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NON-ALGEBRAIC LIMIT CYCLES

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ABSTRACT

For a class of Liénard equation, every solution curve turns out to be non-algebraic. As an immediate corollary of it, the limit cycle of van der Pol equation turns out to be non-algebraic. The paper contains one more result, that is, the Bogdanov-Takens system turns out to have no algebraic limit cycles.

§1. Liénard Equations.

Studying limit cycles has been an important topic in the theory of Dynamical Systems since H.Poincaré first treated it. Van der Pol equation is the most famous example of having a limit cycle, and Liénard equation is known as a generalization of it. Cf.Chapter XI of [Le].

The following are our result and its immediate corollary.

THEOREM. If a polynomial system of Liénard equation

$$\begin{cases} \dot{x} = y \\ \dot{y} = -f(x)y - g(x) \end{cases} \tag{L}$$

satisfies (i) $f, g \neq 0$, (ii) $\deg f \geqq \deg g$, and (iii) $g/f \neq$ constant, then it has no algebraic solution curves.

COROLLARY. The system of van der Pol equation

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\mu(x^2-1)y - x, \quad \mu \neq 0, \end{cases} \tag{P}$$

has no algebraic solution curves.

In particular, the limit cycle of it is not algebraic.

We say that a solution curve on the real plane is algebraic if it is a portion of an algebraic curve. The algebraicity of limit cycles is a very natural notion and it has been studied by several mathematicians. Cf.Bibliography of [Ye]. The existence of non-algebraic limit cycle can be shown by a result of A.Lins-Neto[Ne]. Nonetheless, no one had announced any concrete example of non-algebraic limit cycles.

If the system (L) does not satisfy the condition (ii), then it may possibly have an algebraic limit cycle. Indeed, J.C.Wilson constructed such an example. Cf.§4.

§2. Fundamental Lemma.

The following is a fundamental tool for studying the algebraicity of solution curves.

LEMMA. If a solution curve of a polynomial system

$$\begin{cases} \dot{x} = p(x, y) \\ \dot{y} = q(x, y) \end{cases} \tag{2.1}$$

is a portion of an irreducible algebraic curve $\Phi(x, y)=0$, then there is a polynomial $h(x, y)$ satisfying

$$[p(x, y)\frac{\partial}{\partial x} + q(x, y)\frac{\partial}{\partial y}]\Phi(x, y) = h(x, y)\cdot\Phi(x, y) . \tag{2.2}$$

Conversely, if (2.2) holds, then the algebraic curve is an invariant curve of the system (2.1).

PROOF. By differentiating the equation $\Phi=0$ by the time, we see that the left-hand side of (2.2) is equal to zero on the algebraic curve. So the left-hand side has Φ as a factor. Cf.Theorem Ⅲ.3.1 of [Wa]. Hence the first half is proved.

Conversely, if (2.2) holds, then we obtain

$$\Phi(x^t, y^t) = \Phi(x^0, y^0)\cdot\exp[\int_0^t h(x^s, y^s)ds]$$

for every solution (x^t, y^t). So the algebraic curve $\Phi=0$ is an invariant curve. Hence the second half is proved.□

By using the above lemma, the task of studying algebraic solution curves of (2.1) is transformed into that of solving (2.2).

§3. Proof of the Theorem.

To prove the theorem, we assume that a solution curve of the system (L) is a portion of an irreducible algebraic curve $\Phi=0$. Then, by using the lemma, we obtain

$$[h(x, y) - y\frac{\partial}{\partial x} + f(x)y\frac{\partial}{\partial y} + g(x)\frac{\partial}{\partial y}]\Phi(x, y) = 0 . \tag{3.1}$$

From now, we put

$$\Phi(x,y)=\sum_{j=0}^{k}\Phi_j(x)y^j, \quad \text{where } \Phi_k(x)\neq 0,$$

$$h(x,y)=\sum_{j=0}^{l}h_j(x)y^j,$$

$M=\deg f$, and $N=\deg g$.

Then, by comparing the coefficient of y^{k+j} of (3.1), where j runs from l to 2 in turn, we get

$$h_j(x)=0 \quad \text{for every } 2\leqq j\leqq l.$$

Moreover, by comparing the coefficient of y^{k+1}, we get

$$h_1(x)\Phi_k(x)-\Phi_k'(x)=0.$$

So we obtain

$$h_1(x)=\Phi_k'(x)=0.$$

Thus we attain

$$h(x,y)=h_0(x) \quad \text{and}$$

$$\Phi_k(x)=\text{constant}.$$

By comparing the coefficient of y^j of (3.1), we get

$$-\Phi_{j-1}'+[h_0+jf]\Phi_j+(j+1)g\Phi_{j+1}=0. \tag{3.2}$$

(From now, we omit the variable x to write easily.) Let an integer m be the maximal degree of the polynomials Φ_j and an integer n the maximal suffix attaining it. Then, by putting $j=n$ in (3.2), we get

