbf{R}$.

The system (1.1)-(1.2) is interpreted as a simplified model for thermodynamical phase separation in which $w$ represents the order parameter, $\theta = -\frac{1}{u}$ the (Kelvin) temperature and the free energy functional $F(\theta, w)$ is supposed to be dependent upon the temperature $\theta$ and to be given by the formula

$$F(\theta, w) := \int_\Omega f(\theta, w, \nabla w)dx, \qquad w \in H^1(\Omega), \tag{1.3}$$

$$f(\theta, w, \nabla w) = \{\frac{1}{2}(\nu_o + \nu_1\theta)|\nabla w|^2 + \tau(\theta) + \theta(\hat\beta(w) + \hat g(w)) + \lambda(w)\},$$

where $\hat\beta$ is a proper l.s.c. convex function such that $\partial\hat\beta = \beta$ in $\mathbf{R}\times\mathbf{R}$, $\hat g$ is a primitive of $g$ on $\mathbf{R}$, $\lambda$ is the same as above, $\nu_o \ge 0$, $\nu_1 > 0$ are constants and $\tau : \mathbf{R} \to \mathbf{R}$ is a smooth function.

In some general settings, various models for thermodynamical phase separation phenomena have been proposed and studied for instance by Luckhaus-Visintin [11] and Alt-Pawlow [1,2]. However, in their models the constraint (1.2-3) is not taken account of.

To illustrate our system (1.1)-(1.2), for instance, consider a binary system of alloys with components $A$ and $B$ ocuppying $\Omega$; let $w := w_A$ and $w_B$ be the local concentrations of $A$ and $B$, respectively, such that

$$w_A + w_B = \text{const.};$$

suppose that the free energy functional $F(\theta, w)$ of the Ginzburg-Landau type is of the form (1.3). Then, according to the thermodynamics approach of DeGroot-Mazur [5] and Alt-Pawlow [1,2], we can derive from (1.3) with transformation $u := -1/\theta$, the mass and energy balance equations:

$$\rho(u)_t + \lambda(w)_t + [\frac{1}{2}\nu_o|\nabla w|^2]_t + \nabla \cdot \mathbf{q} = h(t,x) \qquad \text{in } Q, \tag{1.4}$$

$$w_t + \nabla \cdot \mathbf{j} = 0 \qquad \text{in } Q, \tag{1.5}$$

where $\rho(u) = \tau(\theta) - \theta\tau'(\theta)$, $\mathbf{q}$ is the energy flux due to heat and mass transfer, $\mathbf{j}$ is the mass flux of the component $A$ and $h$ is a given heat source. Now suppose further that the fluxes $\mathbf{q}$ and $\mathbf{j}$ are described by the following constitutive relations:

$$\mathbf{q} = \nabla(\frac{1}{\theta}) \ (= -\nabla u) \qquad \text{in } Q, \tag{1.6}$$

$$\mathbf{j} = -\nabla(\frac{\mu}{\theta}) \ (= \nabla(u\mu)) \qquad \text{in } Q, \tag{1.7}$$

where

$$\frac{\mu}{\theta} = \frac{\delta}{\delta w}[\int_\Omega \frac{f(\theta, w, \nabla w)}{\theta} dx] \tag{1.8}$$

and $\frac{\delta}{\delta w}$ denotes the functional derivative with respect to $w$. Since $f(\theta, w, \nabla w)$ includes the non-smooth term $\hat{\beta}(w)$, the right hand side of (1.8) is here understood in the multivalued sense

$$\frac{\delta}{\delta w}[\int_\Omega \frac{f(\theta, w, \nabla w)}{\theta} dx]$$

$$= \{-\nabla \cdot (\frac{\nu_o}{\theta} + \nu_1)\nabla w + \xi + g(w) + \frac{\lambda'(w)}{\theta}; \xi \in L^2(\Omega),\ \xi \in \beta(w) \ \ a.e. \text{ on } \Omega\}, \tag{1.9}$$

