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A note on the heteroclinic Ω-explosions

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Abstract

Palis and Takens have obtained many results about the chaotic dynamics with homoclinic tangency. In 1987, they classified one-parameter families of diffeomorphisms $\{\varphi_\mu : \mu \in \mathbf{R}\}$ on surfaces which display a *homoclinic* Ω*-explosion* at $\mu = 0$ and derived several theorems about it[1]. We also consider one-parameter families of diffeomorphisms which have *heteroclinic tangencies* for $\mu = 0$, are *persistently hyperbolic* for $\mu < 0$ and have *transversal heteroclinic orbits* for positive μ. Using their method, we classified the bifurcations with *heteroclinic* Ω*-explosion.* The theory developed by Palis and Takens regarding the measure of $\{\mu\}$ for which φ_μ is non-hyperbolic can also be applied to our classification.

1 Introduction

There are some excellent results about the bifurcation due to *homoclinic* and *heteroclinic tangencies* such as Smale's homoclinic theorem[2], Robinson's creation of infinitely many sinks in a homoclinic tangency [5], Newhouse theory [3],[4] and the results by Mora, Viana, Benedics and Carleson[6],[7]. All of them studied one-parameter families $\{\varphi_\mu : \mu \in \mathbf{R}\}$ of diffeomorphisms with homoclinic tangency at $\mu = 0$ and in the interval $(-\delta, \delta)$ of a parameter range. It is well-known that those families create infinitely many bifurcations no matter how small this interval may be. Particularly, Palis and Takens successfully in studied the *nonhyperbolic* dynamics (i.e. *bifurcating diffeomorphism*) by investigating those families, and they suggested the possibility of extending their results to *heteroclinic tangencies*[8](Fig.1).

To recall a *heteroclinic point*, let M be a 2-manifold, φ:M $\to$ M a C^2 diffeomorphism and p_1, p_2 ($p_1 \neq p_2$) fixed points for φ. We will be particularly interested in the case where each p_i is a (*hyperbolic*) *saddle fixed point* i.e. $\varphi(p_i)=p_i$ and $d\varphi(p_i)$ has two real eigenvalues λ_i and σ_i with $0 <|\lambda_i|< 1 <|\sigma_i|$. For simplicity we assume that these eigenvalues are positive, so $0 <\lambda_i< 1 <\sigma_i$. We call λ_i (resp. σ_i) the contracting (resp. expanding) eigenvalue. In this situation from the *invariant manifold's theorem*[9] we know that:

- $W^s(p_i)$ and $W^u(p_i)$; the stable and unstable separatrices of p_i, are C^2.

- there are C^1 linearizing coordinates in a neighborhood of p_i, i.e. C^1 coordinates x_1, x_2 such that $p_i = (0,0)$ and such that $\varphi(x_1,x_2) = (\lambda x_1, \sigma x_2)$.

We consider the situation(shown in Fig.1) where the stable manifold $W^s(p_1)$ of p_1 intersects the unstable manifold $W^u(p_2)$ of p_2 transversally at some point x. We call x a *transversal heteroclinic point.* If the separatrices are tangential at x, then we call it a *heteroclinic point of tangency.*

Let us recall a few concepts. The point x is *nonwandering* for φ if for any given neighborhood U of x and any integer $n_0 > 0$, there is an integer n such that $|n| > n_0$ and $\varphi^n(U) \cap U \neq \emptyset$. The union of the nonwandering points is called the *nonwandering set,* which is denoted by $\Omega(\varphi)$. A closed subset $\Lambda \subset \Omega(\varphi)$ is called a *basic set* if it is hyperbolic (see [9] for its definition) and transitive (i.e. it has a dense orbit), and its subset of periodic orbits is dense. *Palis and Takens* defined φ to be *hyperbolic* if $\Omega(\varphi)$ is a hyperbolic set, and *persistently hyperbolic* if every $\tilde{\varphi}$ which is C^2 near φ is also hyperbolic. They showed in [8] that if φ is persistently hyperbolic, then it cannot have cycles and hence is Ω-stable[10].

2 heteroclinic Ω-explosion and its criterions

We say that a one parameter family φ_μ: M $\rightarrow$ M of C^2 diffeomorphisms on a closed surface M has a *heteroclinic Ω-explosion* at $\mu = 0$ if:

(1) For $\mu < 0$, φ_μ is persistently hyperbolic and has two saddle fixed points p_{1_μ}, p_{2_μ} ;

(2) For $\mu = 0$, the nonwandering set $\Omega(\varphi_0)$ consists of $\tilde{\Omega}(\varphi_0) = \lim_{\mu \nearrow 0} \Omega(\varphi_\mu)$ together with heteroclinic orbits of tangency $\mathcal{O}$ associated with a fixed saddle point p_{i_0}, so that $\Omega(\varphi_0) = \tilde{\Omega}(\varphi_0) \cup \mathcal{O}$ and the product of the eigenvalues of $d\varphi_0(p_i)$ is different from 1;

(3) The separatrices have quadratic tangency along $\mathcal{O}$ unfolding generically, i.e., by changing the coordinates, $W^u(p_{i_\mu})$ is given by $x_2 = 0$ and $W^s(p_{j_\mu})$ by $x_2 = ax_1^2 + b\mu$ $(a, b \neq 0)$ in the neighborhood of p_{i_μ}[11].

We shall classify these heteroclinic Ω-explosions according to the following criterions concerning φ_0 (hereafter denoted by φ).

a) *The signs of the eigenvalues* of $d\varphi(p_i)$

The signs of eigenvalues of $d\varphi(p_i)$, λ_i and σ_i, are simply denoted by $(\,,\,)_i$. We consider, in this paper, eight possible combinations, $(+,+)_1$-$(+,+)_2$, $(+,+)_1$-$(-,-)_2$, $(-,-)_1$-$(+,+)_2$, $(-,-)_1$-$(-,-)_2$, $(+,-)_1$-$(+,-)_2$, $(+,-)_1$-$(-,+)_2$, $(-,$

$+)_1$-$(+, -)_2$ and $(-, +)_1$-$(-, +)_2$, which can be summarized as $\lambda_1\sigma_1\lambda_2\sigma_2 > 0$. We say that φ is *orientation-preserving*, resp. *orientation-reversing* if $det(d\varphi) > 0$, resp. $det(d\varphi) < 0$.

b) *The sides of tangencies*

This concerns the positions of $W^s(p_i)$ in the neighborhood of p_j. For example if the configuration of $W^u(p_2)$ and $W^s(p_2)$ in the neighborhood of p_1 is as shown in Fig.2, then we have four cases of the configuration of $W^u(p_1)$ and $W^s(p_1)$ in the neighborhood of p_2 as in Fig.3.

