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The Pierce-Birkhoff Conjecture*

James J. Madden
Deptartment of Mathematics
Louisiana State University
Baton Rouge LA 70803-4918 USA

1. Statement of the Pierce-Birkhoff Conjecture (PBC).
2. Real spectrum; local nature of PBC and relation to singularities.
3. Infinitely near points and the two-dimensional case.
4. Open problems related to the three dimensional case.

Section 1.

Definition. Suppose $X \subseteq \mathbf{R}^n$. A function $\phi : X \to \mathbf{R}$ is called piecewise polynomial—or "PWP" for short—if there is a finite collection $\{P_k\}$ of closed semialgebraic subsets of $\mathbf{R}^n$ and a collection of polynomial functions $\{f_k : \mathbf{R}^n \to \mathbf{R}\}$ such that $X \subseteq \bigcup_k P_k$ and $\phi = f_k$ on $P_k \cap X$. The pair $(\{P_k\}, \{f_k\})$ is called a presentation of ϕ.

Note that $f_k = f_{k'}$ on $P_k \cap P_{k'} \cap X$. ϕ is continuous on X, since the sets $P_k \cap X$ are closed in X. If X is not closed, ϕ may fail to have a continuous extension to $\mathbf{R}^n$.

Definition. Given functions $g_j : X \to \mathbf{R}$, the functions $\bigvee_{j=1}^s g_j$ and $\bigwedge_{j=1}^s g_j$ are defined as follows:

$$\left(\bigvee_{j=1}^{s} g_j\right)(x) = \max\{g_1(x), \ldots, g_s(x)\}$$

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$$\left(\bigwedge_{j=1}^{s} g_j\right)(x) = \min\{g_1(x), \ldots, g_s(x)\}.$$

If $\{h_{ij} : \mathbf{R}^n \to \mathbf{R}\}$ is a finite collection of polynomial functions, then $\bigvee_i \bigwedge_j h_{ij}$ is piecewise polynomial on $\mathbf{R}^n$. Any function of this form (or the restriction to $X \subseteq \mathbf{R}^n$ of any such function) is said to be "sup-inf-polynomial definable (on X)," or "SIPD" for short. It is easy to show that any SIPD function on X is PWP. The converse is not the case in general.

Main definition. We say "the condition of Pierce-Birkhoff holds for X" if every PWP function on X is SIPD. The Pierce-Birkhoff Conjecture is that for all n, the condition of Pierce-Birkhoff holds for $\mathbf{R}^n$, see [G. Birkhoff, R. S. Pierce, Lattice ordered rings, *Anais Acad. Bras. Ci.* **28** (1956), 41–69; *MR* **18** (1957), 191].

The known results are as follows: It is easy to show that the condition of Pierce-Birkhoff holds for $\mathbf{R}$. For $\mathbf{R}^2$, on the other hand, this is a difficult theorem due to L. Mahé, see [L. Mahé, On the Pierce-Birkhoff conjecture, *Rocky Mountain J. Math.* **14**(4), (Fall 1984), 983–5; *Zbl.* **578** (1986), 41008; *MR* **86d**:14020]. For $\mathbf{R}^n$, $n \geq 3$, the Pierce-Birkhoff Conjecture is completely open. Madden and Robson have shown that the condition of Pierce-Birkhoff holds for any smooth compact semialgebraic surface, and Madden has proved an analogue of this result which is valid with an arbitrary real closed field in place of $\mathbf{R}$. (These results have not yet been submitted.) Marshall (to appear in *Canadian J. Math.*) has shown, among other things, that if X is a semialgebraic curve then the condition of Pierce-Birkhoff holds for X if and only if at each singularity of X distinct half-branches have distinct half-tangents. Other variants of the conjecture have been considered; an abstract framework which is useful for formulating these is provided in [J. Madden, Pierce-Birkhoff rings, *Archiv der Math.* **53** (1989), 565–70].

Section 2.

