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Some Aspects of Representations and Algebraic Geometry of Lie Algebras

finiteness criteria for the restriction of $U(\mathfrak{g})$-modules
and applications to Harish-Chandra modules

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Introduction and main results.

Let $\mathfrak{g}$ be a finite-dimensional complex Lie algebra, and $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. The natural increasing filtration $(U_k(\mathfrak{g}))_{k=0,1,..}$ of $U(\mathfrak{g})$ defines a commutative graded ring $\operatorname{gr} U(\mathfrak{g}) = \bigoplus_k U_k(\mathfrak{g})/U_{k-1}(\mathfrak{g})$, which is isomorphic to the symmetric algebra $S(\mathfrak{g})$ of $\mathfrak{g}$ by the Poincaré-Birkhoff-Witt theorem. The identification $S(\mathfrak{g}) = \operatorname{gr} U(\mathfrak{g})$ allows us to relate various objects in (non-commutative) enveloping algebra theory with those in commutative algebra and algebraic geometry for $S(\mathfrak{g})$ and $\mathfrak{g}^* = \operatorname{Spec} S(\mathfrak{g})$, the dual space of $\mathfrak{g}$ (see [2], [4], [14], [17, 18]).

For instance, if H is a $U(\mathfrak{g})$-module generated by a finite-dimensional subspace H_0, we can associate to the pair (H, H_0) a graded $S(\mathfrak{g})$-module of finite type by $\operatorname{gr}(H; H_0) := \bigoplus_k H_k/H_{k-1}$ with $H_k = U_k(\mathfrak{g})H_0$. The annihilator $J(H; H_0)$ of $\operatorname{gr}(H; H_0)$ in $S(\mathfrak{g})$ defines the *associated variety* $\mathcal{V}(\mathfrak{g}; H) \subset \mathfrak{g}^*$ of H, independent of H_0, as the set of common zeros of all the elements of $J(H; H_0)$. The celebrated Hilbert-Serre theorem in commutative ring theory says that this variety $\mathcal{V}(\mathfrak{g}; H)$ supports well the graded $S(\mathfrak{g})$-module $\operatorname{gr}(H; H_0)$ (see Theorem 1.1).

In this paper, we give useful criteria for finitely generated $U(\mathfrak{g})$-modules H to remain finite under the restriction to subalgebras of $U(\mathfrak{g})$, by means of the associated varieties $\mathcal{V}(\mathfrak{g}; H)$. Applying the criteria, we specify among other things, a large class of Lie subalgebras of a semisimple Lie algebra on which all the Harish-Chandra modules are of finite type. This extends a result of Casselmann-Osborne [8] and Joseph [13] on the restriciton of admissible modules to nilpotent Lie subalgebras appearing in the Iwasawa decomposition. Moreover we develop, with the help of Frobenius reciprocity, the finite multiplicity theorems for induced representations of a semisimple Lie group, obtained in our earlier work [20].

Let us now explain our basic ideas and the principal results of this article.

A. For a subalgebra A of $U(\mathfrak{g})$ containing the identity element, let R denote the associated graded subalgebra $\operatorname{gr} A := \bigoplus_{k\geq 0} A_k/A_{k-1}$ of $S(\mathfrak{g})$ with $A_k = A \cap U_k(\mathfrak{g})$. We say that a finitely generated $U(\mathfrak{g})$-module H has the *good restriction* to A if there exists a generating subspace H_0 of H for which the $S(\mathfrak{g})$-module $M := \operatorname{gr}(H; H_0)$ is of finite type over R. It is standard to verify that the original H is finitely generated over A if its restriction to A is good .

We can characterize the $U(\mathfrak{g})$-modules H having the good restriction to a given A, by using the associated varieties. To be specific, we first observe that the graded

$S(\mathfrak{g})$-module $M = \mathrm{gr}(H; H_0)$ is finitely generated over R if and only if the quotient $S(\mathfrak{g})$-module M/R_+M is of finite-dimension, where R_+ denotes the maximal graded ideal of R. Secondly, the Hilbert-Serre theorem (or Hilbert's Nullstellensatz) tells us that $\dim M/R_+M < \infty$ whenever

(VHR0) $$\mathcal{V}(\mathfrak{g}; H) \cap R_+^\# = (0)$$

holds, where $R_+^\#$ denotes the algebraic variety of $\mathfrak{g}^*$ determined by R_+ as the set of common zero points. Furthermore, it is shown that the converse is also true provided that R is Noetherian. (See Proposition 2.1.)

Summing up the above discussion, we obtain the first main result of this paper, as follows.

Theorem I. (see Theorems 2.1 and 2.2(1)) (1) *A finitely generated $U(\mathfrak{g})$-module H has the good restriction to a subalgebra A whenever* (VHR0) *is fulfilled for $R = \mathrm{gr}\, A$. The converse is also true if the ring R is Noetherian.*

(2) *The condition* (VHR0) *guarantees that H is of finite type over A.*

If $A = U(\mathfrak{n})$ for a Lie subalgebra $\mathfrak{n}$ of $\mathfrak{g}$, then the corresponding graded ring $R = S(\mathfrak{n})$ is Noetherian and $R_+^\#$ equals the orthogonal $\mathfrak{n}^\perp$ of $\mathfrak{n}$ in $\mathfrak{g}^*$. Accordingly, one sees from Theorem I that H has the good restriction to $U(\mathfrak{n})$ if and only if $\mathcal{V}(\mathfrak{g}; H) \cap \mathfrak{n}^\perp = (0)$. In this case, we find that, besides the finiteness, H preserves some other invariants under the restriction to $U(\mathfrak{n})$:

Theorem II. (see Theorem 2.2(2)) *If the restriction of an H to $U(\mathfrak{n})$ is good, the Gelfand-Kirillov dimension $d(\mathfrak{n}; H) := \dim \mathcal{V}(\mathfrak{n}; H)$ and the Bernstein degree $c(\mathfrak{n}; H)$ (see* 1.2 *for the definition) of H as a $U(\mathfrak{n})$-module coincide respectively with those $d(\mathfrak{g}; H)$ and $c(\mathfrak{g}; H)$ as a $U(\mathfrak{g})$-module. Furthermore, the variety $\mathcal{V}(\mathfrak{g}; H)$ is carried into $\mathcal{V}(\mathfrak{n}; H)$ by the restriction of linear forms on $\mathfrak{g}$ to $\mathfrak{n}$.*

B. The general results given in **A**, have remarkable applications to Harish-Chandra modules of a semisimple Lie algebra.

Now let $\mathfrak{g}_0$ be a real semisimple Lie algebra, and $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be a Cartan decomposition of $\mathfrak{g}_0$. We denote by $\mathfrak{g}$ the complexified Lie algebra of $\mathfrak{g}_0$, and the complexification of a real vector subspace $\mathfrak{h}_0$ of $\mathfrak{g}_0$ will be denoted by $\mathfrak{h}(\subset \mathfrak{g})$, conventionally. By a Harish-Chandra $(\mathfrak{g}, \mathfrak{k})$-module is meant a finitely generated $U(\mathfrak{g})$-module H on which the subalgebra $U(\mathfrak{k})\mathcal{Z}(\mathfrak{g})$ acts locally finitely, where $\mathcal{Z}(\mathfrak{g})$ denotes the center of $U(\mathfrak{g})$. We regard the variety $\mathcal{V}(\mathfrak{g}; H)$ as a subset of $\mathfrak{g}$ by identifying $\mathfrak{g}^*$ with $\mathfrak{g}$ through the Killing form of $\mathfrak{g}$.

The following two facts are essential for our applications to Harish-Chandra modules.

(1) The associated variety $\mathcal{V}(\mathfrak{g}; H)$ of a Harish-Chandra $(\mathfrak{g}, \mathfrak{k})$-module H is contained in the set $\mathcal{N}(\mathfrak{p})$ of all the nilpotent elements in $\mathfrak{p}$ (Lemma 3.1).

(2) There exists a Harish-Chandra module $\tilde{H}$ for which $\mathcal{V}(\mathfrak{g}; \tilde{H})$ coincides with the whole $\mathcal{N}(\mathfrak{p})$ (Proposition 3.2).

These facts together with Theorem I yield the following

Theorem III. (see Theorem 3.1) *All the Harish-Chandra* $(\mathfrak{g}, \mathfrak{k})$*-modules have the good restriction to a subalgebra* A *of* $U(\mathfrak{g})$ *if* $\mathcal{N}(\mathfrak{p}) \cap R_+^\# = (0)$ *holds for* $R = \operatorname{gr} A$. *The converse is also true when* R *is Noetherian.*

C. Suggested by this theorem, we say that a Lie subalgebra $\mathfrak{n}_0$ is *large* in $\mathfrak{g}_0$ if there exists an inner automorphism x of $\mathfrak{g}_0$ such that

$$(x \cdot \mathfrak{n})^\perp \cap \mathcal{N}(\mathfrak{p}) = (0),$$

or equivalently, each Harish-Chandra $(\mathfrak{g}, \mathfrak{k})$-module has the good restriction to $U(x \cdot \mathfrak{n})$.

We can specify many of large Lie subalgebras of $\mathfrak{g}_0$. At first, the maximal nilpotent Lie subalgebras and also the symmetrizing Lie subalgebras of $\mathfrak{g}_0$ are proved to be large in $\mathfrak{g}_0$ (Propositions 4.1 and 4.2). Theorems I and II applied to the former example cover results of Casselmann-Osborne [8, Th.2.3] and Joseph [13, II, 5.6]. Secondly, it is shown that the largeness of a Lie subalgebra is preserved by the parabolic induction (see 4.2). This means that, if $\mathfrak{h}_0$ is a large Lie subalgebra of the Levi component $\mathfrak{l}_0$ of a parabolic subalgebra $\mathfrak{q}_0 = \mathfrak{l}_0 + \mathfrak{u}_0$, the semidirect product Lie subalgebra $\mathfrak{h}_0 + \mathfrak{u}_0$ is large in $\mathfrak{g}_0$. Here $\mathfrak{u}_0$ is the nilradical of $\mathfrak{q}_0$.