Now, combine (1.4)-(1.5) with (1.6)-(1.9). Then we obtain

$$\rho(u)_t + \lambda(w)_t + [\frac{\nu_o}{2}|\nabla w|^2]_t - \Delta u = h \qquad \text{in } Q, \tag{1.10}$$

and

$$w_t - \Delta(-\nabla \cdot (\nu_1 - \nu_o u)\nabla w + \xi + g(w) - \lambda'(w)u) = 0 \qquad \text{in } Q, \tag{1.11}$$

$$\xi \in \beta(w) \qquad \text{in } Q. \tag{1.12}$$

Therefore, if $\nu_o = 0$ and $\nu_1 = \nu$, or if in (1.10) the term $[\frac{\nu_o}{2}|\nabla w|^2]_t$ is experimentally allowed to be neglected and in (1.11) the coefficient $(\nu_1 - \nu_o u)$ of $\nabla w$ replaced by a positive constant $\nu$, then system (1.1-1)-(1.2-$i$), $i = 1, 3$, is regarded as a simplified form of (1.10)-(1.12). System (1.1)-(1.2) consists of these equations and initial-boundary conditions (1.1-$i$), $i = 2, 3$, and (1.2-$i$), $i$=2,4.

The aim of this paper is to study a weak formulation for system (1.1)-(1.2) in the variational sense, taking advantage of subdifferential techniques in Hilbert spaces.

## 2. Main results

Throughout this note, for a general (real) Banach space $X$ we denote by $|\cdot|_X$ the norm in $X$ and by $X^\star$ the dual space of $X$.

For simplicity we use the notations:

$$(v,w) := \int_\Omega vw dx \qquad \text{for } v, w, \in L^2(\Omega),$$

$$(v,w)_\Gamma := \int_\Gamma vw d\Gamma(x) \qquad \text{for } v, w \in L^2(\Gamma),$$

$$a(v,w) := \int_\Omega \nabla v \cdot \nabla w dx \qquad \text{for } v, w \in H^1(\Omega).$$

Moreover we put

$$H := L^2(\Omega), \qquad V := H^1(\Omega),$$

$$H_o := \{z \in H; \int_\Omega z dx = 0\}, \qquad V_o := V \cap H_o,$$

and denote by $\pi$ the projection from $H$ onto $H_o$, i.e.

$$\pi(z)(x) := z(x) - \frac{1}{|\Omega|}\int_\Omega z(y)dy, \qquad z \in H.$$

Also, $H_o$ is a Hilbert space with $|z|_{H_o} = |z|_H$ as well as $V_o$ with $|z|_{V_o} = |\nabla z|_H$; we use sometimes symbol $(\cdot,\cdot)_o$ for the inner product in $H_o$ and $\langle \cdot,\cdot \rangle_o$ for the duality pairing between $V_o^\star$ and $V_o$.

As usual, identifying $H$ with its dual, we have

$$V \subset H \subset V^\star$$

with dense and compact embeddings. Similarly, identifying $H_o$ with its dual, we have

$$V_o \subset H_o \subset V_o^\star$$

with dense and compact embeddings. Also, we denote by $J_o$ the duality mapping from $V_o$ onto $V_o^\star$ which is defined by the formula

$$\langle J_o z, \eta \rangle_o = a(z,\eta) \qquad \text{for all } z, \eta \in V_o.$$

Therefore, in particular, if $z^\star := J_o z \in H_o$, then $z \in H^2(\Omega)$ and $z$ is the unique solution of the Neumann problem

$$-\Delta z = z^\star \quad \text{in } \Omega, \qquad \frac{\partial z}{\partial n} = 0 \quad \text{on } \Gamma, \qquad \int_\Omega z dx = 0. \tag{2.1}$$

Accordingly, if $\eta \in H^2(\Omega)$ and $\frac{\partial \eta}{\partial n} = 0$ *a.e.* on $\Gamma$, then $J_o[\pi(\eta)] = -\Delta\eta$.

Now, we denote by (P) the system (1.1)-(1.2) mentioned in section 1 and discuss it under the following assumptions (A1)-(A6):

**(A1)** $\rho : \mathbf{R} \to \mathbf{R}$ is an increasing Lipschitz continuous function with Lipschitz continuous inverse $\rho^{-1} : \mathbf{R} \to \mathbf{R}$; we denote by $C_\rho$ a common Lipschitz constant of $\rho$ and $\rho^{-1}$.