We define configurations (B)$-$(D) by replacing p_2 in (b)$-$(d) by p_1.

c) *The mode of connection*

It concerns the distinction that still can be made if the sides of tangencies are prescribed; for example if they are as in (A)-(a) of the above cases, then the global unstable separatrix of p_1 can be connected with the relevant part of $W^s_{loc}(p_2)$ in the two different ways shown in Fig.4:

In all cases which may exist according to the above criterions, we want to determine whether a heteroclinic Ω-explosion can occur, and if so, to determine

-if the ambient manifold M can be orientable or not;

-if the saddle point p_i can be part of a non-trivial basic set or not.

We say that the *expanding*, resp. *contracting, eigenvalue* is *dominating* if the product of the eigenvalues is bigger than one, resp. smaller than one.

We consider, as an example, a one parameter family φ_μ with heteroclinic Ω-explosion, such that the signs of eigenvalues of $d\varphi(p_i)$ $(i = 1, 2)$ are $+\ +$ and such that the sides of tangencies and the mode of connection are as in Fig.4.

In order to prove this we take as before $q \in W^s(p_2)$ and $r \in W^u(p_1)$ both in $\mathcal{O}$ of φ. We can also take a diffeomorphism φ^n from the neighborhood of r to the neighborhood of q for some integer $n > 0$. We see that φ^n is orientation-preserving in the first case in Fig.4 (i.e. the orientation of $\mathcal{B}'$ which is an oriented box near q' is the same as that of $\mathcal{B}$ which is an oriented box near r). From the assumption, φ is an orientation-preserving map. From these it follows that the first case in Fig.4 can occur on any closed surface. However the second case in Fig.4 can not occur if the ambient surface is orientable because φ^n is orientation-reversing in this picture.

We know that a transversal homoclinic orbit and heteroclinic orbit are contained in a horseshoe (that is often called a maximal invariant set or 'non-trivial basic set')[8],[10]. If a saddle fixed point p_i is in a non-trivial basic set Λ, then we express the (local) position of Λ with respect to p_i by the quadrant which is separated by $W^u_{loc}(p_i)$ and $W^s_{loc}(p_i)$ and contains points of $\Lambda \cap U$ for any small neighborhood U of p_i. Furthermore, for each point $x \in \Lambda \cap U$ we can construct a pair of leaves, which are called stable and unstable leaves. For example, if the side of tangencies is as in Fig.2-(A) then the only quadrant where there might be a basic set is the upper right quadrant.

If at least one of the eigenvalues of $d\varphi(p_i)$ is negative, then p_i can not be in a non-trivial basic set, because such a basic set would occupy at least two quadrants.

Neither heteroclinic Ω-explosion nor homoclinic Ω-explosion can occur when *'premature tangencies'*, *'premature creation of horseshoes'* or *'infinitely many circles'*[1] appear.

3 General classification and possible cases

In this section, we systematically go through all the cases as classified by : signs of eigenvalues, side of tangencies, mode of connection, and where necessary, eigenvalue condition. Furthermore we show examples of φ with heteroclinic Ω-explosion for all the possible cases. Since each case realized on an orientable surface can also be realized on a non-orientable surface, we give only those examples on a non-orientable surface that cannot be realized on an orientable surface.

(1) The case $(+,+)_1$-$(+,+)_2$

The sides of tangencies are as in Fig.2. Before classifying them according to the different connections, we observe that (B) (and (b)) cannot occur due to *'premature tangencies'* and that (A) and (D) ((a) and (d)) can be interchanged by replacing φ by φ^{-1}. So we only have to consider the cases (A)-(a), (A)-(c), (A)-(d), and (C)-(c). We examine the connection of the four cases.

We first consider the two modes of connections shown in Fig.5 for the case (A)-(a).

Case (A)-(a)-1 can only occur if the expanding eigenvalue is dominating so that *'premature creation of horseshoes'* don't occur. It can occur when p_i is part of a non-trivial basic set or when p_i is an isolated point in $\tilde{\Omega}(\varphi)$. M can be orientable or non-orientable in this case.

Case (A)-(a)-2 can occur if M is a non-orientable surface. Here p_i is part of a non-trivial basic set to avoid *'infinitely many circles'*. This non-trivial basic set must be in the upper right quadrant. But this quadrant is not allowed. So this case cannot occur.

We next consider the cases shown in Fig.6 for the case (A)-(c).

Case (A)-(c)-1 can occur if M is non-orientable and the expanding eigenvalue is dominating and p_1 and p_2 are not part of non-trivial basic set.

Case (A)-(c)-2 can not be observed for the same reason as in the *case* (A)-(a)-2.

We consider the two cases shown in Fig.7 for the case (A)-(d) too.

Case (A)-(d)-1 can only occur if the expanding eigenvalue is dominating and p_i is isolated in $\tilde{\Omega}(\varphi)$. Therefore it occurs in a non-orientable surface.

Case (A)-(d)-2 can not also occur because of the same reason as in the *case* (A)-(a)-2.

Now we consider the two cases shown in Fig.8 for the case (C)-(c).

Case (C)-(c)-1 can only occur if expanding eigenvalue is dominating and p_i is isolated in $\tilde{\Omega}(\varphi)$. Therefore it occurs in a non-orientable surface.

Case (C)-(c)-2 Again because of the *'infinitely many circles'* it can only occur when p_i is contained in a non-trivial basic set, which must be in the upper left quadrant. The surface M can be orientable or non-orientable in this case.

Here we show examples of all the cases that were not excluded above. Some examples in [8] and [12] were helpful in constructing the following.

Case (A)-(a)-1: p_i is not part of non-trivial basic set.

For example, we start with a diffeomorphism Φ as indicated in Fig.9 with $\Omega(\Phi)$ consisting of two sources and three sinks besides the saddles p_1, p_2.