Thirdly, we say that a Lie subalgebra $\mathfrak{n}_0$ of $\mathfrak{g}_0$ is *quasi-spherical* if there exists a minimal parabolic subalgebra $\mathfrak{q}_{m,0}$ of $\mathfrak{g}_0$ such that $\mathfrak{g}_0 = \mathfrak{n}_0 + \mathfrak{q}_{m,0}$. Such Lie subalgebras give rise to the homogeneous spaces of a semisimple Lie group on which each minimal parabolic subgroup admits an open orbit (see e.g., [3], [5, 6], [15], [16]).

Theorem IV. (see Theorem 4.1) *Any quasi-spherical Lie subalgebra is large in* $\mathfrak{g}_0$.

D. Let G be a connected semisimple Lie group with finite center, and K be a maximal compact subgroup of G. We denote the corresponding Lie algebras by $\mathfrak{g}_0$ and $\mathfrak{k}_0$, respectively. By Harish-Chandra, the admissible Hilbert space G-representations correspond to Harish-Chandra $(\mathfrak{g}, K)$-modules, i.e., such $(\mathfrak{g}, \mathfrak{k})$-modules with compatible K-action, by passing to the K-finite part. On the other side, if (η, E) is a smooth Fréchet representation of a closed subgroup N of G, the space $\mathcal{A}(G; \eta)$ of real analytic sections of associated vector bundle $G \times_N E$, has a natural structure of compatible $(G, U(\mathfrak{g}))$-module (see 5.1).

With the aid of Frobenius reciprocity (cf. Proposition 5.1), Theorems I and III on the restriction of $U(\mathfrak{g})$-modules, allow us to give useful finite multiplicity criteria for analytically induced modules $\mathcal{A}(G; \eta)$ (Theorems 5.3-5.5).

Among other things, we establish the following

Theorem V. (see Theorem 5.5) *Let* N *be a closed subgroup of* G *whose Lie algebra* $\mathfrak{n}_0$ *is large in* $\mathfrak{g}_0$*, and take an* $x \in G$ *for which* $(\mathrm{Ad}(x)\mathfrak{n})^\perp \cap \mathcal{N}(\mathfrak{p}) = (0)$. *Then the intertwining number* $\dim \mathrm{Hom}_{U(\mathfrak{g})}(H, \mathcal{A}(G; \eta))$ *is finite for every Harish-Chandra* $(\mathfrak{g}, K)$*-module* H*, if the restriction of* η *to compact subgroup* $N \cap x^{-1}Kx$ *has the finite multiplicity property.*

This theorem extends one of the principal results in our previous work [20, I, Th.2.12], where we studied the case of certain semidirect product large Lie subalgebras $\mathfrak{n}_0$, through the theory of (K, N)-spherical functions.

The organization of this paper is as follows. We begin with preparing in §1 the notions and fundamental facts which we need throughout this article. §2 gives the theoretical basis of this work. We develop the general theory on restriction of $U(\mathfrak{g})$-modules to subalgebras by using the associated varieties. The criteria for good restriction to subalgebras, are established in various situations in 2.1 and 2.2, and we clarify in 2.3 and 2.4 some important properties of $U(\mathfrak{g})$-modules having the good restriction.

In §3, applying the results of §2 to semisimple Lie algebras $\mathfrak{g}$, we characterize, in relation with the nilpotent variety $\mathcal{N}(\mathfrak{p})$, subalgebras of $U(\mathfrak{g})$ to which all the Harish-Chandra $(\mathfrak{g}, \mathfrak{k})$-modules have the good restriction. The principal result of §3, Theorem 3.1, is presented in much more general setting. §4 is devoted to the specification of large Lie subalgebras of a real semisimple Lie algebra. The last §5 develops finite multiplicity criteria for analytically induced representations of a (semisimple) Lie group, by making use of the results of §§2-4 and a reciprocity of Frobenius type.

An enlarged version of this article, with complete proofs, will appear elsewhere.

1. Associated varieties for finitely generated $U(\mathfrak{g})$-modules.

At first, we equip ourselves with some fundamental facts in commutative algebra and algebraic geometry, and introduce three important invariants: the associated variety, the Bernstein degree and the Gelfand-Kirillov dimension, of finitely generated modules over a complex Lie algebra.

1.1. The Hilbert-Serre theorem. Let V be a finite-dimensional complex vector space. We denote by $S(V) = \bigoplus_{k=0}^{\infty} S^k(V)$ the symmetric algebra of V, where $S^k(V)$ is the subspace of $S(V)$ consisting of all homogeneous elements of degree k. Let $M = \bigoplus_{k=0}^{\infty} M_k$ be a finitely generated, non-zero, graded $S(V)$-module, on which $S(V)$ acts in such a way as $S^k(V)M_{k'} \subset M_{k+k'}$ $(k, k' \geq 0)$. Then it is easy to see that each homogeneous component M_k is finite-dimensional. Set

$$\varphi_M(q) := \dim\,(M_0 + M_1 + \ldots + M_q) \tag{1.1}$$

for each integar $q \geq 0$.

Theorem 1.1. (Hilbert-Serre, see [22, Ch.VII, §12]) (1) *There exists a unique polynomial $\tilde{\varphi}_M(q)$ in q such that $\varphi_M(q) = \tilde{\varphi}_M(q)$ for sufficiently large q.*

(2) *Let $(c(M)/d(M)!)q^{d(M)}$ be the leading term of $\tilde{\varphi}_M$. Then $c(M)$ is a positive integer, and the degree $d(M)$ of this polynomial coincides with the dimension of the associated algebraic variety*

$$\mathcal{V}(M) := \{\lambda \in V^* \mid f(\lambda) = 0 \text{ for all } f \in \mathrm{Ann}_{S(V)}M\}. \tag{1.2}$$

Here, $\mathrm{Ann}_{S(V)}M$ denotes the annihilator of M in $S(V)$, V^* the dual space of V, and $S(V)$ is identified with the polynomial ring over V^* in the canonical way.

Since the annihilator $\mathrm{Ann}_{S(V)}M$ is a graded ideal contained in $S(V)_+ := \bigoplus_{k>0} S^k(V)$, the variety $\mathcal{V}(M)$ is an algebraic cone in V^*. This combined with (2) of the above theorem gives in particular the following corollary, which is one of the keys for studying in §2 the restriction of $U(\mathfrak{g})$-modules to subalgebras.

Corollary 1.1. *A finitely generated, non-zero, graded $S(V)$-module M is finite-dimensional if and only if its associated variety $\mathcal{V}(M)$ equals (0).*

Remark. It is not difficult to deduce this corollary directly from *Hilbert's Nullstellensatz.*

1.2. Associated varieties for $U(\mathfrak{g})$-modules. Let $\mathfrak{g}$ be a finite-dimensional complex Lie algebra, and $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$. For each integer $k \geq 0$, we denote by $U_k(\mathfrak{g})$ the subspace of $U(\mathfrak{g})$ generated by elements $X_1 \dots X_m$ with $m \leq k$ and $X_j \in \mathfrak{g}$ $(1 \leq j \leq m)$. One gets a natural increasing filtration $(U_k(\mathfrak{g}))_{k \geq 0}$ of $U(\mathfrak{g})$ such that

$$U(\mathfrak{g}) = \bigcup_{k=0}^{\infty} U_k(\mathfrak{g}), \ U_k(\mathfrak{g})U_m(\mathfrak{g}) = U_{k+m}(\mathfrak{g}), \ [U_k(\mathfrak{g}), U_m(\mathfrak{g})] \subset U_{k+m-1}(\mathfrak{g}).$$

The associated graded commutative algebra $\mathrm{gr}\, U(\mathfrak{g}) := \bigoplus_{k \geq 0} U_k(\mathfrak{g})/U_{k-1}(\mathfrak{g})$ $(U_{-1}(\mathfrak{g}) := (0))$ is isomorphic to the symmetric algebra $S(\mathfrak{g}) = \bigoplus_{k \geq 0} S^k(\mathfrak{g})$ of $\mathfrak{g}$ in the canonical way. We will identify these two algebras with each other.

Now let H be a finitely generated, non-zero $U(\mathfrak{g})$-module. Take a finite-dimensional generating subspace H_0 of H: $H = U(\mathfrak{g})H_0$. Setting $H_k = U_k(\mathfrak{g})H_0$ for $k = 1, 2, \dots$; $H_{-1} = (0)$, one obtains an increasing filtration $(H_k)_k$ of H such that

$$H = \bigcup_{k=0}^{\infty} H_k, \quad U_m(\mathfrak{g})H_k = H_{k+m}. \tag{1.3}$$

Correspondingly, we have a graded $S(\mathfrak{g})$-module

$$M = \bigoplus_k M_k \quad \text{with} \quad M_k = H_k/H_{k-1}, \tag{1.4}$$

which will be denoted by $\mathrm{gr}(H; H_0)$ because the above filtration of H is determined by H_0. Since $M_k = S^k(\mathfrak{g})M_0$, M is finitely generated over $S(\mathfrak{g})$. So we can define for this M the variety $\mathcal{V}(M) \subset \mathfrak{g}^*$, the integers $c(M)$ and $d(M)$ as in Theorem 1.1. It is easy to see that these quantities are independent of the choice of a generating subspace H_0. Hereafter, we will denote these three invariants of H respectively by $\mathcal{V}(\mathfrak{g}; H)$, $c(\mathfrak{g}; H)$, and by $d(\mathfrak{g}; H)$, emphasizing that H is being considered as a $U(\mathfrak{g})$-module.