**(A2)** $\lambda, \lambda_o : \mathbf{R} \to \mathbf{R}$ are Lipschitz continuous functions and $\lambda_o = \lambda'$; we denote by $C_\lambda$ a common Lipschitz constant of $\lambda$ and $\lambda_o$.

**(A3)** $g : \mathbf{R} \to \mathbf{R}$ is a Lipschitz continuous function; we denote by $C_g$ the Lipschitz constant of $g$.

**(A4)** $\nu$ is a positive constant and $n_o$ is a non-negative constant.

**(A5)** $\beta$ is a maximal monotone graph in $\mathbf{R}\times\mathbf{R}$ with bounded and non-empty interior $int.D(\beta)$ of the domain $D(\beta)$ in $\mathbf{R}$; we put $int.D(\beta) = (\sigma_\star, \sigma^\star)$ for $-\infty < \sigma_\star < \sigma^\star < \infty$ and hence $\overline{D(\beta)} = [\sigma_\star, \sigma^\star]$, and we may assume that $\beta$ is the subdifferential of a non-negative, proper, l.s.c. and convex function $\hat{\beta}$ on $\mathbf{R}$, since the range $R(\beta)$ of $\beta$ is the whole $\mathbf{R}$.

**(A6)** $0 < T < \infty$, $h \in L^2(0,T;H)$, $h_o \in W^{1,2}(0,T;L^2(\Gamma))$ and $u_o \in H$, $w_o \in V$ with $\hat{\beta}(w_o) \in L^1(\Omega)$.

We introduce

$$K(\hat{\beta}) := \{z \in H;\ \hat{\beta}(z) \in L^1(\Omega)\}$$

and

$$K_m(\hat{\beta}) := \{z \in K(\hat{\beta}); \frac{1}{|\Omega|}\int_\Omega z dx = m\} \qquad \text{for each } m \in \mathbf{R}.$$

We next give the weak formulation for (P).

**Definition 2.1.** A couple $\{u, w\}$ of functions $u : [0,T] \to V$ and $w : [0,T] \to H^2(\Omega)$ is called a (weak) solution of (P), if the following conditions (w1)-(w4) are satisfied:

**(w1)** $u \in L^2(0,T;V) \cap L^\infty(0,T;H)$, $\rho(u) \in C_w([0,T];H)$, $C_w([0,T];H)$ being the space of all weakly continuous functions from $[0,T]$ into $H$, $\rho(u)'(= \frac{d}{dt}\rho(u)) \in L^1(0,T;V^\star)$, $w \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;V)$, $w' \in L^2(0,T;V^\star)$ and $\lambda(w)' \in L^1(0,T;V^\star)$;

**(w2)** $\rho(u)(0) = \rho(u_o)$ and $w(0) = w_o$;

**(w3)** for $a.e.$ $t \in [0,T]$ and all $z \in V$,

$$\frac{d}{dt}(\rho(u(t)) + \lambda(w(t)), z) + a(u(t), z) + (n_o u(t) - h_o(t), z)_\Gamma = (h(t), z); \qquad (2.2)$$

**(w4)** for $a.e.$ $t \in [0,T]$,

$$\frac{\partial}{\partial n} w(t) = 0 \qquad a.e. \text{ on } \Gamma, \qquad (2.3)$$

and there is a function $\xi \in L^2(0,T;H)$ such that

$$\xi \in \beta(w) \qquad a.e. \text{ in } Q \qquad (2.4)$$

and

$$\frac{d}{dt}(w(t),\eta)+\nu(\Delta w(t),\Delta\eta)-(g(w(t))+\xi(t)-\lambda'(w(t))u(t),\Delta\eta)=0 \qquad (2.5)$$

for all $\eta \in H^2(\Omega)$ with $\frac{\partial\eta}{\partial n}$ *a.e.* on $\Gamma$, and *a.e.* $t\in[0,T]$.

When it is necessary to indicate the data $h, h_o, u_o, w_o$, we denote problem (P) by (P;$h, h_o, u_o, w_o$).