Suppose $\phi : \mathbf{R}^n \to \mathbf{R}$ is PWP and $(\{P_k\}, \{f_k\})_{k=1}^{m}$ is a presentation of ϕ. We say that $(\{P_k\}, \{f_k\})_{k=1}^{m}$ is a good presentation if for all $i, j \in$

$\{1,2,\ldots,k\}$ there is a polynomial function h_{ij} satisfying

$$h_{ij} \geq \phi \quad \text{on} \quad P_i$$

$$h_{ij} \leq \phi \quad \text{on} \quad P_j.$$

If $(\{P_k\},\{f_k\})_{k=1}^m$ is good, then for any i

$$\bigwedge_j h_{ij} = \phi \quad \text{on} \quad P_i$$

and

$$\bigwedge_j h_{ij} \leq \phi \quad \text{on} \quad P_j,$$

and hence

$$\bigvee_i \bigwedge_j h_{ij} = \phi \quad \text{on} \quad \mathbf{R}^n.$$

This shows that a PWP function is SIPD provided that it has a good presentation. Since the sets P_k may be chosen as small as one likes, it is clear that any obstruction to finding a good presentation must be local, in some sense. In fact, more can be said. Let $\operatorname{Trans}\phi := \{x \in \mathbf{R}^n \mid \phi$ is not polynomial on any neighborhood of $x\}$. (Note that $\operatorname{Trans}\phi \subseteq \bigcup_k \partial P_k$ for any presentation $(\{P_k\},\{f_k\})$ of ϕ.) It is possible to show that if $X \subseteq \mathbf{R}^n$ is any compact set containing no singularity of $\operatorname{Trans}\phi$, then $\phi\,|_X$ is SIPD. Thus, any obstructions to finding a good presentation lies in codimension 2.

My treatment of the real spectrum (below) is intended to be accessable to people who have not thought much about this before. It is not as general as possible, but is adequate for present purposes. For more information, see [J.Bochnak, M.Coste and M.F.Roy, Géométrie algébrique réelle, Ergebnisse der Mathematik und ihher Grenzgebiete, 3. Folge, Band 12, Springer-Verlag, Berlin-Heidelberg-New York, 1987].

Definition. Suppose $X \subseteq \mathbf{R}^n$ is a semialgebraic set. The real spectrum of X, denoted $\widetilde{X}$, is the set of prime filters of closed semialgebraic subsets of X. Given a semialgebraic set $U \subseteq X$ which is relatively open in X, we define $\widetilde{U} := \{\alpha \in \widetilde{X} \mid \forall Y \in \alpha \;\; Y \cap U \neq \emptyset\}$. $\widetilde{X}$ carries the weakest topology in which all the sets $\widetilde{U}$ are open.

Some of this may need a bit of explanation. First of all, what it means for α to be a prime filter of closed semialgebraic subsets of X is:

α is a collection of closed semialgebraic subsets of X,

$V, W \in \alpha \quad \Rightarrow \quad V \cap W \in \alpha$,

$V \supseteq W \in \alpha \quad \Rightarrow \quad V \in \alpha$,

$V \cup W \in \alpha \quad \Rightarrow$ either $V \in \alpha$ or $W \in \alpha$ and

$X \in \alpha$ and $\emptyset \notin \alpha$.

Examples. If $x \in \mathbf{R}^n$, then the set of all closed semialgebraic subsets of $\mathbf{R}^n$ which contain x is a prime filter. If $\gamma : \mathbf{R} \to \mathbf{R}^n$ is an analytic curve, then the set of all closed semialgebraic subsets of $\mathbf{R}^n$ which contain $\gamma([0, \epsilon))$ for some $\epsilon \in \mathbf{R}_{>0}$ is a prime filter.

If $\alpha \in \widetilde{X}$ and $f, g : X \to \mathbf{R}^n$ are semialgebraic functions, then we write

$f(\alpha) = g(\alpha)$ if $f = g$ on some $U \in \alpha$,

$f(\alpha) > g(\alpha)$ if $f > g$ on some $U \in \alpha$,

$f(\alpha) \geq g(\alpha)$ if $f \geq g$ on some $U \in \alpha$.