Definition(cf. [4, III], [17, 18]). For a finitely generated non-zero $U(\mathfrak{g})$-module H, $\mathcal{V}(\mathfrak{g}; H)$, $c(\mathfrak{g}; H)$, and $d(\mathfrak{g}; H)$ $(= \dim \mathcal{V}(\mathfrak{g}; H)$ by Theorem 1.1(2)) are called respectively the *associated variety*, the *Bernstein degree* and the *Gelfand-Kirillov dimension* of H.

2. Restriction of $U(\mathfrak{g})$-modules to subalgebras.

Let A be a subalgebra of $U(\mathfrak{g})$ containing the identitiy element $1 \in U(\mathfrak{g})$. Denote by gr $A = \oplus_{k\geq 0} A_k/A_{k-1}$ with $A_k = A \cap U_k(\mathfrak{g})$, the graded subalgebra of $S(\mathfrak{g}) = \mathrm{gr}\ U(\mathfrak{g})$ associated to A. We say that a finitely generated $U(\mathfrak{g})$-module H has the *good restriction* to A if there exists a finite-dimensional generating subspace H_0 of H for which the associated graded $S(\mathfrak{g})$-module $\mathrm{gr}(H; H_0)$ is finitely generated over gr A.

This section characterizes, by means of the associated varieties, $U(\mathfrak{g})$-modules H having the good restriction to A (Theorem 2.1). We show that such H's are finitely generated over A (Theorem 2.2(1)). Some more properties of these modules H are specified in 2.3.

2.1. Restriction of $S(V)$-modules to graded subalgebras. We first discuss the restriction of graded $S(V)$-modules, where V is any complex vector space of finite dimension. Let $R = \bigoplus_{k\geq 0} R_k$, $R_k \subset S^k(V)$, be a graded subalgebra of $S(V)$ containing the identity element $1 \in S(V)$. $R_+ = \bigoplus_{k>0} R_k$ denotes the maximal homogeneous ideal of R without constant term. We set for any subset Q of $S(V)$,

$$Q^{\#} := \{\lambda \in V^* \mid \ f(\lambda) = 0 \ \text{ for all } \ f \in Q\}. \tag{2.1}$$

Let M be, as in 1.1, a finitely generated, non-zero, graded $S(V)$-module. We consider the following four conditions on M in relation with R:

(a) $\mathcal{V}(M) \cap R_+^{\#} = (0)$, where $R_+^{\#} := (R_+)^{\#}$, and $\mathcal{V}(M) = (\mathrm{Ann}_{S(V)} M)^{\#}$ is the associated variety of M defined in (1.2).
(b) The ideal $\mathrm{Ann}_{S(V)} M + R_+ S(V)$ is of finite codimension in $S(V)$.
(c) The $S(V)$-submodule $R_+ M$ is of finite codimension in M.
(d) M is finitely generated as an R-module.

Then we get the following proposition on the relation among these conditions.

Proposition 2.1. (1) *The condition* (a) (resp. (c)) *is equivalent to* (b) (resp. (d)). *Moreover,* (a) ($\Leftrightarrow$ (b)) *implies* (c) ($\Leftrightarrow$ (d)).

(2) *If the ring R is Noetherian, the four conditions* (a)–(d) *are equivalent with each other.*

Corollary 2.1. *For a vector subspace W of V, set $W^{\perp} = \{\lambda \in V^* \mid \ < \lambda, w >= 0$ for all $w \in W\}$. A finitely generated graded $S(V)$-module M, $\neq (0)$, is of finite type over the subalgebra $S(W)$ if and only if $\mathcal{V}(M) \cap W^{\perp} = (0)$.*

2.2. Good restriction of $U(\mathfrak{g})$-modules. Now, let $\mathfrak{g}$ be any complex Lie algebra, and H be a finitely generated, non-zero $U(\mathfrak{g})$-module. Proposition 2.1 gives the following criterion for H to have the good restriction to a subalgebra of $U(\mathfrak{g})$.

Theorem 2.1. *Let A be a subalgebra of $U(\mathfrak{g})$ containing the identity element $1 \in U(\mathfrak{g})$.*
(1) *The restriction of H to A is good whenever the condition*

$$\mathcal{V}(\mathfrak{g}; H) \cap R_+^\# = (0) \tag{2.2}$$

on algebraic varieties in $\mathfrak{g}^*$ is satisfied. Here $\mathcal{V}(\mathfrak{g}; H)$ is the associated variety of H defined in 1.2, and $R = \mathrm{gr}\ A$ denotes the graded subalgebra of $S(\mathfrak{g})$ associated to A.
(2) *Conversely, if R is Noetherian and if H admits the good restriction to A, one necessarily has (2.2).*

Remark. The condition (2.2) guarantees that the graded $S(\mathfrak{g})$-module $\mathrm{gr}(H; H_0)$ is finitely generated over $R = \mathrm{gr}\ A$ for *every* generating subspace H_0 of H.

Let $\mathfrak{n}$ be a Lie subalgebra of $\mathfrak{g}$. Applying Theorem 2.1 to the case $A = U(\mathfrak{n})$ ($R = S(\mathfrak{n})$ is obviously Noetherian), we obtain immediately the following

Corollary 2.2. *A finitely generated $U(\mathfrak{g})$-module H, $\neq (0)$, has the good restriction to $U(\mathfrak{n})$ if and only if $\mathcal{V}(\mathfrak{g}; H) \cap \mathfrak{n}^\perp = (0)$ holds.*

For later applications in §3, we give here another consequence of Theorem 2.1. Let B, $\ni 1$, be a subalgebra of $U(\mathfrak{g})$, and let $C(B)$ denote the category of finitely generated $U(\mathfrak{g})$-modules H on which B acts locally finitely:

$$\dim Bv < \infty \quad \text{for all } v \in H.$$

We can (and do) take, for such an H, a finite-dimensional B-stable generating subspace $H_0 \subset H$. Set $Q = \mathrm{gr}\ B$. Then it is easily verified that the corresponding graded $S(\mathfrak{g})$-module $M = \mathrm{gr}(H; H_0)$ is annihilated by the maximal graded ideal Q_+ of Q. Hence one gets

$$\mathcal{V}(\mathfrak{g}; H) \subset Q_+^\#. \tag{2.3}$$

Definition. We say that a subalgebra A of $U(\mathfrak{g})$ is *large* relative to B if all the $U(\mathfrak{g})$-module H in the category $C(B)$ have the good restriction to A.

From (2.3) combined with Theorem 2.1, we conclude

Proposition 2.2. *Let B, $Q = \mathrm{gr}\ B$ be as above, and A, $\ni 1$, be a subalgebra of $U(\mathfrak{g})$ for which $R = \mathrm{gr}\ A$ is Noetherian. Then A is large relative to B if and only if*

$$\mathcal{V}_B \cap R_+^\# = (0) \tag{2.4}$$

holds for the subset $\mathcal{V}_B := \cup_H \mathcal{V}(\mathfrak{g}; H)$ of $Q_+^\#$, where H runs over the $U(\mathfrak{g})$-modules in $C(B)$. In particular, so is the case if $Q_+^\# \cap R_+^\# = (0)$.

Remark. It can be interesting to describe the subvariety $\mathcal{V}_B$ of $Q_+^\#$. We will show that $\mathcal{V}_B = Q_+^\#$ holds for the category $C(B)$ of Harish-Chandra modules of a semisimple Lie algebra $\mathfrak{g}$ (see Corollary 3.1).

Now define the double regular representation of $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ on $\mathcal{U} := U(\mathfrak{g})$ by

$$(D_1 \otimes D_2)v = D_1 v\, {}^t D_2 \quad \text{for } D_1,\ D_2 \in U(\mathfrak{g}) \text{ and } v \in \mathcal{U}.$$

Here $D \to {}^t D$ denotes the principal anti-automorphism of $U(\mathfrak{g})$, characterized by ${}^t X = -X$ for $X \in \mathfrak{g}$. Identifying $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ with $U(\mathfrak{g} \oplus \mathfrak{g})$ by the Poincaré-Birkhoff-Witt theorem, we regard $\mathcal{U}$ as a $U(\mathfrak{g} \oplus \mathfrak{g})$-module generated by the identity element $1 \in \mathcal{U}$.

The condition $Q_+^\# \cap R_+^\# = (0)$ in Proposition 2.2 can be related with the good restriction property of this module $\mathcal{U}$, as follows.

Proposition 2.3. *Let A, B ($\ni 1$) be two subalgebras of $U(\mathfrak{g})$. The restriction of $U(\mathfrak{g} \oplus \mathfrak{g})$-module $\mathcal{U}$ to the subalgebra $A \otimes B$ is good if $Q_+^\# \cap R_+^\# = (0)$ is satisfied, where $R = \operatorname{gr} A$ and $Q = \operatorname{gr} B$. The converse is also true if $R \otimes Q$ is Noetherian.*

2.3. Properties of $U(\mathfrak{g})$-modules with good restriction.

The $U(\mathfrak{g})$-modules admitting the good restriction enjoy some nice properties as follows.

Theorem 2.2. *Let H be a finitely generated, non-zero $U(\mathfrak{g})$-module having the good restriction to a subalgebra $A \subset U(\mathfrak{g})$. Then,*

(1) H is finitely generated as an A-module.