**Remark 2.1.** Let $\{u,w\}$ be any solution of (P). Then it follows from (2.5) in (w4) that

$$\frac{d}{dt}(w(t),1)=0 \qquad \text{for } a.e.\ t\in[0,T],$$

whence

$$\int_\Omega w(t,x)dx=\int_\Omega w_o dx \qquad \text{for all } t\in[0,T].$$

Therefore, putting

$$m:=\frac{1}{|\Omega|}\int_\Omega w_o dx, \qquad (2.6)$$

we observe that $w(t)-m\in V_o$ for all $t\in[0,T]$.

Our main results of this paper are stated as follows:

**Theorem 2.1.** *Assume that* $1\le N\le 3$ *and (A1)-(A6) hold, and assume with notation (2.6) that*

$$m\in int.D(\beta), \quad i.e.\ \ \sigma_\star<m<\sigma^\star.$$

*Then (P) has one and only one solution* $\{u,w\}$. *Moreover, the solution* $\{u,w\}$ *has the following bounds:*

$$|u|_{L^\infty(0,T;H)}+|u|_{L^2(0,T;V)}+|w|_{L^\infty(0,T;V)}+|\hat\beta(w)|_{L^\infty(0,T;L^1(\Omega))}+|w'|_{L^2(0,T;V^\star)}$$
$$\le R_o(|u_o|_H,|w_o|_V,|\hat\beta(w_o)|_{L^1(\Omega)},|h|_{L^2(0,T;H)},|h_o|_{L^2(0,T;L^2(\Gamma))}), \qquad (2.7)$$

*where* $R_o:\ \mathbf{R}^5_+\to\mathbf{R}$ *is a function which is bounded on each bounded subset of* $\mathbf{R}^5_+$;

$$|w|_{L^2(0,T;H^2(\Omega))}+|\rho(u)'|_{L^1(0,T;V^\star)}+|\lambda(w)'|_{L^1(0,T;V^\star)}$$
$$\le R_1(\frac{1}{\delta},r(\delta),|u_o|_H,|w_o|_V,|\hat\beta(w_o)|_{L^1(\Omega)},|h|_{L^2(0,T;H)},|h_o|_{L^2(0,T;L^2(\Gamma))}), \qquad (2.8)$$

*where* $R_1:\ \mathbf{R}^7_+\to\mathbf{R}_+$ *is a function which is bounded on each bounded subset of* $\mathbf{R}^7_+$, $\delta$ *is an arbitrary number satisfying*

$$0<\delta<1, \qquad \sigma_\star<m-\delta<m+\delta<\sigma^\star, \qquad (2.9)$$

*and*

$$r(\delta)=\sup\{|r'|;\ r'\in\beta(m-\delta)\cap\beta(m+\delta)\}. \qquad (2.10)$$

**Remark 2.2.** In estimates (2.7) and (2.8), the dependence of the solution $\{u, w\}$ upon functions $\rho$, $\lambda$, $g$ and $\beta$ is not explicitly indicated. However, as will be able to be easily checked, the functions $R_o$ and $R_1$ are chosen so as to be independent of them, as long as the Lipschitz constants $C_\rho$, $C_\lambda$, $C_g$ and the length $\sigma^\star - \sigma_\star$ of $D(\beta)$ vary in a bounded subset of $\mathbf{R}_+$.

**Theorem 2.2.** *Assume that* $1 \le N \le 3$ *and (A1)-(A5) hold. Let* $\{h_n\}, \{h_{on}\}, \{u_{on}\}$ *and* $\{w_{on}\}$ *be bounded sequences in* $L^2(0,T;H)$, $W^{1,2}(0,T;L^2(\Gamma))$, $H$ *and* $V$, *respectively, and assume that* $\{\hat{\beta}(w_{on})\}$ *is bounded in* $L^1(\Omega)$. *Further suppose that as* $n \to \infty$

$$h_n \to h \quad \text{in } L^2(0,T;H), \qquad h_{on} \to h_o \quad \text{in } L^2(0,T;L^2(\Gamma))$$

