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Invariant measures for certain multi-dimensional maps

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Abstract

We investigate singular points of the invariant density for a class of multi-dimensional maps with finite range structure. In particular, we concentrate on maps with countably many discontinuity points which do not satisfy Renyi's condition and do not necessarily satisfy the Markov property. Such maps occur from number theory quite naturally. Under some conditions, we show that indifferent periodic points must be singular points of the invariant density.

1 Introduction

We consider multi-dimensional piecewise smooth maps which are almost expanding. These maps generally do not have the Markov property, but they have a similar structure,which we call a "finite range structure"(FRS) and leads to a nice countable state symbolic dynamics[22]. Many examples of such maps come from number theory (see section 4). The maps we study are typically only C^1-smooth, and they need not satisfy Renyi's condition(uniformly bounded distortion for all iterates).

In [4], sufficient conditions for the existence of absolutely continuous invariant measures were given for systems with FRS. Before [4], analyses of absolutely continuous invariant measures appealed to Renyi's condition and to the Markov property(e.g.[1],[2],[3],[6],[8],[9],[11],[15],[19],[23] ,[25]), both of which may fail(with interesting consequences, as we will see) for systems with FRS. If Renyi's condition holds, then the invariant density obtained is bounded. However without this condition, the invariant density of the finite measure may be unbounded (see section 4).We study singularities of the invariant density (see the definition in section 3) and relate them to the existence of non-hyperbolic periodic orbits.We also provide a sufficient condition for the validity of Rohlin's entropy formula.

We now establish some notation and recapitulate some definitions. We say a map T on a bounded domain $X \subset \mathbf{R}^d$ has a "finite range structure"(FRS) if there exists a countable partition $Q = \{X_a\}_{a \in I}$ of X and a collection of finitely many subsets of X, $\{U_0, U_1, \ldots, U_N\}$ such that

1. each X_a is a measurable, connected subset with piecewise smooth boundary and $intX_a \neq \emptyset$,

2. each U_k has positive Lebesgue measure,

3. for each $X_a, T|_{X_a}$ is injective, of class C^1 with $\det DT|_{X_a} \neq 0$,

4. if $intX_{a_1} \cap int(T^{-1}X_{a_2}) \cap \ldots \cap int(T^{-(n-1)}X_{a_n}) \neq \emptyset$, let $X_{a_1 \ldots a_n} = X_{a_1} \cap T^{-1}X_{a_2} \cap \ldots \cap T^{-(n-1)}X_{a_n}$. Then $T^n X_{a_1 \ldots a_n} = U_k$ for some $k \in \{0, 1, \ldots, N\}$.

Remark A Here a partition means a collection of disjoint sets. In 2, U_k can intersect U_j for $j \neq k$, and in particular one of the U_k can be equal to X. In 3, $\det DT|_{\partial X_a} = 0$ is possible. When we say a function is C^1 on X_a, we mean it agrees on X_a with a C^1 function defined on a neighbourhood of X_a in $\mathbf{R}^d$.

If there exists a constant $C(> 1)$ such that

$$\frac{\sup_{x \in X_{a_1 \ldots a_n}} |\det DT^n(x)|}{\inf_{x \in X_{a_1 \ldots a_n}} |\det DT^n(x)|} < C$$

for all $n > 0$ and all $X_{a_1 \ldots a_n}$, then we say T satisfies Renyi's condition. If $int(X_a \cap TX_b) \neq \emptyset$ implies $X_a \subset TX_b$, then we say T has the Markov property.

In section 3, we explain that indifferent periodic points must be singular points of the invariant density, and we discuss the characterization of non-singular points. In section 4, we apply our theorems to examples on which precise discussions are shown in [21]. In section 5, we consider Rohlin's entropy formula. Proofs of our results of section 3 and of section 5 are given in [21].

2 Notation and preliminary results

We call $X_{a_1 \ldots a_n}$ *a cylinder of rank n with respect to T*. $\mathcal{L}^n$ denotes the family of all cylinders $X_{a_1 \ldots a_n}$ of rank n and $\mathcal{L} \equiv \bigcup_{n=1}^{\infty} \mathcal{L}^n$. For constant $C > 1, X_{a_1 \ldots a_n}$ is called *a R(C.T)-cylinder of rank n* if

$$\frac{\sup_{x \in X_{a_1 \ldots a_n}} |\det DT^n(x)|}{\inf_{x \in X_{a_1 \ldots a_n}} |\det DT^n(x)|} < C.$$

$R(C.T)$ denotes the set of all $R(C.T)$-cylinders. We say that a cylinder $X_{a_1 \ldots a_n}$ satisfies *the local Renyi condition for C* if for all cylinders $X_{b_1 \ldots b_m}$ such that $X_{b_1 \ldots b_m a_1 \ldots a_n} \in \mathcal{L}^{m+n}, X_{b_1 \ldots b_m a_1 \ldots a_n} \in R(C.T)$. We say that T satisfies *the local Renyi condition* if there exists a constant $C(> 1)$ such that $R(C.T)$ is not empty and for all $X_{a_1 \ldots a_n} \in R(C.T)$ satisfies the local Renyi condition for C. We call the constant C *a local Renyi constant*. For $x \in X$ and $n \in \mathbf{N}$, we define

$$C(n, x) \equiv \frac{\sup_{y \in X_{a_1 \ldots a_n}(x)} |\det DT^n(y)|}{\inf_{y \in X_{a_1 \ldots a_n}(x)} |\det DT^n(y)|},$$