$$[h_0+nf]\Phi_n=\Phi_{n-1}'-(n+1)g\Phi_{n+1}. \tag{3.3}$$

Since the right-hand side is of degree less than $m+M-1$, we obtain

$$\deg[h_0+nf]<M \quad \text{and, therefore,}$$

$$\deg[h_0+jf]=M \quad \text{for every } j\neq n.$$

By putting $j=k$ in (3.2), we get

$$\Phi_{k-1}'=[h_0+kf]\Phi_k.$$

Since Φ_k is a non-zero constant, we obtain

$$\deg\Phi_{k-1}=M+1.$$

Similarly, by putting $j=k-1$ in (3.2), we get

$$\Phi_{k-2}'=[h_0+jf]\Phi_{k-1}+kg\Phi_k.$$

So we obtain

$$\deg \Phi_{k-2} = 2(M+1) \ .$$

By repeating the procedure, we attain

$$\deg \Phi_{k-j} = j\,(M+1) \quad \text{for every } 0 \leqq j \leqq k-n \,. \tag{3.4}$$

Thus, by putting $j = k - n$ to (3.4), we get

$$m = r\,(M+1) \ , \quad \text{where } r = k - n \,.$$

By applying (3.4), we see that the right-hand side of (3.3) is of degree at most $m-1$. So we obtain

$$h_0 = -n\,f \ .$$

From now, we assume that Φ is normalized in a sense, that is,

$$\Phi_k = 1 \ .$$

By putting $j = k$ to (3.2), we get

$$\Phi_{k-1}' = r\,f \ .$$

By integrating it, we obtain

$$\Phi_{k-1} = r\,F + R_0 \ ,$$

where $F' = f$. (From now, we denote by R_j an unknown polynomial of degree at most j.) Similarly, by putting $j = k-1$ to (3.2), we get

$$\Phi_{k-2}' = r\,(r-1)\,f\,F + (r-1)\,f\,R_0 + k\,g \ .$$

By integrating it, we obtain

$$\Phi_{k-2} = [\,r\,(r-1)/2]\,F^2 + R_{M+1} \ .$$

By repeating the procedure, we attain

$$\Phi_{k-j} = {}_rC_j\,F^j + R_{(j-1)(M+1)} \quad \text{for every } 0 \leqq j \leqq r \,, \tag{3.5}$$

where ${}_rC_j = r\,!/[\,j\,!\,(r-j)\,!\,]$.

By putting $j = n$ to (3.2), we get

$$\Phi_{n-1}' = (n+1)\,g\,\Phi_{n+1} \ .$$

By putting (3.5) to it, we obtain

$$\Phi_{n-1}' = r\,(n+1)\,g\,F^{r-1} + R_{m-M-2} \ . \tag{3.6}$$

On the other hand, by putting $j = n-1$ to (3.2), we get

$$\Phi_{n-1} = [\,n\,g\,\Phi_n - \Phi_{n-2}'\,]/f \ .$$

By putting (3.5) to it, we obtain

$$\Phi_{n-1} = [\,n\,g\,F^k + R_{m-1}\,]/f \ .$$

By differentiating it, we obtain

$$\Phi_{n-1}{}' = [n\, f\, g'\, F^r + r\, n\, f^2\, g\, F^{r-1} - n\, f'\, g\, F^r + R_{m+M-2}] / f^2 \quad . \tag{3.7}$$

By using (3.6) and (3.7), we attain

$$r\, f^2\, g\, F^{r-1} + n\, [f'\, g - f\, g']\, F^r = R_{m+M-2} \quad .$$

By comparing the term of the maximal degree, we get

$$r\,(M+1) + n\,(M-N) = 0 \quad .$$

Since the second term is not negative, we get

$$m = r\,(M+1) = 0 \quad .$$

So we see that Φ has y as a single variable. Since Φ is irreducible, we obtain

$$\Phi = y + C \quad ,$$

where C = constant. Thus, by putting it to the system (L), we attain

$$g / f = C \quad .$$

Hence we finish proving.□

§4. Unsolved Problem.

In connection with our theorem, we wish to introduce an example of J.C.Wilson[Wi].

EXAMPLE. The following system of Liénard equation

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\mu\,(x^2-1)\,y - (\mu^2/16)\,x^3\,(x^2-4) - x \quad , \quad 0 < |\mu| < 2, \end{cases} \tag{W}$$

has the following algebraic curve as a limit cycle

$$y^2 + (\mu/2)\,x\,(x^2-4)\,y + (x^2-4)\,[(\mu^2/16)\,x^2\,(x^2-4) + 1] = 0 \quad .$$

The above statement can be confirmed by applying the second half of the lemma. Incidentally, we obtain

$$h(x, y) = -(\mu/2)x^2 \ .$$

When $|\mu| \geqq 2$, the algebraic curve turns out to have a singularity, and so it can not be a limit cycle.

The polynomials f and g of the system (W) are of degree two and five respectively, therefore, there is a gap between our theorem and the example. So we wish to propose the following to bridge the gap.