*and*

$$u_{on} \to u_o \quad \text{in } H, \qquad w_{on} \to w_o \quad \text{in } V.$$

*Then we have the following statements (i) and (ii):*

*(i) Suppose that*

$$\sigma_\star < m_n := \frac{1}{|\Omega|}\int_\Omega w_{on}dx < \sigma^\star \qquad \text{for all } n, \tag{2.11}$$

*and*

$$\sigma_\star < m := \frac{1}{|\Omega|}\int_\Omega w_o dx < \sigma^\star.$$

*Let* $\{u_n, w_n\}$ *be the solution of* $(P_n) := (P; h_n, h_{on}, u_{on}, w_{on})$ *for each* $n$ *and* $\{u, w\}$ *be the solution of* $(P) := (P; h, h_o, u_o, w_o)$. *Then, as* $n \to \infty$,

$$u_n \to u \quad \text{in } L^2(0,T;H),$$

$$\rho(u_n) \to \rho(u) \quad \text{weakly in } H \text{ and uniformly in } t \in [0,T],$$

$$w_n \to w \quad \text{in } L^2(0,T;V) \text{ and weakly}^\star \text{ in } L^\infty(0,T;V)$$

*and*

$$w_n' \to w' \quad \text{weakly in } L^2(0,T;V^\star).$$

*(ii) Suppose that (2.11) hlods and*

$$m = \sigma_\star \text{ or } \sigma^\star,$$

*Then, for the solution* $\{u_n, w_n\}$ *of* $(P_n)$, *we have as* $n \to \infty$,

$$w_n \to m \qquad \text{in } C([0,T];H)$$

$$u_n \to u \qquad \text{in } L^2(0,T;H) \text{ and weakly in } L^2(0,T;V)$$

*and*

$$\rho(u_n) \to \rho(u) \quad \text{weakly in } H \text{ and uniformly in } t \in [0,T],$$

*where* $u \in C([0,T];H) \cap W^{1,2}_{loc}((0,T];H) \cap L^\infty_{loc}((0,T];V) \cap L^2(0,T;V)$ *is the unique solution of*

$$\frac{d}{dt}(\rho(u(t)), z) + a(u(t), z) + (n_o u(t) - h_o(t), z)_\Gamma = (h(t), z)$$

$$for\ all\ z \in V,\ a.e.\ t \in [0,T], \tag{2.12}$$
$$u(0) = u_o.$$

**Remark 2.3.** In (ii) of Theorem 2.2, moreover if $h \in L^\infty(Q), h_o \in L^\infty(\Sigma),\ u_o \in L^\infty(\Omega)$ and $m \in D(\beta)$, then the pair $\{u, w\}$, with the solution $u$ of (2.12) and $w = m$, is the solution of (P) in the sense of Definition 2.1. In fact, under such restrictions on the data we see that $u \in L^\infty(Q)$ and hence $\xi := k - g(m) + \lambda'(m)u \in \beta(m)$ on $Q$ for a certain constant $k$. Thus condition (w4) of Definition 2.1 is satisfied.

## 3. Sketch of proofs

(1) (Uniqueness) The uniqueness of the solution of (P) can be proved by using Gronwall's inequality with the help of the following embedding inequalities:

$$|z|_{L^q(\Omega)} \le C_o|\nabla z|_H, \qquad |z|_{L^q(\Omega)} \le \delta|\nabla z|_H + C_\delta|z|_{V_o^*}$$

for all $z \in V_o$ and $1 \le q < 6$, where $C_o$ is a positive constant, and $\delta$ is an arbitrary positive constant with a constant $C_\delta$ dependent only on $\delta$.