where $X_{a_1\dots a_n}(x)$ is the unique cylinder of rank n containing x. As $C(n,x)$ is constant on $X_{a_1\dots a_n}(x)$, we sometimes denote by $C(a_1\dots a_n)$ the constant. For point $x\in X$, if there exists a constant $C>1$ such that for $\forall n>0, \exists i_n(>n)$ so that $C(i_n,x)<C$, we call the point x *a limit point of $R(C.T)$-cylinders*. For $C>1$, we define

$$\mathcal{D}_i^{(C)} \equiv \{X_{d_1\dots d_i}\in \mathcal{L}^i; X_{d_1\dots d_j}\notin R(C.T) \text{ for all } j=1,2,.,i\}$$

$$D_i^{(C)} \equiv \bigcup_{X_{d_1\dots d_i}\in \mathcal{D}_i^{(C)}} X_{d_1\dots d_i},$$

$$\mathcal{B}_i^{(C)} \equiv \{X_{b_1\dots b_i}\in \mathcal{L}^i; X_{b_1\dots b_{i-1}}\in \mathcal{D}_{i-1}^{(C)}, X_{b_1\dots b_i}\in R(C.T)\},$$

and

$$B_i^{(C)} \equiv \bigcup_{X_{b_1\dots b_i}\in \mathcal{B}_i^{(C)}} X_{b_1\dots b_i}.$$

In particular, for $i=0$ we define $D_0=X$. We sometime write ψ_a for $(T|_{X_a})^{-1}$ and $\psi_{a_1\dots a_n}$ for $(T^n|_{X_{a_1\dots a_n}})^{-1}$.

Theorem 2.1 *Let $T: X\rightarrow X$ have a FRS and satisfy the local Renyi condition , and let $Q=\{X_a\}_{a\in I}$ satisfy the generator condition, i.e., $\bigvee_{m=0}^{\infty} T^{-m}Q=\varepsilon(\varepsilon$ is the partition into points). Assume that the local Renyi constant $C(>1)$ satisfies the following:*

1. *(transitivity condition) for all $j\in\{0,1,..,N\}$, there exists a cylinder $X_{a_1\dots a_{s_j}}\in R(C.T)$ such that $X_{a_1\dots a_{s_j}}\subset U_j$ and $T^{s_j}X_{a_1\dots a_{s_j}}=X$,*
2. *$\sum_{n=0}^{\infty}\lambda(D_n^{(C)})<\infty$, where λ is the normalized Lebesgue measure.*

Then there exists a finite , ergodic invariant measure μ which is equivalent to λ, and with respect to μ T is exact.

(cf [4],[20]).

Remark B If we replace the condition 2 by the weaker condition

$$\lim_{n\to\infty}\lambda(D_n^{(C)})=0$$

, then we still have an ergodic invariant measure which is equivalent to λ. This measure need not be finite (although it will be σ-finite)(see[4]).

Remark C The finite measure μ of Theorem 2.1 does not depend on C (in particular the invariant density of μ does not depend on C) (cf [24]).

3 Singularities of the invarinant density

In this section, we assume that T satisfies all assumptions of Theorem 2.1. We say a point $x_0 \in X$ is *an indifferent periodic point* if there exists a $p > 0$ such that $T^p x_0 = x_0$ and $|\det DT^p(x_0)| = 1$. A point $x \in X$ is called *a singular point* of a measurable function f if $\forall \varepsilon > 0$ the essential supremum of $|f|$ on $B_\varepsilon(x)$ is infinite, where $B_\varepsilon(x)$ is a ε-neighbourhood of x.

Theorem 3.1 *Suppose x_0 is an indifferent periodic point, and*

(M) $x_0 \notin (\bigcup_{j=0}^{N} \partial U_j \setminus \bigcup_{a\in I} \partial X_a)$.

Then x_0 is a singular point of the invariant density of μ.

Remark D If T satisfies the Markov property , then $\partial U_j \subset \bigcup_{a\in I} \partial X_a$ for every j and so the condition (M) is automatically satisfied.

The following results are needed to prove Theorem 3.1.

Proposition 3.1 *If x_0 is an indifferent periodic point, then for the local Renyi constant C, $x_0 \in \bigcap_{n=0}^{\infty} D_n^{(C)}$.*

Lemma 3.1 *If x_0 is an indifferent periodic point, then $\lim_{n\to\infty} C(n, x_0) = \infty$.*

Lemma 3.2 *If x_0 is any point such that $\lim_{n\to\infty} C(n, x_0) = \infty$, then for the local Renyi constant C, there exists a number $N_0(C)$ such that $T^n x_0 \in \bigcap_{n=0}^{\infty} D_n^{(C)}$ for all $n \geq N_0(C)$. In particular, if x_0 is a periodic point, then $x_0 \in \bigcap_{n=0}^{\infty} D_n^{(C)}$.*

Let $\mathcal{S}$ be the set of all singular points of the invariant density of μ, and $\mathcal{P}$ be the set of all periodic points for T. We remark that a point x_0 in $\mathcal{P}$ with period p satisfies $|\det DT^p(x_0)| \geq 1$. In fact, the generator condition does not allow the case that $|\det DT^p(x_0)| < 1$. Now we ask , is the converse of Theorem 3.1 true? Some examples in the next section show that the answer is no! A singular point of the invariant density is not necessarily a periodic point (Examples 7,8). A singular point of the invariant density which is periodic is not necessarily an indifferent periodic point(Examples 5,6,7,8). In general, it is unclear how to characterize singular points of the invariant density. For limit points of $R(C.T)$ -cylinders, we can obtain the following answer:

Theorem 3.2 *A limit point x of $R(C.T)$-cylinders is a singular point of the invariant density of μ if and only if*

$$\sum_{n=0}^{\infty} \sum_{X_{d_1\ldots d_n} \in \mathcal{D}_n} |\det D\psi_{d_1\ldots d_n}(x)| = \infty$$

For some class of one-dimensional piecewise C^2-smooth Bernoulli maps, it is possible to characterize completely the singular points of the invariant density by indifferent fixed points, that is, Renyi's condition holds iff there is no indifferent fixed point (Thaler [18]). However, under C^2-smoothness, the existence of an indifferent fixed point leads to an infinite ergodic absolutely continuous invariant measure([17]). On the other hand, in our setting the invariant measure which we obtain is a finite measure. In fact, our one-dimensional example in section 4 which satisfies all assumptions of Theorem 2.1 does not have C^2-smoothness on any neighborhood of an indifferent fixed point(Examples 3,7). In all multi-dimensional examples of the next section, at a singular point x_0 of the invariant density the derivative has at least one eigenvalue of modulus one, and $\sup_{n>0} C(n, x_0)$ is infinite. So we can ask, for example

Question 1 Can a repelling periodic point (i.e.,all eigenvalues of the derivative have modulus strictly greater than one) of a piecewise C^1 map satisfying conditions of Theorem 2.1 , be a singular point of the invariant density?

The following result is a possible tool for approaching Question 1 when the domain of T is one-dimensional.

Corollary 3.1 (One dimensional case) *Let $T^p x_0 = x_0, |(T^p)'(x_0)| > 1$ and assume that there exists $\varepsilon > 0$ such that T restricted to a ε-neighborhood of x_0 is of class C^2. Then $\sup_{n>0} C(n, x_0) \equiv C_0 < \infty$. If $C_0 \leq C$, then conditions of Theorem 3.2 are satisfied ,and so*

$$\sum_{n=0}^{\infty} \sum_{X_{d_1 \dots d_n} \in \mathcal{D}_n} |\det D\psi_{d_1 \dots d_n}(x_0)| < \infty$$

iff x_0 is a non-singular point of the invariant density of μ.

We use Corollary 3.1 for analyze Example 3.

Remark E A possible class of maps satisfying the conditions of Corollary 3.1 are piecewise expanding maps which have smoothness of the class $1 + \alpha$ at the endpoints. Such behavior occurs for Lorenz-type maps.

Question 2 For a repelling periodic point, is $\sup_{n>0} C(n, x_0)$ finite? (In the case of direct product of one-dimensional maps,there is a partial answer (Corollary 3.1)).

Question 3 When $\mathcal{P}$ consists of repelling periodic points, can the invariant density have singular points?

4 Examples and applications

In this section, we will first show some examples which satisfy the assumptions of Theorem 3.1,and thus have ergodic finite invariant measures with unbounded densities whose singular points are indifferent periodic points. Examples 1, 2, and 4 are number theoretical two-dimensional maps. Example 3 is a one-dimensional map which does not relate to number theory. This is also one of examples for which we can verify the condition of Theorem 3.2 to use this result. As we mentioned in section 3, this example suggests that the appearance of indifferent periodic points does not necessarily lead to infiniteness of our invariant measure without (piece-wise) C^2-smoothness. In the case of two-dimensional map, even if C^2-smoothness is valid, from Example 4 we can say the same fact as in the case of one-dimensional map. Next we will show some two-dimensional examples which suggest that the singular points of the invariant densities are not necessarily indifferent periodic points. All of examples 5,6, and 7 have singularities of invariant densities at periodic points with period p which are not indifferent, but at these points the derivative of p-th powers have at least one eigenvalue of modulus one. The last example, 8 shows that singular points of the invariant density are not necessarily periodic.

Example 1 (A skew product two-dimensional map which is related to Diophantine approximation in inhomogeneous linear class)

Let $X = \{(x_1, x_2) \in R^2 : 0 \leq x_2 \leq 1, -x_2 \leq x_1 \leq -x_2 + 1\}$. Define T on X by

$$T(x_1, x_2) = (1/x_1 - [(1 - x_2)/x_1] - [-(x_2/x_1)], -[-(x_2/x_1)] - (x_2/x_1)).$$

The invariant density of the finite invariant measure of T is :

$$h(x_1, x_2) = \frac{1}{2\log 2(1 - {x_1}^2)}$$

([5]), so the singular points of the density $h(x_1, x_2)$ are $(1, 0)$ and $(-1, 1)$. These points are periodic points with period 2 and are indifferent.

Remark F In this example, the local Renyi constant is unique. In fact, under the assumptions of Theorem 3.1, the appearance of the indifferent periodic points gives us to the following condition for the local Renyi constant C: $C \leq \inf\{C(n, x) : n > 0, x \text{ is an indiffrent periodic point }\}$.