We define $\operatorname{supp}\alpha := \{f \in \mathbf{R}[x_1, \ldots, x_n] \mid f(\alpha) = 0\}$. By primality of α, $\operatorname{supp}\alpha$ is a prime ideal. Moreover, for any $f, g \in \mathbf{R}[x_1, \ldots, x_n]$, exactly one of the following holds: $f(\alpha) = g(\alpha)$ or $f(\alpha) > g(\alpha)$ or $f(\alpha) < g(\alpha)$. Thus, α induces a total order on $\mathbf{R}[x_1, \ldots, x_n]/\operatorname{supp}\alpha$.

Definition. Let $\alpha, \beta \in \widetilde{X}$. Then $\langle \alpha, \beta \rangle$ denotes the ideal of $\mathbf{R}[x_1, \ldots, x_n]$ generated by the polynomials f such that both $f(\alpha) \geq 0$ and $f(\beta) \leq 0$.

Theorem. *Suppose $X \subset \mathbf{R}^n$ is semialgebraic and $\phi : X \to \mathbf{R}$ is PWP. Then the following are equivalent:*

1) *ϕ is SIPD,*

2) *$\forall \alpha, \beta \in \widetilde{X} \; \exists h_{\alpha\beta} \in \mathbf{R}[x_1, \ldots, x_n]$ such that $h_{\alpha\beta}(\alpha) \geq \phi(\alpha)$ and*

$h_{\alpha\beta}(\beta) \leq \phi(\beta)$,

3) $\forall \alpha, \beta \in \widetilde{X}$ $\forall f, g \in \mathbf{R}[x_1, \ldots, x_n]$, *if* $f(\alpha) = \phi(\alpha)$ *and* $g(\beta) = \phi(\beta)$ *then* $f - g \in \langle \alpha, \beta \rangle$.

For a proof of this theorem, see [Madden, *op. cit.*]. We shall not repeat it here, but shall be content to make a few comments on it. The proof is based on the compactness of $\widetilde{X}$ and the analogy between condition (2) and the existence of a good presentation. One may think of the elements of $\widetilde{X}$ as limits of shrinking closed semialgebraic sets, and of $\widetilde{X}$ itself as the limit of all semialgebraic partitions of X.

Some additional comments on the geometric meaning of some of the concepts introduced above may be helpful in understanding the theorem. The following assertions follow fairly directly from the definitions. Suppose that X is compact. Then for any $\alpha \in \widetilde{X}$, there is unique a point of X—call it $\operatorname{Cntr}\alpha$ —which is contained in all the sets in α. For any ideal $I \subseteq \mathbf{R}[x_1, \ldots, x_n]$, let $V_{\mathbf{R}}(I)$ denote the set of real zeroes of I. Then $V_{\mathbf{R}}(\operatorname{supp}\alpha)$ is the intersection of the Zariski closures of all the elements of α, and $V_{\mathbf{R}}(\langle \alpha, \beta \rangle)$ is the intersection of the Zariski closures of all the sets $S \cap T$ with $S \in \alpha$ and $T \in \beta$. If $\operatorname{Cntr}\alpha \neq \operatorname{Cntr}\beta$, then $\langle \alpha, \beta \rangle$ contains 1. If $\operatorname{Cntr}\alpha = \operatorname{Cntr}\beta$, then $V_{\mathbf{R}}(\langle \alpha, \beta \rangle)$ has codimension at least 1. It is clear that if ϕ is PWP and $f(\alpha) = \phi(\alpha)$ and $g(\beta) = \phi(\beta)$ for some $f, g \in \mathbf{R}[x_1, \ldots, x_n]$, then $f - g$ must vanish on $V_{\mathbf{R}}(\langle \alpha, \beta \rangle)$ (—compare with condition (3) in the theorem.)

What makes the Pierce-Birkhoff problem difficult is that in general $\langle \alpha, \beta \rangle \neq I(V_{\mathbf{R}}(\langle \alpha, \beta \rangle))$. In many cases there are polynomials outside of $\langle \alpha, \beta \rangle$ which vanish on $V_{\mathbf{R}}(\langle \alpha, \beta \rangle)$. However, if X is smooth and the codimension of $V_{\mathbf{R}}(\langle \alpha, \beta \rangle)$ is exactly 1, then it can be deduced from the Transversal Zeroes Theorem that $\langle \alpha, \beta \rangle = I(V_{\mathbf{R}}(\langle \alpha, \beta \rangle))$. We see again that the difficulties of the Pierce-Birkhoff Conjecture lie in codimension at least 2.