(2) (Existence) For the construction of a solution of (P) we consider the approximate problem $(P)_\mu$ $(=(P_\mu; h, h_o, u_o, w_o))$, with parameter $0 < \mu \le 1$, to find a pair of functions $u_\mu : [0,T] \to V$ and $w_\mu : [0,T] \to H^2(\Omega)$ fulfilling the following conditions $(w1)_\mu$-$(w4)_\mu$:

$(\mathbf{w1})_\mu$ $u_\mu \in W^{1,2}(0,T;H) \cap L^\infty(0,T;V), w_\mu \in W^{1,2}(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega))$;

$(\mathbf{w2})_\mu$ $u_\mu(0) = u_o$ and $w_\mu(0) = w_o$;

$(\mathbf{w3})_\mu$ for $a.e.\ t \in [0,T]$ and all $z \in V$,

$$(\rho(u_\mu)'(t) + \lambda(w_\mu)'(t), z) + a(u_\mu(t), z) + (n_o u_\mu(t) - h_o(t), z)_\Gamma = (h(t), z); \tag{3.1}$$

$(\mathbf{w4})_\mu$ for $a.e.\ t \in [0,T]$,

$$\frac{\partial w_\mu(t)}{\partial n} = 0 \qquad a.e.\ \text{on } \Gamma, \tag{3.2}$$

and there is a function $\xi_\mu \in L^2(0,T;H)$ such that

$$\xi_\mu \in \beta(w_\mu) \qquad a.e.\ \text{on } Q \tag{3.3}$$

and

$$\begin{aligned}(w_\mu'(t), \eta) - \mu(w_\mu'(t), \Delta\eta) + \nu(\Delta w_\mu(t), \Delta\eta)&\\ -(g(w_\mu(t)) - \lambda'(w_\mu(t))u_\mu(t) + \xi_\mu(t), \Delta\eta) = 0&\end{aligned} \tag{3.4}$$

for all $\eta \in H^2(\Omega)$ with $\dfrac{\partial\eta}{\partial n} = 0$ $a.e.$ on $\Gamma$ and $a.e.\ t \in [0,T]$.

Besides we reformulate $(\mathrm{P})_\mu$ as a system of evolution equations including subdifferential operators. For this purpose, let us introduce convex functions $\varphi$ on $H_o$ and $\psi^t$, $t \le t \le T$, on $H$ as follows:

$$\varphi(z) := \begin{cases} \frac{\nu}{2}|\nabla z|_H^2 + \int_\Omega \hat{\beta}(z+m)dx & \text{if } z \in V_o \text{ and } \hat{\beta}(z+m) \in L^1(\Omega), \\ \infty & \text{otherwise,} \end{cases} \tag{3.5}$$

where

$$m := \frac{1}{|\Omega|}\int_\Omega w_o dx,$$

and

$$\psi^t(z) := \begin{cases} \frac{1}{2}|\nabla z|_H^2 + \frac{n_o}{2}|z|_{L^2(\Gamma)}^2 - (h_o(t), z)_\Gamma & \text{if } z \in V, \\ \infty & \text{otherwise.} \end{cases} \tag{3.6}$$

We then consider the subdifferential $\partial\varphi$ of $\varphi$ in $H_o$ and the subdifferential $\partial\psi^t$ of $\psi$ in $H$. It is easy to see that

(i) $z^\star \in \partial\varphi(z)$ if and only if $z^\star \in H_o$, $z \in V_o \cap (K_m(\hat{\beta}) - m)$ and

$$(z^\star, v - z)_o \le \nu a(z, v - z) + \int_\Omega \hat{\beta}(v+m)dx - \int_\Omega \hat{\beta}(z+m)dx$$

$$\text{for all } v \in V_o \cap (K_m(\hat{\beta}) - m);$$

(ii) $\partial\psi^t$ is singlevalued, and $z^\star = \partial\psi^t(z)$ if and only if $z^\star \in H$, $z \in V$ and

$$(z^\star, v) = a(z, v) + (n_o z - h_o(t), v)_\Gamma \qquad \text{for all } v \in V.$$

For each $\mu \in (0,1]$, problem $(\mathrm{P})_\mu$ has at most one solution and we have:

**Lemma 3.1.** *Let $\sigma_\star < m < \sigma^\star$, and $\lambda_1(r) := \lambda(r+m)$ and $g_1(r) := g(r+m)$ for $r \in \mathbf{R}$. Then a pair $\{u_\nu, w_\mu\}$ of functions is a solution of $(P)_\mu$ if and only if the pair $\{u_\mu, v_\mu\}$ with $v_\mu := w_\mu - m$ is a solution of the problem $(P)'_\mu$ defined below:*