(2) Assume that $A = U(\mathfrak{n})$ for some Lie subalgebra $\mathfrak{n}$ of $\mathfrak{g}$ (see Corollary 2.2*). By (1), H is of finite type over $U(\mathfrak{n})$, and so one can define the associated variety $\mathcal{V}(\mathfrak{n}; H)$, Bernstein degree $c(\mathfrak{n}; H)$, and Gelfand-Kirillov dimension $d(\mathfrak{n}; H)$ of H as a $U(\mathfrak{n})$-module as well as those as a $U(\mathfrak{g})$-module. These two kinds of invariants have the relations*

$$c(\mathfrak{g}; H) = c(\mathfrak{n}; H), \quad d(\mathfrak{g}; H) = d(\mathfrak{n}; H), \tag{2.5}$$

and hence

$$\dim \mathcal{V}(\mathfrak{g}; H) = \dim \mathcal{V}(\mathfrak{n}; H). \tag{2.6}$$

Moreover one has

$$p^* \mathcal{V}(\mathfrak{g}; H) \subset \mathcal{V}(\mathfrak{n}; H), \tag{2.7}$$

where $p^* : \mathfrak{g}^* \to \mathfrak{n}^*$ denotes the restriction map of linear forms.

The following is a direct consequence of Theorem 2.2(2).

Corollary 2.3. *If a finitely generated $U(\mathfrak{g})$-module H has the good restriction to $U(\mathfrak{n})$, the Gelfand-Kirillov dimension $d(\mathfrak{g}; H)$ does not exceed* $\dim \mathfrak{n}$.

We now give two more consequences of Theorems 2.1 and 2.2:

Corollary 2.4. *Let I be a right ideal of $U(\mathfrak{g})$ such that $I \neq U(\mathfrak{g})$. For a finitely generated $U(\mathfrak{g})$-module H, the factor space H/IH is finite-dimensional if $\mathcal{V}(\mathfrak{g}; H) \cap (\mathrm{gr}\ I)^{\#} = (0)$, where* $\mathrm{gr}\ I = \bigoplus_k I_k/I_{k-1}$ *with* $I_k = U_k(\mathfrak{g}) \cap I$.

Corollary 2.5. *Let $\mathfrak{n}$ be a Lie subalgebra of $\mathfrak{g}$, and H be a finitely generated $U(\mathfrak{g})$-module satisfying the condition $\mathcal{V}(\mathfrak{g}; H) \cap \mathfrak{n}^{\perp} = (0)$. Then, the $\mathfrak{n}$-homology groups $H_k(\mathfrak{n}, H)$ $(k = 0, 1, \dots)$ of H (see e.g., [7] for the definition) are all finite-dimensional.*

Let I be a non-trivial right ideal of $U(\mathfrak{g})$. We denote by N_I the left normalizer of I in $U(\mathfrak{g})$:

$$N_I = \{D \in U(\mathfrak{g}) \mid \ DI \subset I\}. \tag{2.8}$$

For any $U(\mathfrak{g})$-module H, the factor space H/IH becomes an N_I-module.

We conclude this section with an interesting generalization of Corollary 2.4, as follows.

Proposition 2.4. *Let B be a subalgebra of N_I containing the identity element. Denote by* $\mathrm{gr}\ I$ (resp. $\mathrm{gr}\ B$) *the graded ideal* (resp. *graded subalgebra*) *of $S(\mathfrak{g})$ associated to I* (resp. B). *For a finitely generated $U(\mathfrak{g})$-module H, H/IH is of finite type over B whenever the variety $\mathcal{V}(\mathfrak{g}; H) \cap (\mathrm{gr}\ I)^{\#} \cap (\mathrm{gr}\ B)_{+}^{\#}$ reduces to (0). Here $(\mathrm{gr}\ B)_{+}$ denotes the maximal graded ideal of* $\mathrm{gr}\ B$.

This proposition actually includes Corollary 2.4 as a special case $B = \mathbf{C}1$.

An application of the proposition will be given in §3 for semisimple Lie algebras $\mathfrak{g}$.

3. Nilpotent varieties in $\mathfrak{p}$ and good restriction of Harish-Chandra modules.

Until the end of §4, let $\mathfrak{g}$ be a complex semisimple Lie algebra. In this section, applying the results of §2 we characterize, in relation with nilpotent varieties in $\mathfrak{p}$, subalgebras of $U(\mathfrak{g})$ to which all the Harish-Chandra $(\mathfrak{g}, \mathfrak{k})$-modules have the good restriction, where $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a symmetric decomposition of $\mathfrak{g}$. The main results here are stated in Theorems 3.1 and 3.2.

Although our principal interest lies in the applications to Harish-Chandra modules, we proceed here in more general situation as much as possible.

3.1. Associated varieties for $U(\mathfrak{g})$-modules in $C(\mathfrak{k}, \mathcal{Z})$. Let $\mathfrak{k}$ be any Lie subalgebra of $\mathfrak{g}$, and $\mathcal{Z} = \mathcal{Z}(\mathfrak{g})$ denotes the center of $U(\mathfrak{g})$. Set $B(\mathfrak{k}, \mathcal{Z}) = U(\mathfrak{k})\mathcal{Z}(\mathfrak{g})$, and we consider as in 2.2 the category $C(\mathfrak{k}, \mathcal{Z}) := C(B(\mathfrak{k}, \mathcal{Z}))$ of locally $B(\mathfrak{k}, \mathcal{Z})$-finite, finitely generated $U(\mathfrak{g})$-modules.

A Lie subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ is said to be *symmetrizing* if it is the set of fixed points of an involutive automorphism of $\mathfrak{g}$. In this case, the $U(\mathfrak{g})$-modules in $C(\mathfrak{k}, \mathcal{Z})$ will be called *Harish-Chandra* $(\mathfrak{g}, \mathfrak{k})$-*modules*. This category of Harish-Chandra modules is enjoying an essential role in representation theory of real semisimple Lie groups (see e.g., [9, 19]).

On the other hand, the category $C(\mathfrak{k}, \mathcal{Z})$ for a Borel subalgebra $\mathfrak{k}$, includes the highest weight modules (cf. [10, Chap.7]).

We now study the associated varieties $\mathcal{V}(\mathfrak{g}; H)$ of $U(\mathfrak{g})$-modules H in $C(\mathfrak{k}, \mathcal{Z})$. Identifying $\mathfrak{g}^*$ with $\mathfrak{g}$ through the Killing form of $\mathfrak{g}$, we regard $\mathcal{V}(\mathfrak{g}; H)$ as a variety in $\mathfrak{g}$. For a subset $\mathfrak{s}$ of $\mathfrak{g}$, let $\mathcal{N}(\mathfrak{s})$ denote the set of nilpotent elements of $\mathfrak{g}$ contained in $\mathfrak{s}$.

Lemma 3.1. (cf. [18, Cor.5.13]) *Let $Q(\mathfrak{k}, \mathcal{Z}) = \mathrm{gr}\ B(\mathfrak{k}, \mathcal{Z})$ be the graded subalgebra of $S(\mathfrak{g})$ corresponding to $B(\mathfrak{k}, \mathcal{Z}) = U(\mathfrak{k})\mathcal{Z}(\mathfrak{g})$. Then the variety $Q(\mathfrak{k}, \mathcal{Z})^{\#}_{+}$ (see* (2.1)) *is contained in $\mathcal{N}(\mathfrak{p})$, and hence, by* (2.3), *it holds that*

$$\mathcal{V}(\mathfrak{g}; H) \subset Q(\mathfrak{k}, \mathcal{Z})^{\#}_{+} \subset \mathcal{N}(\mathfrak{p}) \tag{3.1}$$

for every $U(\mathfrak{g})$-module H in the category $C(\mathfrak{k}, \mathcal{Z})$. Here $\mathfrak{p} := \mathfrak{k}^{\perp}$ denotes the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form of $\mathfrak{g}$.

It shoud be noted that $\mathfrak{p}$ is an $(\mathrm{ad}\ \mathfrak{k})$-stable subspace of $\mathfrak{g}$.

For symmetrizing $\mathfrak{k}$ we can construct a Harish-Chandra $(\mathfrak{g}, \mathfrak{k})$-module $\tilde{H}$ whose associated variety $\mathcal{V}(\mathfrak{g}; \tilde{H})$ is exactly the whole nilpotent variety $\mathcal{N}(\mathfrak{p})$. For this, we need the following

Proposition 3.1. *Let $\mathfrak{k}$ be a Lie subalgebra of $\mathfrak{g}$ such that $\mathfrak{k} \cap \mathfrak{p} = (0)$ for $\mathfrak{p} = \mathfrak{k}^{\perp}$.*

(1) *One has $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ as* $(\mathrm{ad}\ \mathfrak{k})$*-modules.*

(2) *The $U(\mathfrak{g})$-module*

$$\tilde{H} := U(\mathfrak{g})/U(\mathfrak{g})(\mathfrak{k} + U(\mathfrak{g})^{K}_{+}) \tag{3.2}$$

lies in the category $C(\mathfrak{k}, \mathcal{Z})$, and its associated variety is described as

$$\mathcal{V}(\mathfrak{g}; \tilde{H}) = (S(\mathfrak{p})^{K})^{\#}_{+} \cap \mathfrak{p}. \tag{3.3}$$

Here $U(\mathfrak{g})^{K}_{+}$ (resp. *$S(\mathfrak{p})^{K}$*) *denotes the set of elements D in $\mathfrak{g}U(\mathfrak{g})$* (resp. *in $S(\mathfrak{p})$*) *such that* $(\mathrm{ad}\, X)D = 0$ *for all $X \in \mathfrak{k}$.*

A nilpotent element $X \in \mathfrak{p}$ is called *normal* if there exists an element $T \in \mathfrak{k}$ and a non-zero complex number β such that $[T, X] = \beta X$. Let $\mathcal{N}_{nor}(\mathfrak{p})$ denote the set of normal nilpotent elements in $\mathfrak{p}$.