We describe another instance in which $\langle \alpha, \beta \rangle = I(V_{\mathbf{R}}(\langle \alpha, \beta \rangle))$. (This will be used in the next section.) For any $x \in \mathbf{R}^n$ and $Y \subseteq \mathbf{R}^n$, let $\overrightarrow{xY} := \{\lambda(y - x) \mid \lambda \in \mathbf{R}_{\geq 0}, y \in Y\}$. Assume X is compact. If $\alpha \in \widetilde{X}$, it is possible to show that there is $t \in \mathbf{R}^n$ such that $\bigcap_{Y \in \alpha} \overrightarrow{(\operatorname{Cntr}\alpha)Y} =$

$\{\lambda t \mid \lambda \in \mathbf{R}_{\geq 0}\}$. We call $D(\alpha) := \{\lambda t \mid \lambda \in \mathbf{R}_{\geq 0}\}$ the "tangent ray of α". If $\operatorname{Cntr}\alpha = x = \operatorname{Cntr}\beta$ and $D(\alpha) \neq D(\beta)$, then n independent linear functions can easily be found in $\langle \alpha, \beta \rangle$, and hence $\langle \alpha, \beta \rangle = \mathbf{m}_x$, ($\mathbf{m}_x =$ the maximal ideal at x). (Note: If X is a manifold, but is not given as a subset of $\mathbf{R}^n$, we may view $D(\alpha)$ as an element in the tangent space at $\operatorname{Cntr}\alpha$.)

Section 3.

It is possible to use quadratic transformations to prove that the Pierce-Birkhoff condition holds for any smooth semialgebraic surface. We describe how to do this in the present section, after first reviewing some facts about quadratic transforms of surfaces.

Suppose that X is an algebraic variety and P is a point of X. We write $P' \succ P$ to indicate that $P' \in \pi^{-1}(P) \subset X'$, where $\pi : X' \to X$ is the blow-up with center P. If there is a sequence $P^{(k)} \succ P^{(k-1)} \succ \ldots \succ P^{(0)} = P$, then we call $P^{(k)}$ an "infinitely near point of P". The algebra associated with points infinitely near to P when P is a regular point of a surface is a classical topic in algebraic geometry. Zariski found a way of formulating it in modern ideal-theoretic language.* We shall sketch the relevant part of the theory.

Let P be a regular point in a surface, and let a sequence $P^{(k)} \succ P^{(k-1)} \succ \ldots \succ P^{(0)} = P$ be given. Let $(\mathcal{O}^{(i)}, \mathbf{m}^{(i)})$ denote the local ring at $P^{(i)}$. Recall that $\mathcal{O}^{(i+1)}$ is a localization of $\mathcal{O}^{(i)}[y_i/x_i]$, where x_i and y_i

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* See [O. Zariski, Polynomial ideals defined by infinitely near points, *Amer. J. Math.* **60** (1938), 151–204]. This was one of Zariski's earliest attempts to express notions of algebraic geometry in ideal-theoretic language. Lately, there has been renewed interest in this theory; see [C. Huneke, Complete ideals in two-dimensional regular local rings, *Commutative Algebra*, Proc. Microprogram, MSRI Publication No. 15, Springer-Verlag, Berlin-Heidelberg-New York (1989), 325–38], [J. Lipman, On complete ideals in regular local rings, *Algebraic Geometry and Commutative Algebra* in honor of M. Nagata (1987), H. Hijikata (ed.), Kinokuniya Co., Tokyo, 203–31], and [M. Spivakovsky, Valuations in function fields of surfaces, *Amer. J. Math.* **112** (1990), 107–56].