**$(\mathrm{P})'_\mu$** *Find a pair $\{u_\mu, v_\mu\}$ of functions satisfying the following conditions $(w1)'_\mu$-$(w4)'_\mu$:*

**$(\mathrm{w1})'_\mu$** *$u_\mu \in W^{1,2}(0,T;H) \cap L^\infty(0,T;V)$ and $v_\mu \in W^{1,2}(0,T;H_o) \cap L^\infty(0,T;V_o)$;*

**$(\mathrm{w2})'_\mu$** *$u_\mu(0) = u_o$ and $v_\mu(0) = v_o := w_o - m$;*

**$(\mathrm{w3})'_\mu$** for *a.e.* $t \in [0,T]$,

$$\rho(u_\mu)'(t) + \lambda_1(v_\mu)'(t) + \partial\psi^t(u_\mu(t)) = h(t); \tag{3.7}$$

**$(\mathrm{w4})'_\mu$** for *a.e.* $t \in [0,T]$,

$$(J_o^\star + \mu I)v'_\mu(t) + \partial\varphi(v_\mu(t)) + \pi[g_1(v_\mu(t)) - \lambda'_1(v_\mu(t))u_\mu(t)] \ni 0. \tag{3.8}$$

We can prove Lemma 3.1 by using the following lemma which is concerned with the Lagrange multipliers of elliptic variational inequalities.

**Lemma 3.2.** *Let $\sigma_\star < m < \sigma^\star$ and $\ell$ be any element of $H$. Consider the following two problems $(M_m)$ and $(M_m)'$:*

$(\mathbf{M}_m)$ *Find a function $z_m \in K_m(\hat{\beta}) \cap V$ such that*

$$\nu a(z_m, z_m - \eta) + \int_\Omega \hat{\beta}(z_m)dx \le (\ell, z_m - \eta) + \int_\Omega \hat{\beta}(\eta)dx \qquad \text{for all } \eta \in K_m(\hat{\beta}) \cap V.$$

$(\mathbf{M}_m)'$ *Find a function $z_m \in K_m(\hat{\beta}) \cap H^2(\Omega)$, $\gamma_m \in \mathbf{R}$ and $\xi_m \in H$ such that*

$$-\nu\Delta z_m + \xi_m = \ell + \gamma_m \qquad \text{in } \Omega$$

*and*

$$\xi_m \in \beta(z_m) \qquad \text{a.e. on } \Omega, \qquad \frac{\partial z_m}{\partial n} = 0 \qquad \text{a.e. on } \Gamma.$$

*Then $(M_m)'$ has a solution $\{z_m, \xi_m, \gamma_m\}$ and the function $z_m$ is the unique solution of $(M_m)$. Moreover, $\gamma_m$ can be chosen so that*

$$|\gamma_m| \le 4M^5(1 + |\ell|_H), \tag{3.9}$$

*where $M = \max\{\frac{1}{\delta}, r(\delta), \sigma^\star - \sigma_\star, |\Omega|, \frac{1}{|\Omega|}\}$ for $\delta$ and $r(\delta)$ satisfying (2.9) and (2.10); $z_m$ satisfies that*

$$(-\Delta z_m, \xi_m) \ge 0 \tag{3.10}$$

*and*

$$\nu|\Delta z_m|_H \le |\ell|_H + |\gamma_m||\Omega|^{\frac{1}{2}}. \tag{3.11}$$

For the detail proof of Lemma 3.2 we refer to [9; Proposition 5.1]. Thanks to the additional term $\mu v'_\mu$ problem $(P_\mu)'$, hence $(P_\mu)$, is uniquely solved in the Hilbert spaces $H$ and $H_o$ by applying time-dependent subdifferential techniques evolved in [4, 10]. In fact, we have the following result.