Let l (resp. $\tilde{l}$) be a curve segment from $W^u(p_1)$ to $W^s(p_2)$ (resp. from $W^s(p_1)$ to $W^u(p_2)$), and $\mathcal{U}$ (resp. $\tilde{\mathcal{U}}$) be a small neighborhood of l (resp. $\tilde{l}$), and let σ_μ (resp. $\tilde{\sigma_\mu}$) be a 1-parameter family of diffeomorphisms with support in $\mathcal{U}$ (resp. $\tilde{\mathcal{U}}$) (so that for $\mu \leq -1$, σ_μ (resp. $\tilde{\sigma_\mu}$) is the identity and for $\mu > -1$, it pushes $W^u(p_1)$ (resp. $W^s(p_i)$) down so that for $\mu = 0$ we have a generically unfolding tangency of $\sigma_\mu(W^u(p_1))$ and $W^s(p_2)$ (resp. $\tilde{\sigma_\mu}(W^s(p_1))$ and $W^u(p_2)$)). Here we can obtain φ_μ defined by $\varphi_\mu = \tilde{\sigma_\mu} \circ \sigma_\mu \circ \Phi$ as in Fig. 10.

Case (A)-(a)-1: p_i is part of non-trivial basic set. Non-trivial basic sets are obtained from the horseshoes by modification of their stable manifolds and unstables manifolds. We can obtain φ_μ as in Fig. 11.

Case (C)-(c)-1: This example is given in the projective plane $P^2 = D^2 \cup \mathcal{M}$ where the boundary of D^2 is identified with the boundary of the Möbius band $\mathcal{M}$ as in Fig. 12.

Case (C)-(c)-2; We can realize this case by the use of a 'type-3 horseshoe' [Palis & Takens, 1987] in the projective plane as in Fig. 13.

(2) $(+,+)_1$-$(-,-)_2$ ($(-,-)_1$-$(+,+)_2$ is obtained replacing φ with φ^{-1}) and $(-,-)_1$-$(- -)_2$

When the eigenvalues of $d\varphi(p_i)$ are negative, we can take two possibilities of the sides of tangencies in the neighborhood p_i, (we may assume that $i = 2$) as in Fig. 14.

The case (b') cannot occur because of *'premature tangencies'*. For $(+,+)_1$-$(+,-)_2$, we only have the cases (A)-(a') and (C)-(a').

We consider the two modes of connection shown in Fig.15 for the case (A)-(a'):

Case (A)-(a')-1 can only be observed if the expanding eigenvalue is dominating, because otherewise *'premature creation of horseshoes'* would occur. Therefore it can occur when p_2 is an isolated point in $\tilde{\Omega}(\varphi)$ whether p_1 is part of a non-trivial basic set or not. M can be orientable or non-orientable in this case.

Case (A)-(a')-2 Here p_2 is part of a non-trivial basic set to avoid *'infinitely many circles'*. This must be in the upper right quadrant. However this quadrant is not allowed. So this case cannot occur.

The above reasoing applied to the cases for (C)-(a') implies that the only possible case is the one shown in Fig. 16.

On any surface M *case* (C)-(a') can occur if the expanding eigenvalue is dominating and p_1 and p_2 are not part of a non-trivial basic set.

When the eigenvalues of both $d\varphi(p_1)$ and $d\varphi(p_2)$ are negative, heteroclinic Ω-explosion can not occur because, as in the case of homoclinic Ω-explosion[1], premature creation of horseshoes occurs whether the expanding eigenvalue is dominating or not.

We show an example of the *Case* (A)-(a')-1 in Fig. 17.

We can also obtain an example of the *Case* (C)-(a') by slightly changing Fig. 17 so that p_1 is not part of a non-trivial basic set.

(3) The case $(+,-)_1$-$(+,-)_2$

In this case there are two possible combinations of the sides of tangencies as in Fig. 19. The case (b") cannot occur because of *'premature tangencies'*. So we may only consider the connection of (A")-(a") as in Fig. 20.

Case (A")-(a")-1 can only occur if expanding eigenvalue is dominating and p_i is isolated in $\tilde{\Omega}(\varphi)$. It can occur in any surface.

Case (A")-(a")-2 can occur that M is non-orientable and if the expanding eigenvalue is dominating. Neither p_1 or p_2 can be part of non-trivial basic set.

The other cases can not generate heteroclinic Ω-explosion.

We show some examples for the above two cases in Fig.21-22.

The oter three cases $(+,-)_1$-$(-,+)_2$, $(-,+)_1$-$(+,-)_2$ and $(-,+)_1$-$(-,+)_2$ can be treated essentially in the same way as **(3)**. This completes our classification of the heteroclinic Ω-explosion.

One of the theorems in [1] says that there is a set of $\mu \geq 0$ for which the nonwandering set of φ_μ is not hyperbolic but that it is very *'small'* in following sense.

Let φ_μ be a family with homoclinic Ω-explosion at $\mu = 0$. If $d^s(\Lambda)+d^u(\Lambda)< 1$, where Λ is the basic set of φ_0 associated with the homoclinic tangency, then

$$\lim_{\delta \to 0} \frac{m(\mu \in [0, \delta] : \varphi_\mu \text{ is not persistently hyperbolic})}{\delta} = 0$$

where m denotes Lebesgue measure and d^s (resp. d^u) is the stable (resp. unstable) limit capacity. They suggest that the same theory will hold for heteroclinic Ω-explosions. We are now trying to it and apply the theorem to our classification of heteroclinic Ω-explosions.