Example 2 (A real two-dimensional map which is related to a complex continued fraction expansion)

Let $X = \{z = x_1\alpha + x_2\overline{\alpha} : -(1/2) \leq x_1, x_2 \leq 1/2\}(\alpha = 1 + i)$ and define T on X by $Tz = 1/z - [1/z]_1$, where $[z]_1$ denotes $[x_1 + 1/2]\alpha + [x_2 + 1/2]\overline{\alpha}$ for a complex number $z = x_1\alpha + x_2\overline{\alpha}$. Let the index set I be ; $I = \{n\alpha + m\overline{\alpha} : m, n \in \mathbf{Z}\}\backslash\{0\}$. T induce a continued fraction expansion of $z \in X$,

$$z = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots a_n + \cdots}}},$$

where each a_i is contained in I.(Figure (1)). T has indifferent periodic points± 2, $\pm 2i$, and the invariant density which was obtained by S.Tanaka([16]) has singularities at these points and no others.(Figure(2)). He showed by his own method that finiteness and ergodicity of the invariant measure. On the other hand, T satisfies all assumptions of Theorem 2.1 and 3.1.Further dynamical properties (for example weak Bernoulli property) are discussed in [20]. So about this example, we omit further details.

Example 3 (A one-parameter family of maps on the interval $[0, 1]$)

Let $X = [0, 1]$ and for α with $0 < \alpha < 1$ define

$$f_\alpha(x) = \begin{cases} \frac{x}{(1 - x_\alpha)^{1/\alpha}} & \text{on } X_0 = [0, (1/2)^{1/\alpha}) \\ \frac{x}{(1/2)^{1/\alpha} - 1} + \frac{1}{1 - (1/2)^{1/\alpha}} & \text{on } X_1 = [(1/2)^{1/\alpha}, 1] \end{cases}$$

In the case of a direct product of one-dimensional maps, if one of the Invariant densities of the maps gives an infinite invariant measure, the direct product has an infinite invariant measure, too. The next example is defined by using a one-dimensional map with an infinite invariant measure, but the map itself has a finite invariant measure with unbounded density.

Example 4 (Two-dimensional map which is related to Brun's algorithm,"B map")

First, we define a one-dimensional map $T_1 : [0, 1] \to [0, 1]$ by

$$T_1(x) = \begin{cases} \frac{x}{1-x} & \text{on } X_0 = [0, 1/2) \\ \frac{1}{x} - 1 & \text{on } X_1 = [1/2, 1] \end{cases}$$

(Figure (3)).

Now ,we define Brun's map T. Let $X = \{(x_1, x_2) \in \mathbf{R}^2; \leq x_2 \leq x_1 \leq 1\}$, and let for $i \in \{0, 1, 2\}, X_i = \{(x_1, x_2) \in X; x_i + x_1 \geq 1 \geq x_{i+1} + x_1\}$, where we put $x_0 = 1$ and $x_3 = 0$. T is defined by

$$T(x_1, x_2) = \begin{cases} (T_1(x_1), \frac{x_2}{1-x_1}) & \text{on } X_0 \\ (T_1(x_1), \frac{x_2}{x_1}) & \text{on } X_1 \\ (\frac{x_2}{x_1}, T_1(x_1)) & \text{on } X_2 \end{cases}$$

(Figure(4)). Note that $T(0,0) = (0,0)$ and $|\det DT(0,0)| = 1$, so $(0,0)$ is the indifferent fixed point for T, and T is a piecewise C^2-map. This map is one of examples of Markovian MCF algorithm with the weak convergence property (Lagarias [7]), and Schweiger determined the unique absolutely continuous invariant measure which is ergodic. This invariant measure is finite and the invariant density $h(x_1, x_2)$ is the following([12], [13],[14]);

$$h(x_1, x_2) = \frac{1}{2x_1(1+x_2)}.$$

So $(0,0)$ is the only singular point of $h(x_1, x_2)$. In fact, T satisfies all assumptions of Theorem 2.1 and 3.1.

Example 5 (A modification of Brun's map with a finite partition)

Now we define a modification of Brun's map such that two pieces of the partition touch at the fixed point 0. The domain X is the same as in Example 4. We devide X_0 into two pieces $X_\alpha = \{(x_1, x_2) \in X_0; x_1 \geq 2x_2\}$ and $X_\beta = \{(x_1, x_2) \in X_0; x_1 < 2x_2\}$, and define T^* on the se pieces by

$$T^*(x_1, x_2) = \begin{cases} (T_1(x_1), \frac{2x_2}{1-x_1}) & \text{on } X_\alpha \\ (T_1(x_1), \frac{2x_2 - x_1}{1-x_1}) & \text{on} X_\beta \end{cases}$$

On X_1 and X_2, T^* is defined as in Example 4. This changing in the definition of T^* allows us to have the non-indifferent fixed point 0. In fact $|\det DT^*(0,0)| = 2$.

We remark that the dynamical properties of T^* in which we are interested are not changed essentially[21], and still the fixed point 0 is a singular point of the invariant density.

Example 6 (A modification of Brun's map with a countable partition)

Let the domain X be as in Example 4, and let devide X_0 into countably many pieces,

$$X_{\alpha_k} = \{(x_1, x_2) \in X_0; \frac{2x_1}{k+2} \leq x_2 < \frac{2x_1}{k+1}\} (k > 0).$$

On each X_{α_k} , define $T^{**}(x_1, x_2) = (\frac{x_1}{1-x_1}, \frac{k(k+1)x_2/2 - x_1}{1-x_1})$. On X_1 and X_2, the definition of T^{**} are the same as in Example 4.