We now arrive at

Proposition 3.2. (1) *Let $\mathfrak{k}$, $\mathfrak{p} = \mathfrak{k}^{\perp}$, and $\tilde{H}$ be as in* Proposition 3.1. *Then it holds that*

$$\mathcal{N}_{nor}(\mathfrak{p}) \subset \mathcal{V}(\mathfrak{g}; \tilde{H}) \subset \mathcal{N}(\mathfrak{p}). \tag{3.4}$$

(2) *Assume $\mathfrak{k}$ be symmetrizing. Then one has $\mathfrak{k} \cap \mathfrak{p} = (0)$, and the equalities hold in* (3.4). *Hence $\tilde{H}$ is a Harish-Chandra $(\mathfrak{g}, \mathfrak{k})$-module such that $\mathcal{V}(\mathfrak{g}; \tilde{H}) = \mathcal{N}(\mathfrak{p})$.*

The following is an immediate consequence of Lemma 3.1 and Proposition 3.2(2).

Corollary 3.1. (see Remark to Proposition 2.2) *Assume that $\mathfrak{k}$ is symmetrizing, and let $\mathcal{V}_{B(\mathfrak{k},\mathcal{Z})}$ be the subset of $Q(\mathfrak{k}, \mathcal{Z})^{\#}_{+}$ defined in* Proposition 2.2, *where $B(\mathfrak{k}, \mathcal{Z}) = U(\mathfrak{k})\mathcal{Z}(\mathfrak{g})$ and $Q(\mathfrak{k}, \mathcal{Z}) = \mathrm{gr}\ B(\mathfrak{k}, \mathcal{Z})$ as before. Then one has*

$$\mathcal{V}_{B(\mathfrak{k},\mathcal{Z})} = Q(\mathfrak{k}, \mathcal{Z})^{\#}_{+} = \mathcal{N}(\mathfrak{p}). \tag{3.5}$$

Remark. It is interesting to describe the associated varieties $\mathcal{V}(\mathfrak{g}; H)$ for important Harish-Chandra $(\mathfrak{g}, \mathfrak{k})$-modules H. We can achieve this for the discrete series of a semisimple Lie group by an elementary method based on Hotta-Parthasarathy's work [11] (see also [21]). The details will be discussed elsewhere.

3.2. Characterization of large subalgebras relative to $B(\mathfrak{k}, \mathcal{Z})$. Let A, $\ni 1$, be a subalgebra of $U(\mathfrak{g})$, and $\mathfrak{k}$ be a Lie subalgebra of $\mathfrak{g}$. Consider the following two conditions on A in relation to $\mathfrak{k}$.

(NPRO) $\mathcal{N}(\mathfrak{p}) \cap R_+^\# = (0)$, where $\mathfrak{p} = \mathfrak{k}^\perp$ and $R = \mathrm{gr}\ A$.

(ALKZ) A is large relative to $B(\mathfrak{k}, \mathcal{Z})$, i.e., all the $U(\mathfrak{g})$-modules H in the category $C(\mathfrak{k}, \mathcal{Z})$ have the good restriction to A. So, in this case, these modules H have the properties specified in 2.3.

Getting together the results in 2.2 and 3.1, we find a close relation between these conditions as follows, which is one of the most important results of this article.

Theorem 3.1. *For A and $\mathfrak{k}$ as above, the condition* (NPRO) *always implies* (ALKZ). *Moreover, these two conditions are equivalent with each other if $R = \mathrm{gr}\ A$ is Noetherian and if $\mathfrak{k}$ is symmetrizing.*

As a special case, we obtain the following criterion.

Theorem 3.2. ($\mathfrak{k}$: symmetrizing, $A = U(\mathfrak{n})$) *All the Harish-Chandra $(\mathfrak{g}, \mathfrak{k})$-module have the good restriction to a Lie subalgebra $\mathfrak{n}$ of $\mathfrak{g}$ if and only if there does not exist any non-zero nilpotent element of $\mathfrak{g}$ orthogonal to $\mathfrak{k} + \mathfrak{n}$ with respect to the Killing form:*

$$\mathcal{N}((\mathfrak{k} + \mathfrak{n})^\perp) = \mathfrak{n}^\perp \cap \mathcal{N}(\mathfrak{p}) = (0). \tag{3.6}$$

By applying Proposition 2.4, one gets another consequence of the condition (NPRO) as in

Proposition 3.3. *Let $\mathfrak{k}$, A be as in* Theorem 3.1, *and let I be a proper, right ideal of $U(\mathfrak{g})$ such that $A/A \cap I$ is finite-dimensional. If* (NPRO) *is satisfied, the factor space H/IH is finitely generated as a $\mathcal{Z}(\mathfrak{g})$-module for every locally $\mathfrak{k}$-finite, finitely generated $U(\mathfrak{g})$-module H.*

4. Large Lie subalgebras of a real semisimple Lie algebra.

Let $\mathfrak{g}_0$ be, throughout this section, a real semisimple Lie algebra, and $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the Cartan decomposition of $\mathfrak{g}_0$ determined by an involution θ. We write $\mathfrak{h}$ ($\subset \mathfrak{g}$) for the complexification of a real vector subspace $\mathfrak{h}_0$ of $\mathfrak{g}_0$ by dropping the subscript '0'.

A Lie subalgebra $\mathfrak{n}_0$ of $\mathfrak{g}_0$ is said to be *large* in $\mathfrak{g}_0$ if there exists an element $x \in \mathrm{Int}(\mathfrak{g}_0)$ for which the subalgebra $U(x\cdot\mathfrak{n})$ is large in $U(\mathfrak{g})$ relative to $B(\mathfrak{k},\mathcal{Z}) = U(\mathfrak{k})\mathcal{Z}(\mathfrak{g})$ (see (ALKZ) in 3.2). This amounts to, thanks to Theorem 3.2, a simple geometric condition:

$$(x\cdot\mathfrak{n})^{\perp} \cap \mathcal{N}(\mathfrak{p}) = (0) \quad \text{for some } x \in \mathrm{Int}(\mathfrak{g}_0). \tag{4.1}$$

Here $\mathrm{Int}(\mathfrak{g}_0)$ denotes the group of inner automorphisms of $\mathfrak{g}_0$. Notice that the largeness of a Lie subalgebra does not depend on the choice of a $\mathfrak{k}_0$, since such $\mathfrak{k}_0$'s are conjugate with each other by inner automorphisms.

This section specifies many of large Lie subalgebras of $\mathfrak{g}_0$, and we find that every quasi-spherical Lie subalgebra (cf. [3], [15]) is large in $\mathfrak{g}_0$.

4.1. Two kinds of typical large Lie subalgebras. Let $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{a}_{p,0} + \mathfrak{u}_{m,0}$ be an Iwasawa decomposition of $\mathfrak{g}_0$. Here is the first important example of large Lie subalgebras of $\mathfrak{g}_0$.

Proposition 4.1. *The maximal nilpotent Lie subalgebra $\mathfrak{u}_{m,0}$ is large in $\mathfrak{g}_0$.*

The above proposition, together with Theorem 2.2, covers the results of Casselman-Osborne[8, Th.2.3] and Joseph[13, II, 5.6] on the restriction of Harish-Cahndra modules to $\mathfrak{u}_m$.

Secondly, let $\mathfrak{h}_0$ be any symmetrizing Lie subalgebra of $\mathfrak{g}_0$ defined by an involutive automorphism σ of $\mathfrak{g}_0$. Then there exists an inner automorphism y of $\mathfrak{g}_0$ such that $\sigma_y := y \circ \sigma \circ y^{-1}$ commutes with the Cartan involution θ. Let $\mathfrak{g}_0 = y\cdot\mathfrak{h}_0 \oplus \mathfrak{s}_0$ be the eigenspace decomposition of $\mathfrak{g}_0$ by σ_y. Take a maximal abelian subspace $\mathfrak{a}_{ps,0}$ of $\mathfrak{p}_0 \cap \mathfrak{s}_0$, and an element $X' \in \mathfrak{a}_{ps,0}$ which is regular in the sense: $\dim \mathrm{Ker}(\mathrm{ad}\, X')$ is minimal among the elements of $\mathfrak{a}_{ps,0}$. Then one has a Cartan decomposition of $\mathfrak{g}_0$ with respect to $y\cdot\mathfrak{h}_0$ as

$$\mathfrak{g}_0 = (\mathfrak{k}_0 + x'y\cdot\mathfrak{h}_0) \oplus \mathfrak{a}_{ps,0}, \tag{4.2}$$

where $x' = \exp(\mathrm{ad}\, X')$, and $\mathfrak{a}_{ps,0}$ is orthogonal to $\mathfrak{k}_0 + x'y\cdot\mathfrak{h}_0$ with respect to the Killing form. See [20, I, Lemma 1.9] for the proof of (4.2). We thus deduce

$$(x'y\cdot\mathfrak{h})^{\perp} \cap \mathcal{N}(\mathfrak{p}) = \mathcal{N}(\mathfrak{a}_{ps}) = (0), \tag{4.3}$$

because the elements of $\mathfrak{a}_{ps}$ are semisimple, and so this gives the second typical example of large Lie subalgebras.

Proposition 4.2. *Any symmetrizing subalgebra $\mathfrak{h}_0$ is large in $\mathfrak{g}_0$.*

This allows us to deduce the finite multiplicity theorem [1] for the quasi-regular representation on $L^2(G/H)$, associated to a semisimple symmetric space G/H.