**Proposition 3.1.** *In addition to all the conditions of Theorem 2.1, assume that $u_o \in V$. Then, for each $\mu \in (0, 1]$, problem $(P)_\mu$ has one and only one solution $\{u_\mu, w_\mu\}$. Moreover, the solution $\{u_\mu, w_\mu\}$ satisfies the bounds of the following type:*

$$|u_\mu|_{C([0,T];H)} + |\nabla u_\mu|_{L^2(0,T;H)} + |w'_\mu|_{L^2(0,T;V^*)}$$

$$+\mu|w'_\mu|^2_{L^2(0,T;H)} + |w_\mu|_{L^\infty(0,T;V)} + |\hat{\beta}(w_\mu)|_{L^\infty(0,T;L^1(\Omega))}$$

$$\le \tilde{R}_o(|u_o|_H, |w_o|_V, |\hat{\beta}(w_o)|_{L^1(\Omega)}, |h|_{L^2(0,T;H)}, |h_o|_{L^2(0,T;L^2(\Gamma))}),$$

*where $\tilde{R}_o : \mathbf{R}^5_+ \to \mathbf{R}_+$ is a function which is independent of $\mu$ and bounded on each bounded subset of $\mathbf{R}^5_+$;*

$$|w_\mu|_{L^2(0,T;H^2(\Omega))} + |\rho(u_\mu)'|_{L^1(0,T;V^*)} + |\lambda(w_\mu)'|_{L^1(0,T;V^*)}$$

$$\leq \tilde{R}_1(\frac{1}{\delta}, r(\delta), |u_o|_H, |w_o|_V, |\hat{\beta}(w_o)|_{L^1(\Omega)}, |h|_{L^2(0,T;H)}, |h_o|_{L^2(0,T;L^2(\Gamma))}),$$

*where* $\tilde{R}_1 : \mathbf{R}^7_+ \to \mathbf{R}_+$ *is a function which is independent of* $\mu$ *and bounded on each bounded subset of* $\mathbf{R}^7_+$, $\delta$ *is an arbitrary number satisfying (2.9) and* $r(\delta)$ *is a constant given by (2.10).*

By the above proposition we obtain a solution $\{u, w\}$ of (P), passing to the limit in $\mu \to 0$, and see that the solution satisfies estimates (2.7) and (2.8).

(3) (Proof of Theorem 2.2) The assertions of Theorem 2.2 follow easily from estimates for the solution of (P) in Theorem 2.1.

**Remark 3.1.** In this paper the domain $D(\beta)$ of $\beta$ is supposed to be bounded in $\mathbf{R}$. However this is not essential for the assertions of Theorems 2.1 and 2.2. For instance, our results can be extended to the case when $int.D(\beta) \neq \emptyset$ and there are constants $k_\beta > 0$ and $k'_\beta > 0$ such that

$$|\beta(r)| \geq k_\beta |r| - k'_\beta \quad \text{for all } r \in D(\beta);$$

note that under this condition we may assume that

$$\hat{\beta}(r) \geq \hat{k}_\beta |r|^2 \qquad \text{for all } r \in D(\hat{\beta}),$$

where $\hat{k}_\beta > 0$ is a certain constant.

**Application.** As a typical example of maximal monotone graphs $\beta$ in $\mathbf{R}\times\mathbf{R}$ arising in the context of phase separation (cf. [3]), we consider an increasing smooth function $\beta^c : (0,1) \to \mathbf{R}$ defined by

$$\beta^c(w) := c \log \frac{w}{1-w}$$

with positive real parameter $c$. Also, as an example of non-smooth $\beta$, we consider the subdifferential $\beta^0$ of the indicator function of the interval $[0,1]$ in $\mathbf{R}$, which is the limit of $\beta^c$ as $c \to 0$ in the sense of maximal monotone graphs in $\mathbf{R}\times\mathbf{R}$.

By virtue of Theorem 2.1, problem (P) with $\beta = \beta^c$ ($c \geq 0$) has one and only one solutioon $\{u^c, w^c\}$, provided that $u_o \in H$, $w_o \in V$ with $0 < m < 1$ and $\log \frac{w_o}{1-w_o} \in L^1(\Omega)$, $h \in L^2(0,T;H)$ and $h_o \in W^{1,2}(0,T;L^2(\Gamma))$. Moreover, it easily follows from the estimates (2.7),(2.8) and the uniqueness of solutions to (P) that as $c \to 0$, the solution $\{u^c, w^c\}$ converges to the solution $\{u^0, w^0\}$ in the similar sense as in (i) of Theorem 2.2.

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