are appropriately chosen generators of $\mathbf{m}^{(i)}$. (To allay any questions: we assume that $\mathcal{O}^{(k)}$ is 2-dimensional.) Given $f \in \mathcal{O}^{(0)}$, put $f^{(0)} := f$ and inductively define $f^{(i+1)} :=$ [redacted]x_i[redacted]$^{-\operatorname{ord}_i(f^{(i)})} f^{(i)} \in \mathcal{O}^{(i+1)}$. The so-called "effective multiplicities" of the sequence $P^{(k)} \succ P^{(k-1)} \succ \ldots \succ P^{(0)} = P$ are the integers $e_i := \min\{\operatorname{ord}_i(f^{(i)}) \mid f^{(i)} \in \mathbf{m}^{(i)}, i = 0, 1, 2, \ldots, k\}$. Given $f \in \mathcal{O}^{(0)}$, the so-called "virtual transforms" of f are the elements

$$V^{(i)}(f) := \left(\prod_{j=0}^{i-1} \text{[redacted]}\, x_j \rfloor^{-e_j}\right) f.$$

In general, $V^{(i)}(f)$ is not an element of $\mathcal{O}^{(i)}$. It can be shown that $I(P^{(k)}) := \{f \in \mathbf{m}^{(0)} \mid V^{(i)}(f) \in \mathbf{m}^{(i)}, i = 1, 2, \ldots, k\}$ is a simple ideal primary to $\mathbf{m}^{(0)}$ and that $P^{(k)} \mapsto I(P^{(k)})$ is a bijection between points infinitely near P and simple ideals primary to $\mathbf{m}^{(0)}$. (This theory works when relativized to the real coordinate ring of the set of real points of a surface, but since the intent of the present talk is just to convey impressions, I won't say anything more explicit.)

Now assume that $X_{\mathbf{R}}$ is the set of real points of a surface X, and for simplicity assume that $X_{\mathbf{R}}$ is a compact manifold of (real) dimension 2. Take any $\alpha \in \widetilde{X_{\mathbf{R}}}$ with $V_{\mathbf{R}}(\operatorname{supp}\alpha) \neq \operatorname{Cntr}\alpha$ and $\operatorname{Cntr}\alpha$ regular. Let $\pi : X' \to X$ be the blow-up of $\operatorname{Cntr}\alpha$. There is an element of $\widetilde{X'_{\mathbf{R}}}$—call it α'—consisting of the closed semialgebraic subsets of $X'_{\mathbf{R}}$ which contain some set of the form $X'_{\mathbf{R}} \cap \pi^{-1}(S \setminus \{P\})$ with $S \in \alpha$. We may iterate this construction, and we get a sequence of real spectrum points $\alpha = \alpha^{(0)}, \alpha^{(1)}, \ldots$ whose centers form a sequence of infinitely near points: $\ldots \succ \operatorname{Cntr}\alpha^{(1)} \succ \operatorname{Cntr}\alpha^{(0)}$.

If β is another point of $\widetilde{X_{\mathbf{R}}}$ with $V_{\mathbf{R}}(\operatorname{supp}\beta) \neq \operatorname{Cntr}\beta$, $\operatorname{Cntr}\beta = \operatorname{Cntr}\alpha$ and $\pm D(\alpha) = \pm D(\beta)$ (notation in the last paragraph of the previous section), then $\operatorname{Cntr}\beta' = \operatorname{Cntr}\alpha'$. Therefore, if $\langle\alpha, \beta\rangle$ is not the maximal ideal at $\operatorname{Cntr}\alpha$, the sequences $\ldots \succ \operatorname{Cntr}\alpha^{(1)} \succ \operatorname{Cntr}\alpha^{(0)}$ and $\ldots \succ \operatorname{Cntr}\beta^{(1)} \succ \operatorname{Cntr}\beta^{(0)}$ agree for some initial terms.