4.2. Inheritance of the largeness by parabolic induction. Let $\mathfrak{q}_0$ be any parabolic subalgebra of $\mathfrak{g}_0$, and $\mathfrak{q}_0 = \mathfrak{l}_0 + \mathfrak{u}_0$ with $\mathfrak{l}_0 = \mathfrak{q}_0 \cap \theta\mathfrak{q}_0$, be its Levi decomposition. Since the Levi component $\mathfrak{l}_0 = (\mathfrak{k} \cap \mathfrak{l}_0) + (\mathfrak{p} \cap \mathfrak{l}_0)$ is reductive, one can define large Lie subalgebras of $\mathfrak{l}_0$ just in the same way.

The largeness of Lie subalgebras is preserved by parabolic induction.

Lemma 4.1. *If $\mathfrak{h}_0$ is a large Lie subalgebra of $\mathfrak{l}_0$, the semidirect product Lie subalgebra $\mathfrak{h}_0 + \mathfrak{u}_0$ is large in $\mathfrak{g}_0$.*

Thanks to this lemma, we can generalize Proposition 4.2 to

Proposition 4.3. (cf. [20]) *Let $\mathfrak{h}_0$ be a symmetrizing subalgebra of the Levi factor $\mathfrak{l}_0$ of a parabolic subalgebra $\mathfrak{q}_0 = \mathfrak{l}_0 + \mathfrak{u}_0$. Then $\mathfrak{h}_0 + \mathfrak{u}_0$ is large in $\mathfrak{g}_0$.*

This proposition actually contains Proposition 4.2 as a special case $\mathfrak{q}_0 = \mathfrak{g}_0$.

Using this proposition, we can recover our finite multiplicity theorems for induced representations of semisimple Lie groups, given in [20, I]. See 5.4 for the details.

4.3. Quasi-spherical Lie subalgebras. Let $\mathfrak{q}_{m,0} = \mathfrak{m}_0 + \mathfrak{a}_{p,0} + \mathfrak{u}_{m,0}$ be a minimal parabolic subalgebra of $\mathfrak{g}_0$, where $\mathfrak{m}_0$ denotes the centralizer of $\mathfrak{a}_{p,0}$ in $\mathfrak{k}_0$. We say that a Lie subalgebra $\mathfrak{n}_0$ of $\mathfrak{g}_0$ is *quasi-spherical* if there exists a $z \in \mathrm{Int}(\mathfrak{g}_0)$ such that $z\mathfrak{n}_0 + \mathfrak{q}_{m,0} = \mathfrak{g}_0$. This is equivalent to saying that, if G is a connected Lie group with Lie algebra $\mathfrak{g}_0$, the analytic subgroup of G corresponding to $\mathfrak{n}_0$ has an open orbit on the maximal flag variety G/Q_m, where Q_m denotes a minimal parabolic subgroup of G.

It is easy to verify that the large Lie subalgebras specified in 4.1-4.2 are all quasi-spherical.

The following theorem is the principal result of this section.

Theorem 4.1. *Quasi-spherical Lie subalgebras are always large in $\mathfrak{g}_0$.*

Remark. One can see from Theorem 3.2, coupled with a recent result of Bien-Oshima, that the converse is also true in the above theorem if $\mathfrak{n}_0$ is algebraic in $\mathfrak{g}_0$, i.e., $\mathfrak{n}_0$ is the Lie algebra of an algebraic subgroup N of G, where G is a semisimple algebraic group with Lie algebra $\mathfrak{g}_0$.

In fact, it is easy to deduce from our Theorem 3.2 that, if $\mathfrak{n}_0$ is large in $\mathfrak{g}_0$, the induced representations $\mathrm{Ind}_N^G(\eta)$ have the finite multiplicity property for all finite-dimensional N-representations η (see 5.4; for this, $\mathfrak{n}_0$ need not to be algebraic). A result of Bien-Oshima assures that, under the above assumption, these representations $\mathrm{Ind}_N^G(\eta)$ are of multiplicity finite only when $\mathfrak{n}_0$ is quasi-spherical.

5. Finite multiplicity theorems for induced representations.

Let G be any connected Lie group with Lie algebra $\mathfrak{g}_0$ (not necessarily semisimple), and A, $\ni 1$, be a subalgebra of $U(\mathfrak{g})$ with $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbf{R}} \mathbf{C}$. Following the idea of induced representations, we can associate, to any given Fréchet A-module E, an analytically induced G- and $U(\mathfrak{g})$-module $\Gamma(G\uparrow A; E)$ (see 5.1).

This section makes clear what we can know about these modules $\Gamma(G \uparrow A; E)$ by applying our results in §§2-4 (see Theorems 5.1 and 5.2). Moreover, for semisimple G, we largely develop and simplify our previous work [20] on the finiteness of multiplicities in induced representations, by making use of the associated varieties of Harish-Chandra modules (see Theorems 5.3-5.5).

5.1. Analytically induced modules $\Gamma(G\uparrow A; E)$ and $\mathcal{A}(G;\eta)$. We begin with the precise definition of our induced modules. Let A be as above, and E be an A-module with Fréchet space structure on which the elements of A act as continuous linear operators.

We then define $\Gamma = \Gamma(G\uparrow A; E)$ to be the space of all E-valued, real analytic functions f on G satisfying

$$R_D f(x) = {}^tD\cdot f(x) \tag{5.1}$$

for $D \in {}^tA$ and $x \in G$. Here $D \to {}^tD$ is the principal anti-automorphism of $U(\mathfrak{g})$ (see 2.2), and $D \to R_D$ identifies $U(\mathfrak{g})$ with the algebra of left invariant differential operators on G. The group G acts on Γ by left translation L:

$$L_g f(x) = f(g^{-1}x) \quad (g \in G). \tag{5.2}$$

The $U(\mathfrak{g})$-action on Γ, gained by differentiation, will be denoted again by L. We call $(L, \Gamma(G\uparrow A; E))$ the *G-representation* or *$U(\mathfrak{g})$-module analytically induced from E.*

If (η, E) is a smooth Fréchet representation (cf. [20, I, 2.1]) of a closed subgroup N of G, the real analytic functions $f : G \to E$ such that

$$f(gn) = \eta(n)^{-1} f(g) \text{ for } (n, g) \in N \times G,$$

form a G-submodule, say $\mathcal{A}(G;\eta)$, of $\Gamma(G \uparrow U(\mathfrak{n}); E)$. Here $\mathfrak{n}$ is the complexified Lie algebra of N, and E is viewed as a $U(\mathfrak{n})$-module through differentiation. In this sense our $\Gamma(G\uparrow A; E)$'s include the group theorical (analytically) induced modules $\mathcal{A}(G;\eta)$.

Now let H be a $U(\mathfrak{g})$-module. We discuss $U(\mathfrak{g})$-homomorphisms from H to $\Gamma = \Gamma(G\uparrow A; E)$ and especially the intertwining number

$$I_{U(\mathfrak{g})}(H, \Gamma) := \dim \operatorname{Hom}_{U(\mathfrak{g})}(H, \Gamma). \tag{5.3}$$

When H is irreducible, $I_{U(\mathfrak{g})}(H, \Gamma)$ gives the multiplicity of H in Γ as $U(\mathfrak{g})$-submodules.

Fix an element $x \in G$. If T is a $U(\mathfrak{g})$-homomorphism from H to Γ,

$$\iota_x(T)(v) := (Tv)(x) \quad (v \in H) \tag{5.4}$$

gives rise to a linear map $\iota_x(T)$ from H to E. It is easily verified that $\iota_x(T)$ commutes with the actions of $xA := \mathrm{Ad}(x)A \subset U(\mathfrak{g})$ as

$$\iota_x(T) \circ D = (x^{-1}D) \circ \iota_x(T) \tag{5.5}$$

for all $D \in xA$, where $x^{-1}D = \mathrm{Ad}(x)^{-1}D$. Moreover, $\iota_x(T) = 0$ implies $T = 0$, since Tv $(v \in H)$ are real analytic functions on connected G.

We have thus obtained a half part of the Frobenius reciprocity for induced modules, as follows.

Proposition 5.1. *Let H, $\Gamma = \Gamma(G \uparrow A; E)$ and $x \in G$ be as above. The assignment $T \to \iota_x(T)$ defined in (5.4) gives an injective linear map*

$$\iota_x : \mathrm{Hom}_{U(\mathfrak{g})}(H, \Gamma) \hookrightarrow \mathrm{Hom}_{xA}(H, E_x), \tag{5.6}$$

where E_x stands for the Fréchet space E viewed as an (xA)-module by $D \cdot e = (x^{-1}D)e$ $(e \in E)$.

This proposition allows us to give in the succeeding subsections criteria for the finiteness of intertwining numbers $I_{U(\mathfrak{g})}(H, \Gamma)$ by means of the associated varieties of H and A.

5.2. Finite multiplicity criteria, I. First, observe that the vector space $\mathrm{Hom}_{xA}(H, E_x)$ in (5.6) is finite-dimensional if so are both A-module E and factor space H/I_xH with $I_x := (\mathrm{Ann}_{xA}E_x)U(\mathfrak{g})$. Corollary 2.4 together with Proposition 5.1 gives the following finiteness criterion, which is the first important result of this section.

Theorem 5.1. *Let H be a finitely generated $U(\mathfrak{g})$-module. The intertwining number $I_{U(\mathfrak{g})}(H, \Gamma)$ from H to an analytically induced $U(\mathfrak{g})$-module $\Gamma = \Gamma(G \uparrow A; E)$ is finite whenever two conditions:*

$$\dim E < \infty, \tag{5.7}$$

and

$$\mathcal{V}(\mathfrak{g}; H) \cap x^{-1} \cdot R_+^\# = (0) \quad \textit{for some } x \in G, \tag{5.8}$$

are satisfied. Here $\mathcal{V}(\mathfrak{g}; H)$ is the associated variety of H, $R_+^\#$ with $R = \mathrm{gr}\ A$, is the algebraic variety of $\mathfrak{g}^*$ defined in 2.1, and G acts on $\mathfrak{g}^*$ through the coadjoint representation.