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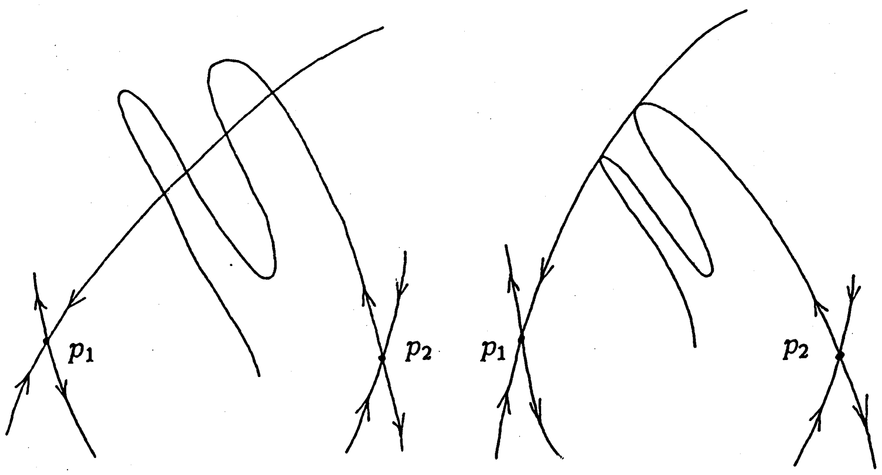