Example 7 (A product of one-dimensional maps with a contable partition)

Let $X = [0,1]^2$, and define T on X by

$$T(x_1, x_2) = (T_1(x_1), T_2(x_2)),$$

where

$$T_1(x_1) = \frac{x_1(2-\sqrt{x_1})^2}{4(1-\sqrt{x_1})^2} - [\frac{x_1(2-\sqrt{x_1})^2}{4(1-\sqrt{x_1})^2}]$$

and $T_2(x_2) = 2x_2 - [2x_2]$. (Here $[x]$ denotes the Gauss part of x.) $T(0,0) = (0,0), |\det DT(0,0)| = 2$, so $(0,0)$ is a non-indifferent fixed point. First we remark the properties of T_1. $T_1(0) = 0, (T_1)'(0) = 1$ and $T_1''(0) = \infty$. 0 is an indifferent fixed point of T^1, and on any neighbourhood of 0, T_1 is only of class C^1, not of class C^2. In fact T_1 has a finite invariant measure with a unbounded density $h(x_1) = 1/\sqrt{x_1}$(Thaler [17]).The indifferent fixed point 0 is exactly the singular point of the invariant density $h(x_1)$. Since the invariant measure of T_2 is the Lebesgue measure, the invariant density $h(x_1, x_2)$ of T is given by $h(x_1, x_2) = 1/\sqrt{x_1}$, and this gives us a finite invariant measure. Hence this example shows that a periodic point which is a singular point of the invariant density is not necessarily indifferent and singular points of the invariant density are not necessarily periodic.

Example 8 (A product of one-dimensional maps with countably many cylinders in $\mathcal{D}_1$)

Let $X = [0,1]^2$, and we define a product of the one-dimensional map $f_\alpha(0 < \alpha < 1)$ in Example 3 and the Gauss transformation which is related to the simple continued fraction expansion, that is T is defined by

$$T(x_1, x_2) = \begin{cases} (\frac{x_1}{(1-x_1{}^\alpha)^{1/\alpha}}, \frac{1}{x_2} - l) & \text{on } X_{(0,l)}(l \in \mathbf{N}) \\ (\frac{x}{(1/2)^{1/\alpha}-1} + \frac{1}{1-(1/2)^{1/\alpha}}, \frac{1}{x_2} - l) & \text{on } X_{(1,l)} \end{cases}$$

where $X_{(0,l)} = \{(x_1, x_2) \in X; 0 \le x_1 < \frac{1}{2^{1/\alpha}}, \frac{1}{l+1} \le x_2 \le \frac{1}{l}\}$ and $X_{(1,l)} = \{(x_1, x_2) \in X; (\frac{1}{2})^{1/\alpha} \le x_1 \le 1, \frac{1}{l+1} \le x_2 < \frac{1}{l}\}$.

T has an invariant density $h_1(x_1)h_2(x_2)$, where $h_1(x_1)$ is the invariant density of f_α and $h(x_2)$ is the invariant density of the Gauss transformation,(it is well-known that $h_2(x_2) = \frac{1}{\log 2(1+x_2)}$.) We have already known from Theorem 3.1 that h_1 gives a finite invariant measure and at the indifferent fixed point of f_α; 0, h_1 is unbounded.On the other hand, the Gauss transformation satisfies Renyi's condition,so $h_2(x_2)$ is bounded from above and below. As a result, we can obtain a finite ergodic invariant measure whose density is unbounded on $\{0\} \times [0,1]$.

5 Rohlin's entropy formula

When Renyi's condition is satisfied Rohlin's entropy formula is true ([13]). In our setting, the invariant densty has singular points, however the entropy formula is still true under some conditions

Theorem 5.1 (Rohlin's entropy formula) *Let T satisfy all assumptions of Theorem 2.1.Assume further*

1. $\log|\det DT(\ )| \in \mathcal{L}^1(X, \lambda)$

2. $\sharp\mathcal{D}_1 < \infty$

3. *there is a constant* $K > 0$ *such that*

$$\sup_{X_a \in \mathcal{B}_1} (\sum_{n=0}^{\infty} \sum_{X_{d_1 \ldots d_n} \in \mathcal{D}_n} \inf_{x \in T^n X_{d_1 \ldots d_n} \cap X_a} |\det D\psi_{d_1 \ldots d_n}(x)|) < K$$

4. *there is a number* $l > 0$ *such that* $\sup_x C(n,x) = O(n^l)$.

Then $h(T) = \int_X \log|\det DT(x)| d\mu(x)$.

Remark G

If $\frac{d\mu}{d\lambda}|_{B_1}$ is bounded from above, then the condition 3 is valid. We can verify the condition 3 explicitly for our new class of examples (Examples 1,2,3),so we can apply the theorem for these examples(Cf.[20]).

Lemma 5.1 *Under the assumptions 1,2, and 3, we have*

$$H(Q) \equiv -\sum_{a \in I} \mu(X_a) \log \mu(X_a) < \infty.$$

Lemma 5.2 *Under the assumptions1,2,and 3, we have*

$$\log|\det DT(\ )| \in \mathcal{L}^1(X, \mu).$$

Lemma 5.3 *4 allows us to have*

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{1}{\lambda(X_{a_1 \ldots a_n}(x))} = \lim_{n\to\infty} \frac{1}{n} \log|\det DT^n(x)|.$$