CONJECTURE. Even if the system (L) satisfies $2\deg f \geqq \deg g > \deg f$, it has no algebraic limit cycles.

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§5. APPENDIX- The Bogdanov-Takens System.

The proposed problem in §4 is difficult to answer. In this section, we give an answer of the problem in case of $\deg f = 1$ and $\deg g = 2$. In this case, by transforming the variables (x, y, t), the system can be rewritten into the following form

$$\begin{cases} \dot{x} = y \\ \dot{y} = \mu_1 + \mu_2 y + x^2 + x y \ . \end{cases} \tag{BT}$$

The above system is called Bogdanov-Takens system in Bifurcation Theory. The system can possibly have a single limit cycle if the coefficients μ_1, μ_2 satisfy a suitable condition. Cf.§7.3 of [GH] and [LRW].

The following is our result in this section.

THEOREM. The Bogdanov-Takens system (BT) has no algebraic limit cycles.

PROOF. To prove the theorem, we assume that the system (BT) has an algebraic limit cycle. Before we begin the proof, we remark that the system (BT) can be transformed into the system

$$\begin{cases} \dot{x} = y - x \\ \dot{y} = -f(x)\,y - g(x) \ , \end{cases} \tag{5-1}$$

where

$f(x) = x + c_0$ and

$g(x) = d_1 x + d_0$.

Since the system (BT) has an algebraic limit cycles, so does the system (5-1). Since the region surrounded by the limit cycle must contain a singularity, we can assume without loss of generality that the origin is a singularity, i.e. $d_0 = 0$, surrounded by the limit cycle.

We denote by $\Phi=0$ the algebraic limit cycle. Then, by using the lemma, we obtain

$$[h(x,y)-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial x}+f(x)y\frac{\partial}{\partial y}+g(x)\frac{\partial}{\partial y}]\Phi(x,y)=0 \quad . \tag{5-2}$$

From now, we will proceed with the proof by using the same notation and method as in §3. By comparing the coefficient of y^j of (5.2), we get

$$-\Phi_{j-1}{}'+x\Phi_j{}'+[h_0+jf]\Phi_j+(j+1)g\Phi_{j+1}=0 \quad . \tag{5-3}$$

By using the same method of §3, we attain

$m=2r$ and

$h=-nf-2r$.

Since the algebraic curve $\Phi=0$ is not a straight line, we obtain

$m=2r\neq 0$.

As in §3, we assume

$\Phi_k=1$.

By putting $j=k$ to (5-3), we get

$\Phi_{k-1}{}'=rf-2r$.

So we obtain

$\Phi_{k-1}=rF-2rx+R_0$,

where $F'=f$. Similarly, by putting $j=k-1$ to (5-3), we get,

$\Phi_{k-2}{}'=r(r-1)fF-3r(r-1)x^2+R_1$.

So we obtain

$\Phi_{k-2}=[r(r-1)/2]F^2-r(r-1)x^3+R_2$.

By repeating the procedure, we attain

$$\Phi_{k-j}={}_rC_jF^j-2^{2-j}j\,{}_rC_jx^{2(j-1)+1}+R_{2(j-1)} \quad \text{for every } 0\leqq j\leqq r, \tag{5.4}$$

where ${}_rC_j=r!/[j!(r-j)!]$.

By putting $j=n$ to (5-3), we get

$\Phi_{n-1}{}'=x\Phi_n{}'-2r\Phi_n+(n+1)g\Phi_{n+1}$.

By putting (5-4) to it, we obtain

$$\Phi_{n-1}{}'=rxfF^{r-1}-2rF^r+r(n+1)gF^{r-1}+2^{2-r}rx^{m-1}+R_{m-2} \quad . \tag{5-5}$$

On the other hand, by putting $j=n-1$ to (5-3), we get

$$\Phi_{n-1}=[n\,g\,\Phi_n+x\,\Phi_{n-1}'-\Phi_{n-2}']/[f+2r]\ .$$

By putting (5-4) to it, we obtain

$$\Phi_{n-1}=[n\,g\,F^r+R_m]/[f+2r]\ .$$

By differentiating it, we obtain

$$\Phi_{n-1}'=[n\,f\,g'\,F^r+r\,n\,f^2\,g\,F^{r-1}-n\,f'\,g\,F^r+R_m]/[f+2r]^2\ . \tag{5-6}$$

By using (5-5) and (5-6), we attain

$$r[g+x\,f-2F+2x]f^2F^{r-1}+n[f'\,g-f\,g']F^r=R_m\ .$$

By comparing the term of the maximal degree, we get

$$d_1-c_0+2=0\ .$$

So, by using the lemma, we can easily confirm that the algebraic curve

$$y+(1/2)x^2+(c_0-2)x=0$$

is an invariant curve of the system (5-1). Clearly, the curve must get across the limit cycle. It is a contradiction. Thus the system (BT) has no algebraic limit cycles. Hence we finish proving.□

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