Figure 1: Transversal heteroclinic points & heteroclinic points of tangency

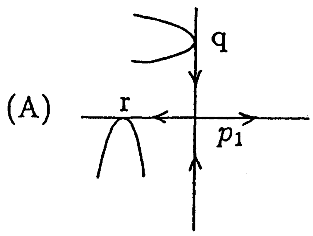


Figure 2: The configuration of tangencies in the neighborhood of p_1

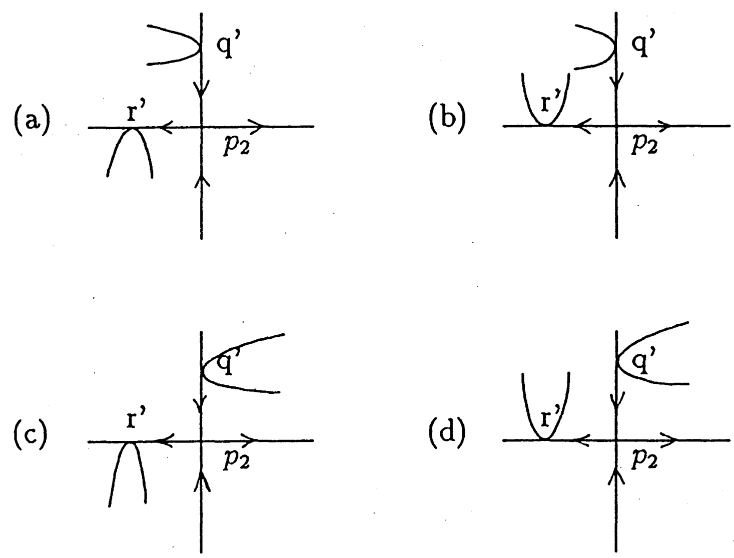


Figure 3: The configuration of tangencies in the neighborhood of p_2

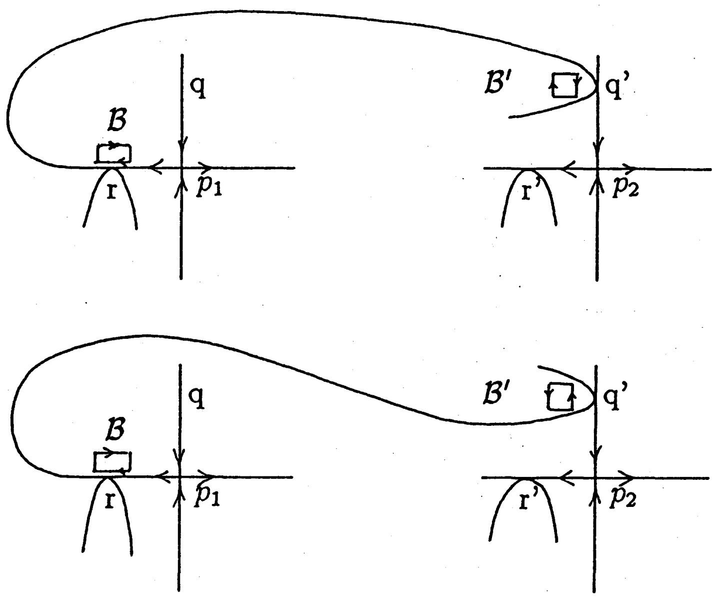


Figure 4: The mode of connection

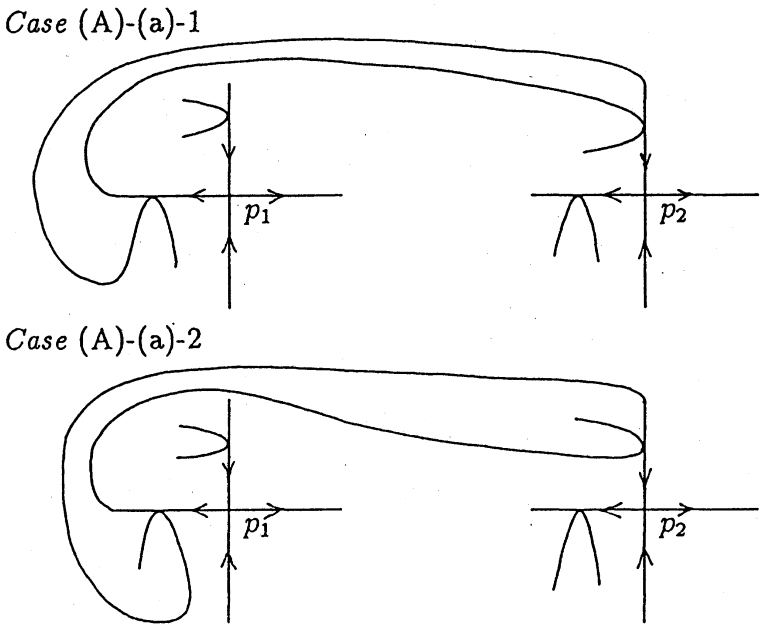


Figure 5: The *case* (A)-(a)-1 and the *case* (A)-(a)-2

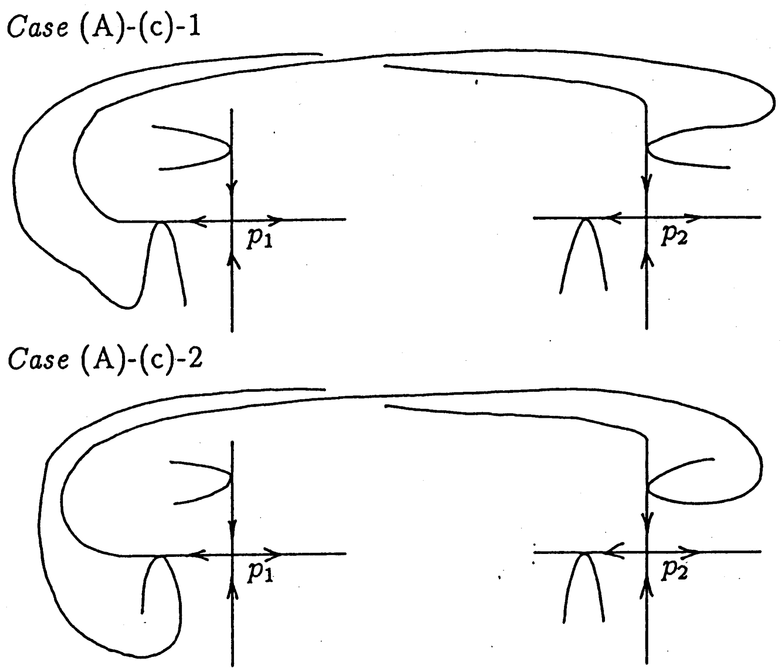