6 Appendix

Let define for $C > 1$,

$$\mathcal{B}_1^{(C)} = \{X_b \in \mathcal{L}^1; X_b \in R(C.T) \text{ and } X_b \text{ satisfies the local Renyi condition}\}$$

$$\mathcal{D}_1 = \mathcal{L}^1 \backslash \mathcal{B}_1^{(C)}.$$

$$\mathcal{B}_2^{(C)} = \{X_{b_1 b_2} \in \mathcal{L}^2; X_{b_1} \in \mathcal{D}_1^{(C)}, X_{b_2} \in \mathcal{B}_1^{(C)}\}$$

$$\mathcal{D}_2^{(C)} = \{X_{d_1 d_2} \in \mathcal{L}^2; X_{d_1}, X_{d_2} \in \mathcal{D}_1^{(C)}\},$$

and inductively define

$$\mathcal{B}_n^{(C)} = \{X_{b_1 \ldots b_n} \in \mathcal{L}^n; X_{b_1 \ldots b_{n-1}} \in \mathcal{D}_{n-1}^{(C)}, X_{b_n} \in \mathcal{B}_1^{(C)}\}$$

$$\mathcal{D}_n^{(C)} = \{X_{d_1...d_n} \in \mathcal{L}^n; X_{d_1...d_i} \in \mathcal{D}_i^{(C)} for i = 1, 2, ..n\}.$$

Notice that

$$\mathcal{B}_n^{(C)} \subset \{X_{b_1...b_n} \in \mathcal{L}^n; X_{b_1...b_{n-1}} \in \mathcal{D}_{n-1}^{(C)}, X_{b_1...b_n} \in R(C.T)\}.$$

Under the above new definition of $\mathcal{D}_n$ and $\mathcal{B}_n$, still we have Theorem 2.1 and Theorem 3.1. Here we show sketch of the proof(see [4]).

Let $T_R : \bigcup_{i=1}^{\infty} B_i \to \bigcup_{i=1}^{\infty} B_i$ be the jump transformation, that is, $T_R x = T^i x$ for $x \in B_i$. The index set of the partition with respect to T_R is $J = \bigcup_{n=1}^{\infty}\{(a_1 \ldots a_n) \in I^n; X_{a_1...a_n} \in \mathcal{B}_n\}$. So each cylinders with respect to T_R have sequences of symbols in $J, (\alpha_1 \ldots \alpha_n), \alpha_i \in J$.

Theorem 6.1 (cf[4]) *Let T satisfy all assumptions of Theorem 2.1 under the new definitions of $\mathcal{D}_n$ and $\mathcal{B}_n$. Then T_R still satisfies all conditions for the existence of a finite ergodic invariant measure with a bounded density, that is the following conditions are valid:*

1. *(generator condition) the partition for T_R with the index set J is a generating partition,*
2. *(transitivity condition) each U_j contains a cylinder $X_{\alpha_1...\alpha_{s_j}}$ such that $T_R^{s_j} X_{\alpha_1...\alpha_{s_j}}$ X,*
3. *Renyi's condition.*

For 1, we have to show that the σ-algebra generated by cylinders with respect to T_R coincides with the one with respect to T. Under the new definition, we have still $D_n = B_{n+1} \cup D_{n+1}$, and $D_n = \bigcup_{k=1}^{\infty} B_{n+k}$ (mod 0), so it is almost the same as in [4] to show that every cylinder with respect to T is a disjoint union of cylinders with respect to T_R. To show this, we used only the "local Renyi condition". In fact, if $X_{a_1...a_n} \in \mathcal{D}_n$, we can show immediately. If $X_{a_1...a_n} \notin \mathcal{D}_n$, then still there is a maximal number $k_0 \in [1, n]$ such that $X_{a_1...a_{k_0}}$ is a cylinder with respect to T_R. If $X_{a_{k_0+1}...a_n} \notin \mathcal{D}_{n-k_0}$, then there is a l with $0 < l < n$ such that $X_{a_{k_0+1}...a_l} \in \mathcal{B}_{l-(k_0+1)+1}$ and the local Renyi condition allows us to have $X_{a-1...a_l} \in \mathcal{L}_{T_R}$, i.e.,$X_{a_1...a_l}$ is a cylinder with respect to T_R. This contradicts to the maximality of k_0. So $X_{a_{k_0+1}...a_n} \in \mathcal{D}_{n-k_0}$. Thus $X_{a_{k_0+1}...a_n} = \bigcup_{k=1}^{\infty} B_{n-k_0+k} \cap X_{a_{k_0+1}...a_n}$ and hence $X_{a_1...a_n} = X_{a_1...a_{k_0}} \cap T^{-k_0}(\bigcup_{k=1}^{\infty} B_{n-k_0+k} \cap X_{a_{k_0+1}...a_n})$. For 2 and 3, it is immediate to show the ergodicity of T with respect to λ, T-invariance of μ and finiteness of μ as in [4].