For a subalgebra $B, \ni 1$, of $U(\mathfrak{g})$, let $C(B)$ be as in 2.2 the category of locally B-finite, finitely generated $U(\mathfrak{g})$-modules. The above theorem together with (2.6) immediately gives

Corollary 5.1. *Let $R = \mathrm{gr}\ A$, $Q = \mathrm{gr}\ B$ be the graded subalgebras of $S(\mathfrak{g})$ associated to subalgebras A, $B \subset U(\mathfrak{g})$ respectively. If there exists an element $x \in G$ such that $R_+^\# \cap x \cdot Q_+^\# = (0)$, the intertwining number $I_{U(\mathfrak{g})}(H, \Gamma(G \uparrow A; E))$ is finite for every $U(\mathfrak{g})$-module H in $C(B)$ and for every A-module E of finite dimension.*

5.3. Estimation of the multiplicities. Let $\mathfrak{k}$ be a Lie subalgebra of $\mathfrak{g}$, and H be a $U(\mathfrak{g})$-module in the category $C(B)$ with $B = U(\mathfrak{k})$. Take a finite-dimensional, B-stable, generating subspace H_0. By noting that B is generated by 1 and $\mathfrak{k}$ as algebra, it is easy to see that the subspaces $H_k = U_k(\mathfrak{g})H_0$ $(k = 0, 1, \ldots)$ are all B-stable. Hence the corresponding graded $S(\mathfrak{g})$-module $M = \mathrm{gr}(H; H_0) = \bigoplus_{k\geq 0} M_k$ with $M_k = H_k/H_{k-1}$, admits a natural B-module structure. Write this B-action on M by

$$B \times M \ni (D, v) \to D \diamond v \in M,$$

in order to distinguish it from the original $S(\mathfrak{g})$-action. One finds from the definition,

$$X \diamond Dv - D(X \diamond v) = ((\mathrm{ad}\, X)D)v \tag{5.9}$$

for $X \in \mathfrak{k}$ and $D \in S(\mathfrak{g})$.

With (5.6) in mind, we can give, by using this $(S(\mathfrak{g}), B)$-module M, an upper bound of the intertwining number $I_A(H, E) = \dim \mathrm{Hom}_A(H, E)$ as in

Proposition 5.2. *Let H, $B = U(\mathfrak{k})$ be as above, and A, $\ni 1$, be a subalgebra of $U(\mathfrak{g})$. One has for any A-module E,*

$$I_A(H, E) \leq \sum_{k=0}^{\infty} I_{A\cap B}(M_k/((A \cap B) \diamond (R_+M)_k), E). \tag{5.10}$$

where $R = \mathrm{gr}\ A$ and $(R_+M)_k := R_+M \cap M_k$.

This together with (5.6) immediately gives the following theorem.

Theorem 5.2. *The intertwining number $I_{U(\mathfrak{g})}(H, \Gamma)$ from H in $C(B)$, $B = U(\mathfrak{k})$, to an analytically induced $U(\mathfrak{g})$-module $\Gamma = \Gamma(G \uparrow A; E)$ is bounded by*

$$\min_{x\in G}\{\sum_{k=0}^{\infty} I_{xA\cap B}(M_k/((xA \cap B) \diamond ((xR_+)M)_k), E_x)\}. \tag{5.11}$$

Remarks. (1) By setting $\mathfrak{k} = (0)$, or $B = \mathbf{C}1$, one finds that this theorem recovers Theorem 5.1. In fact, in this case (5.11) turns to be

$$\dim E \times \{\min_{x\in G}(\dim\ (M/(xR_+)M))\},$$

which is finite under the assumptions (5.7) and (5.8) (see Proposition 2.1).

(2) If $A = U(\mathfrak{n})$ for a Lie subalgebra $\mathfrak{n}$ of $\mathfrak{g}$, one finds from the Poincaré- Birkhoff-Witt theorem that $xA \cap B = U(x\cdot\mathfrak{n} \cap \mathfrak{k})$ with $x\cdot\mathfrak{n} = \mathrm{Ad}(x)\mathfrak{n}$. Hence, in view of (5.9) we have

$$(xA \cap B) \diamond ((xR_+)M)_k = ((xR_+)M)_k = ((x\cdot\mathfrak{n})M)_k$$

in (5.11).

5.4. Finite multiplicity criteria, II: case of semisimple Lie groups. Now assume G be a connected semisimple Lie group with finite center, and let K be a maximal compact subgroup of G. In this subsection we apply the results of 5.2 and 5.3 to Harish-Chandra modules for G.

By keeping the notation in §4, $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ with $\mathfrak{k}_0 = \mathrm{Lie}(K)$, denotes a Cartan decomposition of $\mathfrak{g}_0 = \mathrm{Lie}(G)$. Let H be a Harish-Chandra $(\mathfrak{g}, \mathfrak{k})$-module (see 3.1). Assume that the compact group K acts on H in such a way as

$$\dim\{Kv\} < \infty,$$

and

$$(d/dt)_{t=0}(\exp tX)v = Xv$$

for $v \in H$ and $X \in \mathfrak{k}_0$, where $\{Kv\}$ stands for the K-submodule of H generated by v. Such an H is called a *Harish-Chandra* $(\mathfrak{g}, K)$-*module.* Observe that, since our K is connected, the above two conditions assure the compatibility of $\mathfrak{g}$ and K actions:

$$k \cdot Xv = (k \cdot X) \cdot k^{-1}v$$

for $k \in K$ and $X \in \mathfrak{g}$, where $k \cdot X = \mathrm{Ad}(k)X$.

We note that, if a Harish-Chandra $(\mathfrak{g}, \mathfrak{k})$-module H appears in some $\Gamma = \Gamma(G \uparrow A; E)$ as a $U(\mathfrak{g})$-submodule, H necessarily has the $(\mathfrak{g}, K)$-module structure inherited from Γ. A fundamental theorem of Harish-Chandra says that the (irreducible) Harish-Chandra $(\mathfrak{g}, K)$-modules correspond to the (irreducible) admissible representations of G, by passing to the K-finite part (see e.g., [19, Chap.8]). From these two reasons we concentrate on the $(\mathfrak{g}, K)$-modules from now on.

Definition. Let $\Gamma = \Gamma(G \uparrow A; E)$ and $\mathcal{A} = \mathcal{A}(G; \eta)$ be the induced G- and $U(\mathfrak{g})$-modules defined in 5.1. We say that Γ (resp. $\mathcal{A}$) has the *finite multiplicity property* if the intertwining number $I_{U(\mathfrak{g})}(H, \Gamma)$ (resp. $I_{U(\mathfrak{g})}(H, \mathcal{A})$) is finite for every Harish-Chandra $(\mathfrak{g}, K)$-module H.

Remarks. (1) Any $U(\mathfrak{g})$-homomorphism from H to Γ or to $\mathcal{A}$ commutes with the K-actions by virtue of the connectedness of K.

(2) When η is a finite-dimensional unitary representation of a closed subgroup N, the assignment $H \to I_{U(\mathfrak{g})}(H, \mathcal{A})$ gives an upper bound of the multiplicity function for G-representation $L^2\text{-}\mathrm{Ind}_N^G(\eta)$ unitarily induced from η. Here H runs over the Harish-Chandra $(\mathfrak{g}, K)$-modules associated with irreducible unitary representations of G. See [20, I, §3] for the details.

Here is our first application to semisimple group G, of the general results in 5.2-5.3, which follows immediately from Corollaries 3.1 and 5.1.

Proposition 5.3. *The induced module* $\Gamma(G \uparrow A; E)$ *has the finite multiplicity property for any finite-dimensional* A*-module* E*, if there exists an* $x \in G$ *for which* $\mathcal{N}(\mathfrak{p}) \cap x \cdot (\mathrm{gr}\, A)_+^\# = (0)$ (cf. (NPRO) in 3.2). *Here* $\mathcal{N}(\mathfrak{p})$ *is the totality of nilpotent elements in* $\mathfrak{p}$.

As a special case, we gain

Corollary 5.2. *If* $\mathfrak{n}_0$ *is a large Lie subalgebra of* $\mathfrak{g}_0$ *(see §4), the conclusion of* Proposition 5.3 *is true for* $A = U(\mathfrak{n})$ *with* $\mathfrak{n} = \mathfrak{n}_0 \otimes_{\mathbf{R}} \mathbf{C}$.

In view of the large Lie subalgebras specified in §4, one may realize that this corollary has numbers of applications.

Remark. For quasi-spherical Lie subalgebras $\mathfrak{n}_0$ (see 4.3), Bien-Oshima recently got a result similar to the above corollary. But our method here is completely different from theirs.

Now let (η, E) be a smooth Fréchet representation of a closed subgroup N of G, and consider the induced module $\mathcal{A}(G;\eta)$. For a Harish-Chandra $(\mathfrak{g}, K)$-module H, take a finite-dimensional, K-stable generating subspace H_0 of H. Then the associated graded $S(\mathfrak{g})$-module $M = \mathrm{gr}(H; H_0) = \bigoplus_k M_k$ has a natural K-module structure.

We can estimate the intertwining number $I_{U(\mathfrak{g})}(H, \mathcal{A}(G;\eta))$ from H to $\mathcal{A}(G;\eta)$:

Theorem 5.3. *For each* $x \in G$, *one has the inequality*

$$I_{U(\mathfrak{g})}(H, \mathcal{A}(G;\eta)) \leq \sum_{k=0}^{\infty} I_{K\cap xNx^{-1}}(M_k/((x\cdot\mathfrak{n})M)_k, E_x), \tag{5.12}$$

where $((x\cdot\mathfrak{n})M)_k = M_k \cap (x\cdot\mathfrak{n})M$, *and* (η_x, E_x) *is the representation of* xNx^{-1} *on* E *defined by* $\eta_x(xnx^{-1}) = \eta(n)$ $(n \in N)$.