I have proved the following:

Theorem. *If $\langle\alpha, \beta\rangle$ is primary to the maximal ideal at* $\operatorname{Cntr}\alpha$ *then $\langle\alpha, \beta\rangle$ is the simple complete ideal corresponding to the sequence of infinitely near* *points* $\mathrm{Cntr}\,\alpha^{(k)} \ldots \succ \mathrm{Cntr}\,\alpha^{(1)} \succ \mathrm{Cntr}\,\alpha^{(0)}$, *where* k *is the least integer for which* $D(\alpha^{(k)})$ *and* $D(\beta^{(k)})$ *are distinct.*

I will not attempt to describe the proof, which depends on Zariski's theory. I would like to sketch the proof of the following

Corollary. *Every regular real algebraic surface* $X_{\mathbf{R}}$ *satisfies the condition of Pierce-Birkhoff.*

Proof sketch. Suppose that $\phi : X_{\mathbf{R}} \to \mathbf{R}$ is PWP, $\alpha, \beta \in \widetilde{X_{\mathbf{R}}}$ and $\phi(\alpha) = f(\alpha)$ and $\phi(\beta) = g(\beta)$ for some f, g in the real coordinate ring of $X_{\mathbf{R}}$. We need to show $g - f \in \langle\alpha, \beta\rangle$. In remarks in section 2, we indicated that this was immediate except in the case when $\langle\alpha, \beta\rangle$ is primary to a maximal ideal which properly contains it. In the difficult case $\mathrm{rad}\langle\alpha, \beta\rangle$ corresponds to the point $\mathrm{Cntr}\,\alpha \in X_{\mathbf{R}}$. Take a small closed disk D about $\mathrm{Cntr}\,\alpha$, within which $\mathrm{Trans}\,\phi$ is a finite union of half-branches $B_1, \ldots, B_s$ of curves emanating from $\mathrm{Cntr}\,\alpha$ which are disjoint except at $\mathrm{Cntr}\,\alpha$. We can assume there are at least three such half-branches by adding a new one if needed, and that the half-branches are numbered in order as one travels around $\mathrm{Cntr}\,\alpha$. Consecutive half-branches bound closed semialgebraic "wedges", $W_1, \ldots, W_s$, with W_1 between B_s and B_1, etc. Suppose the numbering has been choicen so that $W_1 \in \alpha$ and $W_{m+1} \in \beta$ and the tangent rays of $B_1, \ldots, B_m$ are equal to $D(\alpha)(=$ the common tangent ray of α and of $\beta)$. The key idea of the proof is that the configuration fo the wedges is preserved by blowing up $\mathrm{Cntr}\,\alpha$, provided that $D(\alpha') = D(\beta')$. Indeed, consider the the sequence of points $\mathrm{Cntr}\,\alpha^{(k)} \ldots \succ \mathrm{Cntr}\,\alpha^{(1)} \succ \mathrm{Cntr}\,\alpha$, where k is the least integer for which $D(\alpha^{(k)})$ and $D(\beta^{(k)})$ are distinct. The proper transforms of the (curves corresponding to the) half-branches $B_1, \ldots, B_m$ must pass through these points, so by the theorem above, a function in the coordinate ring which vanishes on one of the sets $B_1, \ldots, B_m$ belongs to $\langle\alpha, \beta\rangle$. Now assume $\phi = f_i$ on W_i (f_i in the coordinate ring). Then $f_{i+1} - f_i$ vanishes on B_i. As $g - f = f_{m+1} - f_1 = (f_{m+1} - f_m) + (f_m - f_{m-1}) + \ldots + (f_2 - f_1)$, we have the desired result.

Section 3.

The difficulty with making this approach work in higher dimensions is that the correspondence between infinitely near points and ideals becomes much more complicated. The extension of Zariski's theory to higher dimensions was a problem he himself posed in the paper mentioned above, but very little seems to have been accomplished in the interim. Some interesting results are in [J. Lipman, On complete ideals in regular local rings. *Algebraic Geometry and Commutative Algebra* in honor of M. Nagata (1987), H. Hijikata (ed.), Kinokuniya Co., Tokyo, 203–31]. Fortunately, the results discussed above do not seem to me to depend on any of the parts of Zariski's theory which are known to fail in higher dimensions, e.g. unique factorization of complete ideals in dimension 2.

I believe that there are examples in dimension 3 in which we can hold α fixed and maintain the condition that k is the least integer for which $D(\alpha^{(k)})$ and $D(\beta^{(k)})$ are distinct and yet cause $\langle \alpha, \beta \rangle$ to vary depending on the position of $D(\beta^{(k)})$.