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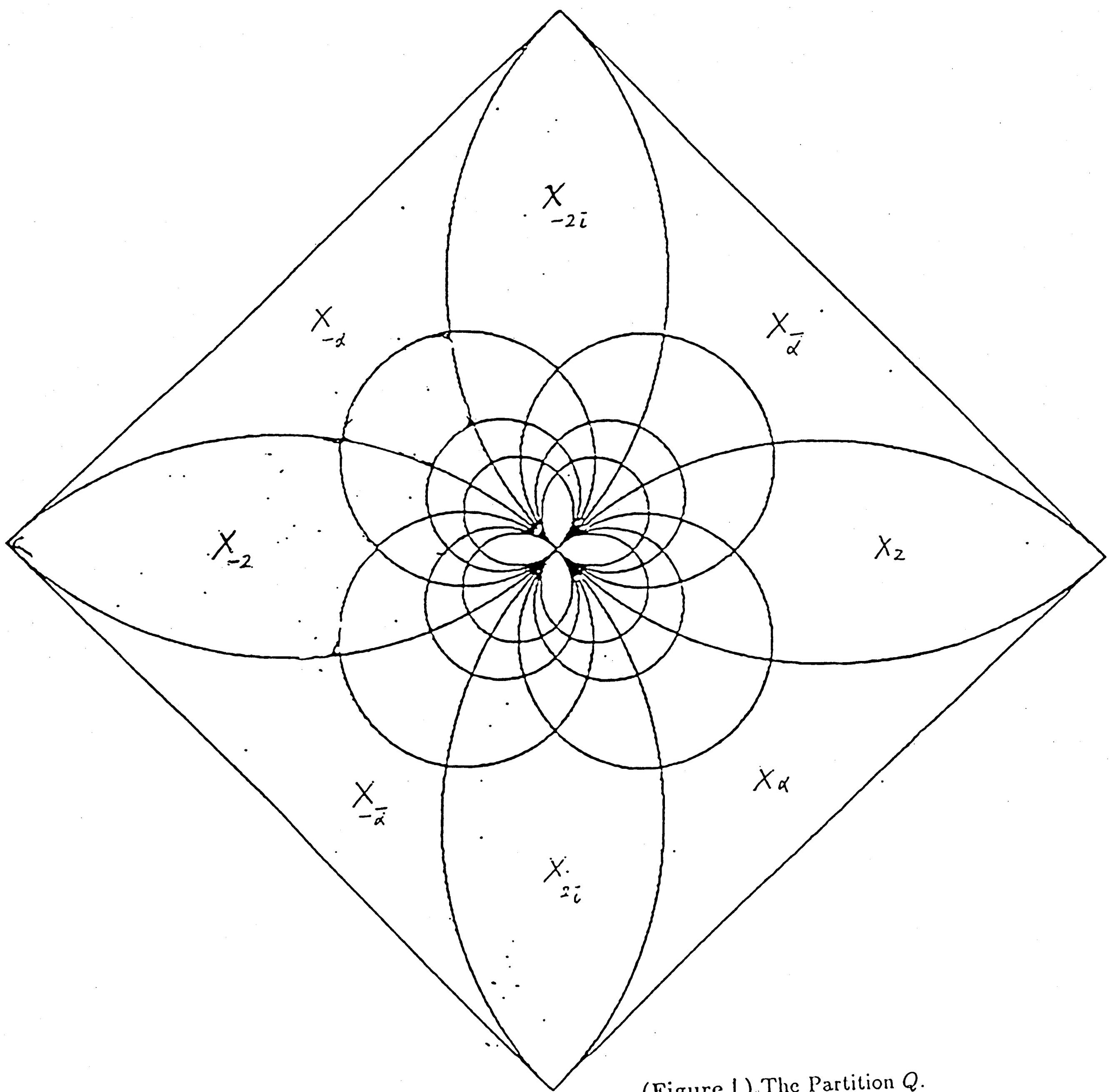
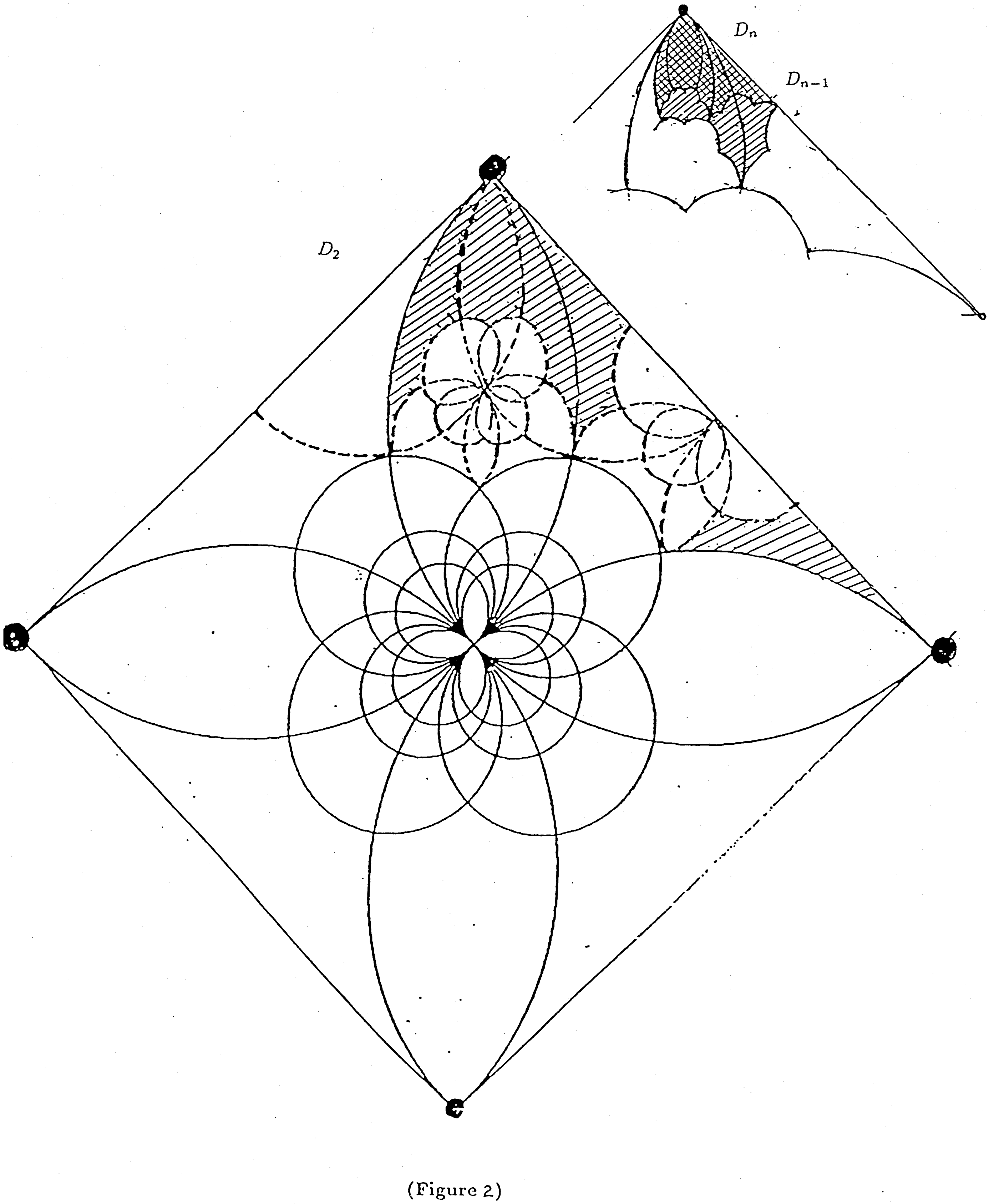
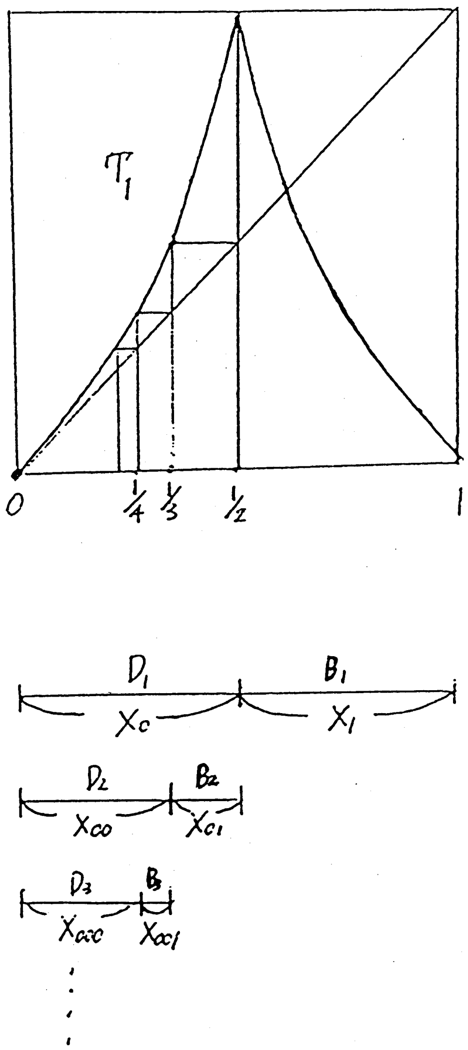