Figure 6: The *case* (A)-(c)-1 and the *case* (A)-(c)-2

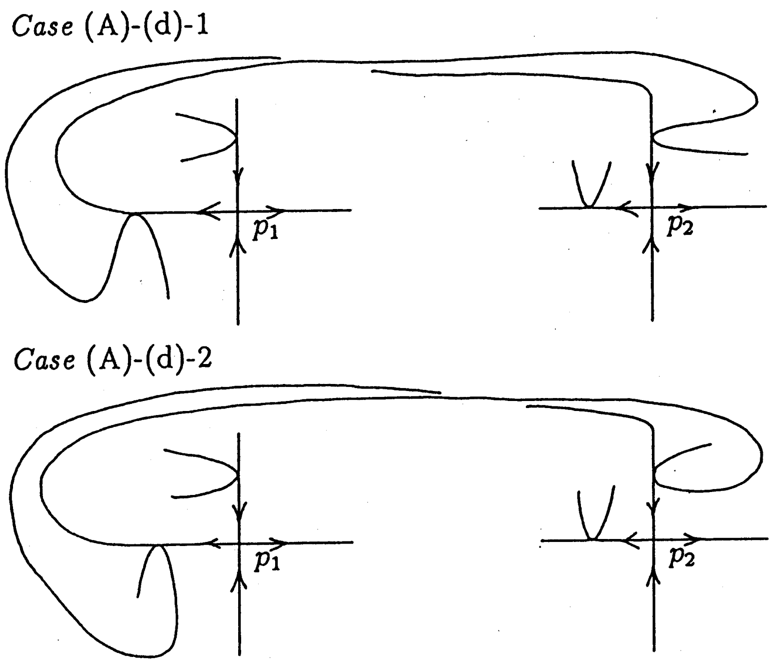


Figure 7: The *case* (A)-(d)-1 and the *case* (A)-(d)-2

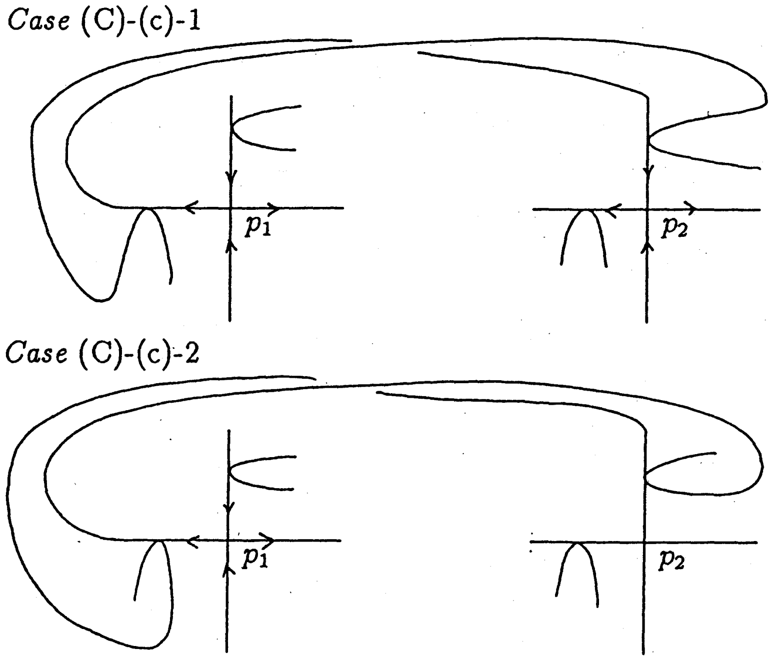


Figure 8: The *case* (C)-(c)-1 and the *case* (C)-(c)-2

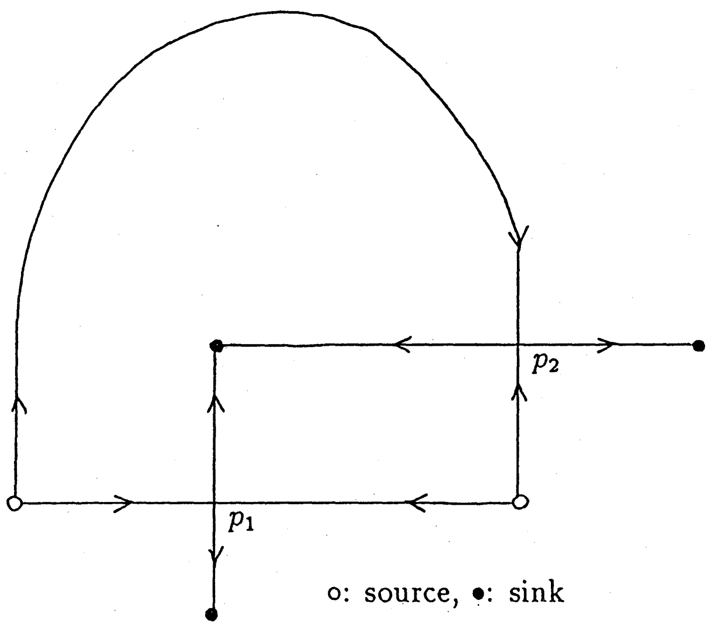


Figure 9: Dynamics of Φ

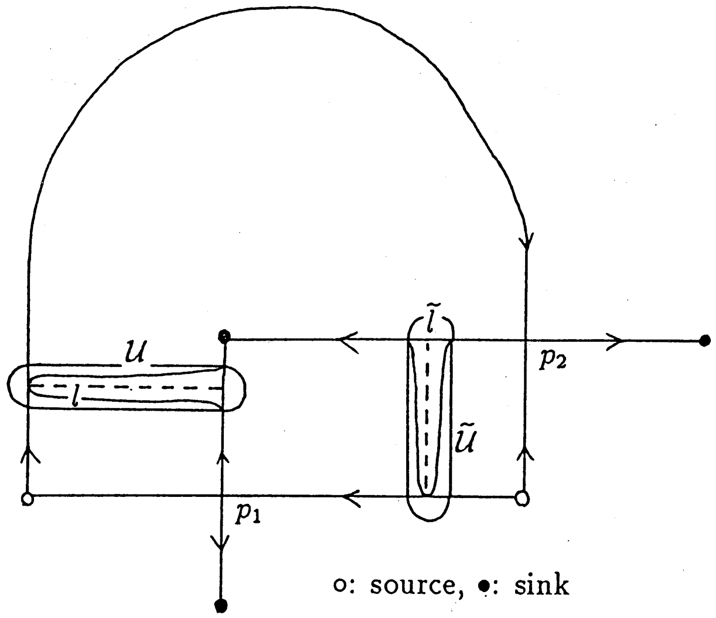


Figure 10:

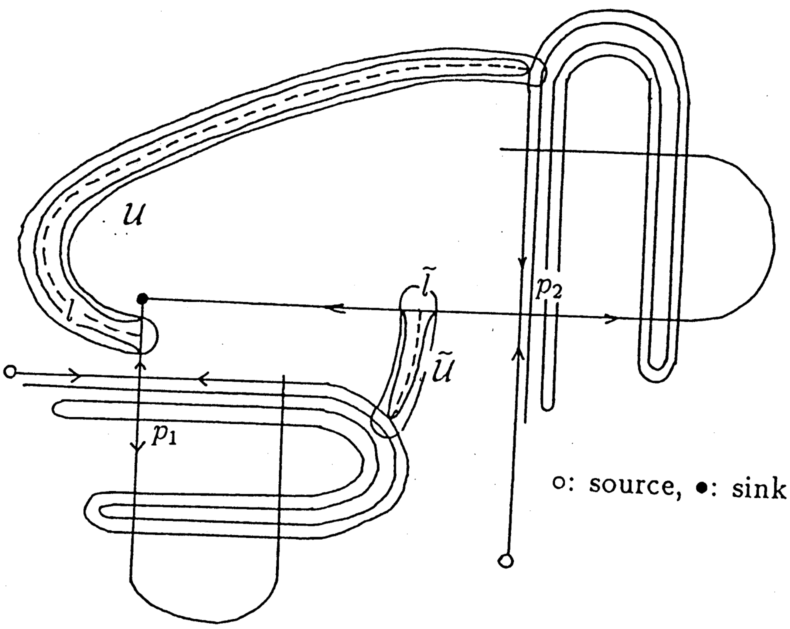


Figure 11:

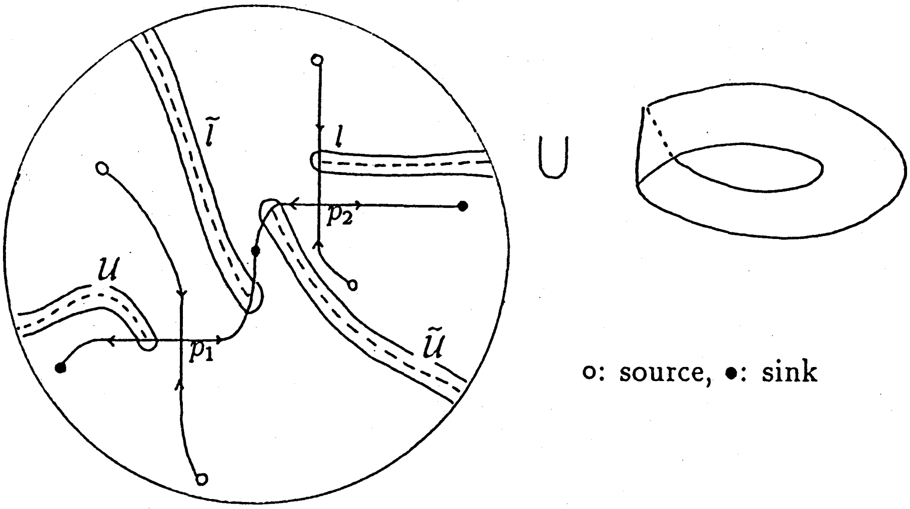


Figure 12:

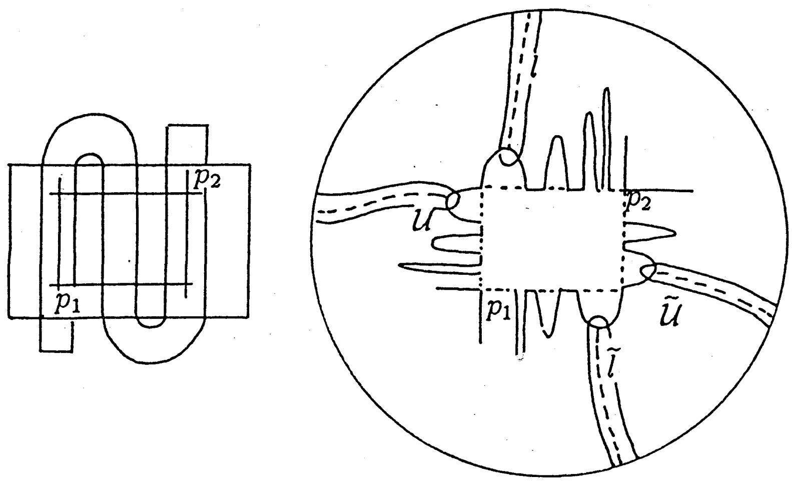


Figure 13:

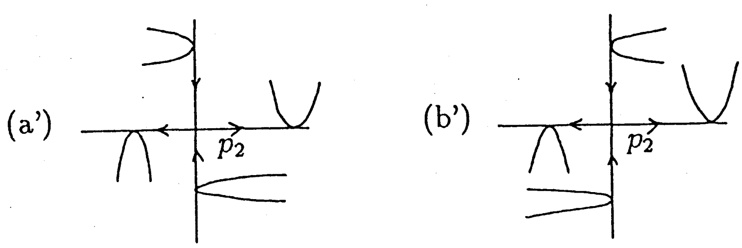
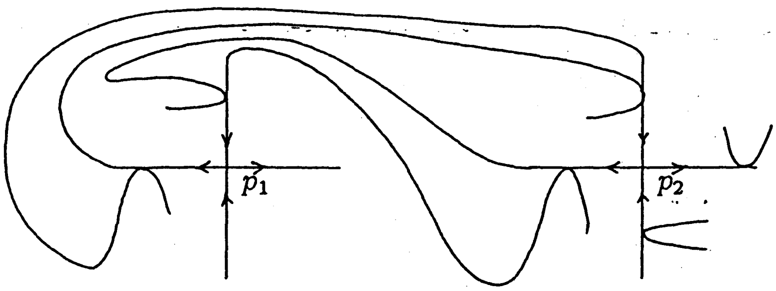


Figure 14: The configuration of tangencies in the neighborhood of p_2

Case (A)-(a')-1



Case (A)-(a')-2

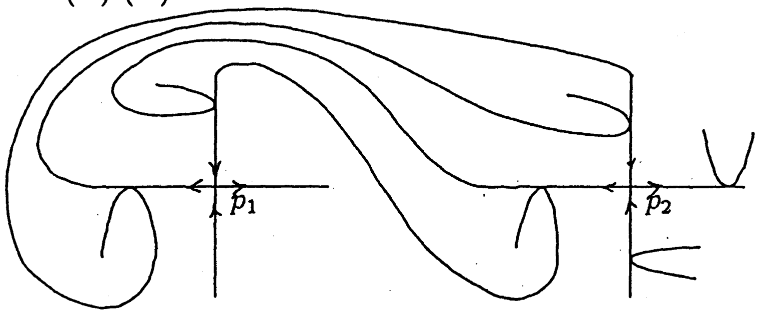


Figure 15: The *case* (A)-(a')-1 and the *case* (A)-(a')-2

Case (C)-(a')

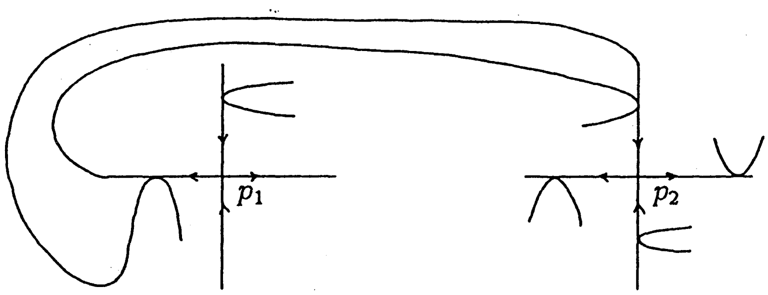


Figure 16: The *case* (C)-(a')

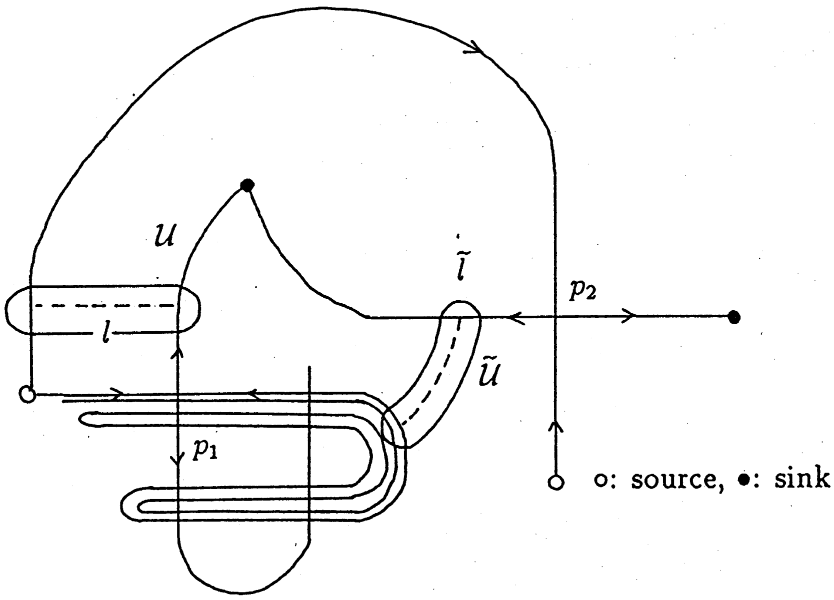


Figure 17: The *case* (A)-(a')-1

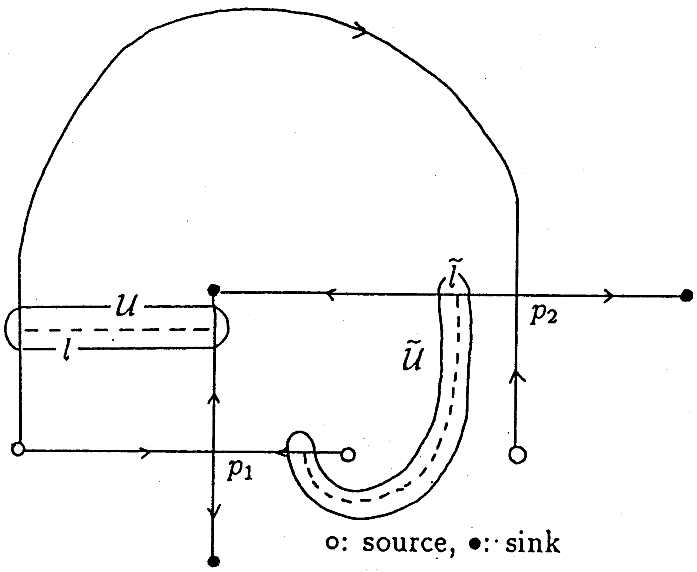


Figure 18: The *case* (C)-(a')