This theorem enables us to deduce useful criteria for the finiteness of intertwining numbers $I_{U(\mathfrak{g})}(H, \mathcal{A}(G;\eta))$, which are applicable even to infinite-dimensional (η, E)'s.

To be specific, fix an $x \in G$, and let Π denote the set of equivalence classes of irreducible finite-dimensional representations of $K \cap xNx^{-1}$. Then the locally finite $(K \cap xNx^{-1})$-module $M/(x\cdot\mathfrak{n})M = \bigoplus_k M_k/((x\cdot\mathfrak{n})M)_k$ is decomposed into a direct sum of the irreducibles as

$$M/(x\cdot\mathfrak{n})M \simeq \oplus_{\gamma\in\Pi}[m_\gamma]V_\gamma,$$

where V_γ is an irreducible $(K\cap xNx^{-1})$-module of class γ, and m_γ denotes the multiplicity of γ in $M/(x\cdot\mathfrak{n})M$.

One finds that (5.12) is rewritten as

$$I_{U(\mathfrak{g})}(H, \mathcal{A}(G;\eta)) \leq \sum_{\gamma\in\Pi} m_\gamma I_{K\cap xNx^{-1}}(V_\gamma, E_x). \tag{5.13}$$

The sum in the right hand side is finite if and only if there exists a finite subset F of Π for which

$$m_\gamma = 0 \;\text{ or }\; I_{K\cap xNx^{-1}}(V_\gamma, E_x) = 0 \;\text{ for } \gamma \notin F, \tag{5.14}$$

and

$$I_{K\cap xNx^{-1}}(V_\gamma, E_x) < \infty \;\text{ for } \gamma \in F. \tag{5.15}$$

The above discussion coupled with Corollary 2.2 leads us to the following

Theorem 5.4. *Under the above notation, the intertwining number $I_{U(\mathfrak{g})}(H, \mathcal{A}(G;\eta))$ from a Harish-Chandra module H to an induced $U(\mathfrak{g})$-module $\mathcal{A}(G;\eta)$ takes finite value if there exists an $x \in G$ such that*

$$\mathcal{V}(\mathfrak{g}; H) \cap (x\cdot\mathfrak{n})^{\perp} = (0), \tag{5.16}$$

and that

$$I_{K\cap xNx^{-1}}(V_\gamma, E_x) < \infty \quad \textit{holds} \tag{5.17}$$

for every irreducible constituent V_γ of $M/(x\cdot\mathfrak{n})M$. Here $M = \mathrm{gr}(H; H_0)$, and $\mathcal{V}(\mathfrak{g}; H)$ denotes the associated variety of H.

From this theorem, we immediately deduce an interesting criterion for $\mathcal{A}(G;\eta)$ to be of multiplicity finite, as follows.

Theorem 5.5. *Let N be a closed subgroup of G whose Lie algebra $\mathfrak{n}_0$ is large in $\mathfrak{g}_0$, and take an element $x \in G$ such that $(x\cdot\mathfrak{n})^{\perp} \cap \mathcal{N}(\mathfrak{p}) = (0)$. Then, for a smooth Fréchet representation (η, E) of N, the induced module $\mathcal{A}(G;\eta)$ has the finite multiplicity property if so is the restriction of η to the compact subgroup $x^{-1}Kx \cap N$.*

This theorem extends one of the principal results in our previous work, [20, I, Th.2.12], where we studied the case of semidirect product large Lie subalgebras $\mathfrak{n}_0 = \mathfrak{h}_0 + \mathfrak{u}_0$ specified in Proposition 4.3, through the theory of (K, N)-spherical functions. Interesting applications are found in [20, II] for reduced generalized Gelfand-Graev representations.

5.5. Relation with K-harmonic polynomials on $\mathfrak{p}$. We conclude this article by relating the $(K \cap xNx^{-1})$-module $M/(x\cdot\mathfrak{n})M$ in Theorems 5.3 and 5.4, with K-harmonic polynomials on $\mathfrak{p}$.

As in §3, regard the elements of $S(\mathfrak{p})$ as polynomial functions on $\mathfrak{p}$ through the Killing form of $\mathfrak{g}$. An element $f \in S(\mathfrak{p})$ is called *K-harmonic* if f is annihilated by every $\mathrm{Ad}(K)$-invariant, constant coefficient differential operator on $\mathfrak{p}$ without constant term. Let $\mathcal{H}(\mathfrak{p})$ denote the totality of K-harmonic polynomials on $\mathfrak{p}$. It is easily observed that $\mathcal{H}(\mathfrak{p})$ is a graded K-submodule of $S(\mathfrak{p})$: $\mathcal{H}(\mathfrak{p}) = \oplus_{k\geq 0}\mathcal{H}^k(\mathfrak{p})$, where $\mathcal{H}^k(\mathfrak{p}) := \mathcal{H}(\mathfrak{p}) \cap S^k(\mathfrak{p})$ is K-stable.

A result of Kostant and Rallis (cf. [12, p.381]) says that the multiplication $(h, j) \to hj$ $(h \in \mathcal{H}(\mathfrak{p}),\ j \in S(\mathfrak{p})^K)$ gives a K-isomorphism

$$\mathcal{H}(\mathfrak{p}) \otimes S(\mathfrak{p})^K \simeq S(\mathfrak{p}), \tag{5.18}$$

where $S(\mathfrak{p})^K$ is the algebra of $\mathrm{Ad}(K)$-fixed elements of $S(\mathfrak{p})$. This implies

$$S(\mathfrak{p}) = \mathcal{H}(\mathfrak{p}) \oplus (\mathcal{H}(\mathfrak{p}) \otimes S(\mathfrak{p})^K_+) \tag{5.19}$$

as K-modules, with $S(\mathfrak{p})^K_+ = S(\mathfrak{p})^K \cap \mathfrak{p}S(\mathfrak{p})$ as before. The linear projection from $S(\mathfrak{p})$ to $\mathcal{H}(\mathfrak{p})$ along this decomposition will be denoted by α.

For any Harish-Chandra $(\mathfrak{g}, K)$-module H, we can and do take a finite-dimensional generating subspace $H_0 \subset H$ of the form

$$H_0 = \oplus_{\delta \in \Phi} H(\delta) \tag{5.20}$$

for a finite subset Φ of $\hat{K}$ (= the unitary dual of K), where $H(\delta)$ denotes the δ-isotypic component of H. Noting that H_0 is stable under K and $U(\mathfrak{g})^K$, one sees that the associated graded $(S(\mathfrak{g}), K)$-module $M = \mathrm{gr}(H; H_0) = \bigoplus_k M_k$ is annihilated by $\mathfrak{k}$ and $S(\mathfrak{p})^K_+$. Hence it follows from (5.19) that $M = \mathcal{H}(\mathfrak{p})M_0$ and that

$$\beta : \mathcal{H}(\mathfrak{p}) \otimes M_0 \ni h \otimes v \to hv \in M \tag{5.21}$$

gives a surjective K-homomorphism (cf. [17, Proof of Prop.5.5]). Note that $H_0 \simeq M_0$ as K-modules.

Now let N be any closed subgroup of G with complexified Lie algebra $\mathfrak{n}$. We set

$$\mathcal{H}(\mathfrak{p}; \mathfrak{n}) = \mathcal{H}(\mathfrak{p})/\alpha(p[\mathfrak{n}]S(\mathfrak{p})), \tag{5.22}$$

where $p[\mathfrak{n}]$ denotes the image of $\mathfrak{n}$ by the projection $\mathfrak{g} \to \mathfrak{p}$ along $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Note that $\mathcal{H}(\mathfrak{p}; \mathfrak{n})$ is a $(K \cap N)$-module.

We can relate $(K \cap N)$-module $M/\mathfrak{n}M$ with $\mathcal{H}(\mathfrak{p}; \mathfrak{n})$ as follows.

Proposition 5.4. (1) *The K-homomorphism β in* (5.21) *naturally induces a surjective $(K \cap N)$-module map*

$$\mathcal{H}(\mathfrak{p}; \mathfrak{n}) \otimes M_0 \to M/\mathfrak{n}M. \tag{5.23}$$

(2) *If $\mathcal{N}(\mathfrak{p}) \cap \mathfrak{n}^\perp = (0)$, the space $\mathcal{H}(\mathfrak{p}; \mathfrak{n})$ is finite-dimensional.*

This proposition, combined with Theorem 5.3, allows us to estimate the intertwining number $I_{U(\mathfrak{g})}(H, \mathcal{A}(G; \eta))$ in (5.12) by means of $\mathcal{H}(\mathfrak{p}; \mathfrak{n})$ and $H_0 \simeq M_0$ as in

Corollary 5.3. *Let H be a Harish-Chandra $(\mathfrak{g}, K)$-module and $\mathcal{A}(G; \eta)$ be the G- and $U(\mathfrak{g})$-module analytically induced from a smooth N-representation (η, E). Then one has*

$$I_{U(\mathfrak{g})}(H, \mathcal{A}(G; \eta)) \leq I_{K \cap xNx^{-1}}(\mathcal{H}(\mathfrak{p}; x \cdot \mathfrak{n}) \otimes H_0, E_x) \tag{5.24}$$

for each $x \in G$. Here $M = \mathrm{gr}(H; H_0)$ with H_0 in (5.20), *and xNx^{-1} acts on $E_x = E$ by $xnx^{-1} \to \eta(n)$ $(n \in N)$.*

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