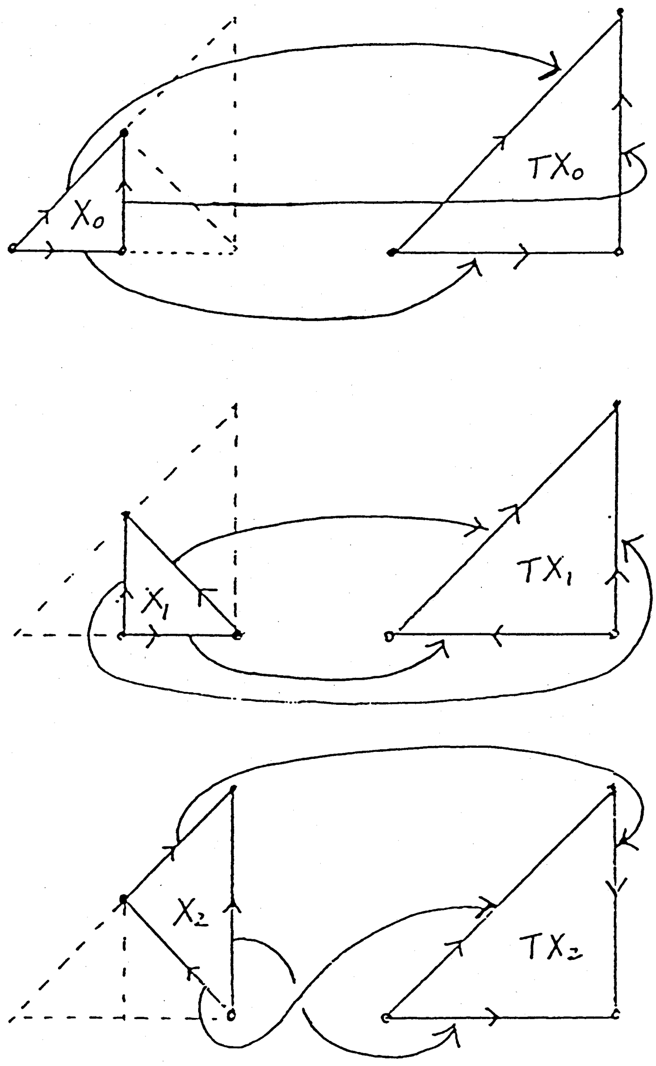
(Figure 1) The Partition Q.
$\mathcal{D}_1 = \{X_\alpha, X_{-\alpha}, X_{\bar{\alpha}}, X_{-\bar{\alpha}}, X_{2i}, X_{-2i}, X_2, X_{-2}\}$



(Figure 2)



(Figure 3)



(Figure 4)