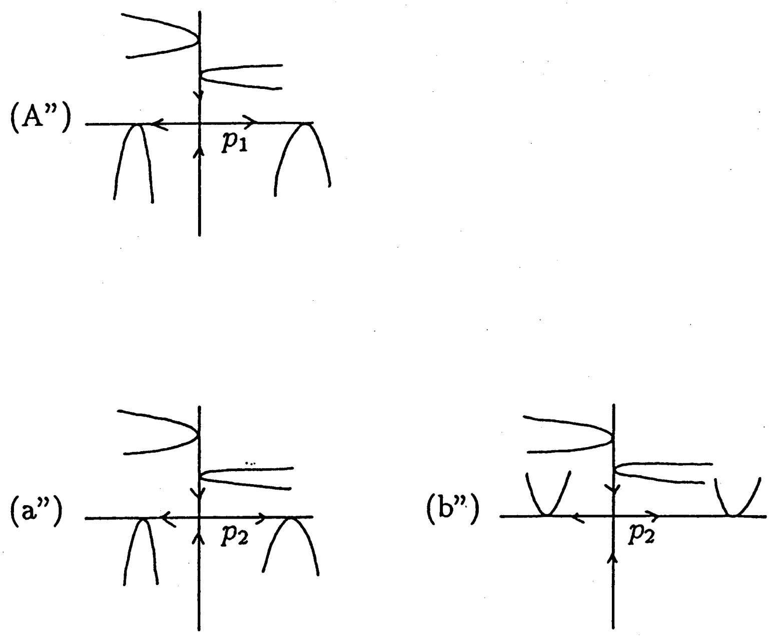


Figure 19: The configurations of tangencies in the neighborhood of p_i

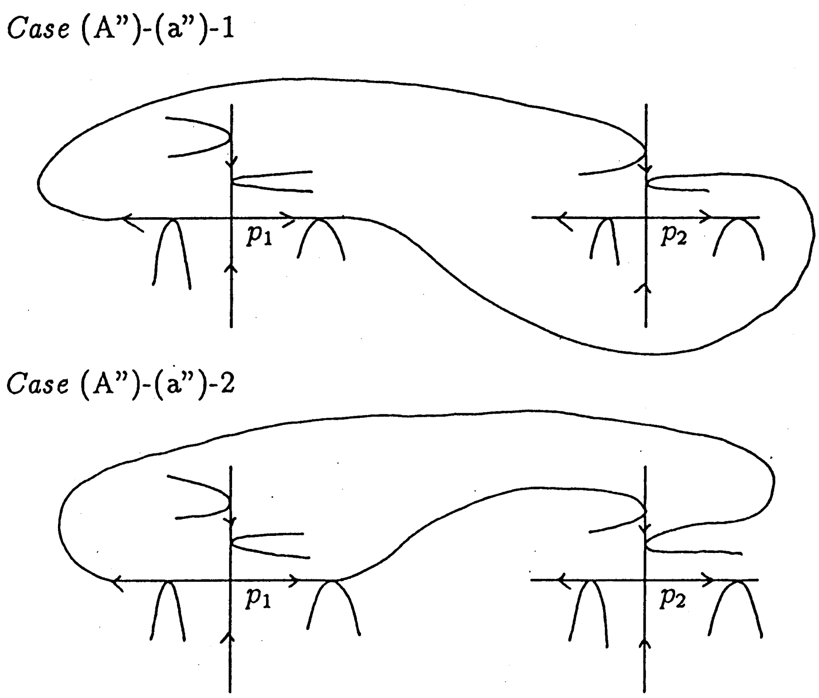


Figure 20: The *case* (A")-(a")-1 and the *case* (A")-(a")-2

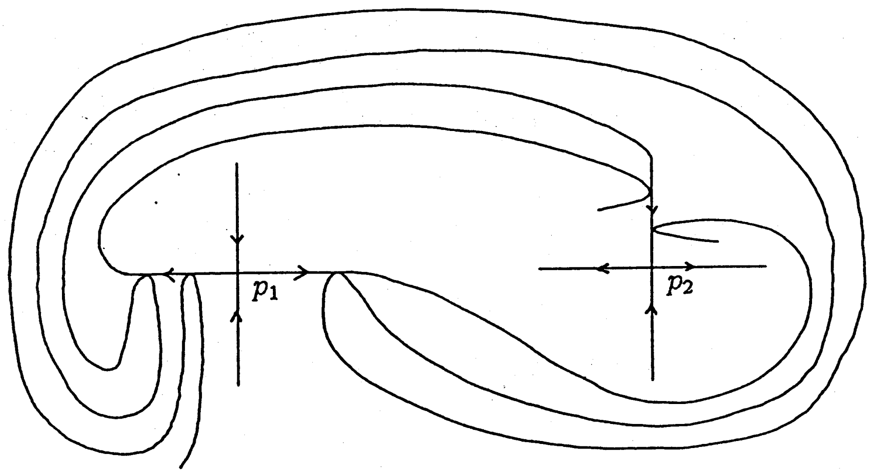


Figure 21: The *case* (A")-(a")-1

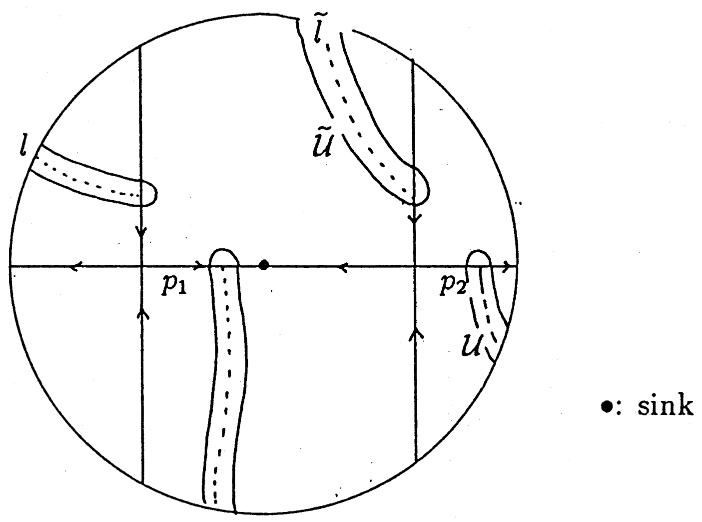


Figure 22: The *case* (A")-(a")-2