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A Note on Minimal Models

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A model M is said to be *minimal* if there is no proper elementary submodel of M. We consider the size of an indiscernible set in a minimal model. In [2] Shelah showed that if a theory T is ω-stable then there is no infinite indiscernible set in a minimal model of T. On the other hand Marcus [1] constructed a theory having a minimal (and prime) model with an infinite indiscernible set. The theory is stable but non-superstable. In this note we show the following theorem:

THEOREM. *Let T be superstable and let A be any set. Then there is no minimal model over A which has an infinite set of indiscernibles over A.*

1. Notation

We fix a countable stable theory T. We usually work in a big model $\mathbb{C}$ of T. Our notations are fairly standard. $A, B, ...$ are used to denote small subsets of $\mathbb{C}$. $\bar{a}, \bar{b}, ...$ are used to denote finite sequences of elements in $\mathbb{C}$. $\varphi, \psi, ...$ are used to denote formulas (with parameter). $p, q, ...$ are used to denote types (with parameter). The nonforking extension of a stationary types p to the domain A is denoted by $p|A$. The type of a over A is denoted by $tp(a/A)$. $R^\infty(p)$ is the infinity rank of a type p. We simply write $R^\infty(a/A)$ instead of $R^\infty(tp(a/A))$. The set of realizations of a type p (resp. a formula φ) in a model M is denoted by p^M (resp. φ^M).

2. Theorem and Proof

First we prove the following lemma:

Lemma. *Let T be superstable and let A be any set. Let $I = \{a\} \cup J$ be an infinite Morley sequence of some stationary type $p \in S(A)$. Let M be a model containing $I \cup A$. Suppose that B is a maximal set satisfying $J \subset B \subset M$ and $B \downarrow_A a$. Then B is an elementary submodel of M.*

Proof: For the simplicity of the notation, we may assume that $A = \emptyset$. Take any consistent formula $\varphi(x, \bar{b}_0)$ over B. By the Tarski criterion it is enough to see that φ is satisfied by B. By the superstability of T we can pick an element b of φ^M such that $R^\infty(b/B)$ is minimal.

Claim. *b is independent from a over B.*

Proof: Take a formula $\theta(x,\bar{b_1}) \in tp(b/B)$ such that $R^\infty(b/B) = R^\infty(\theta)$. Without loss of generality, we can assume that $\bar{b_0} \subset \bar{b_1}$. Suppose that b and a are not independent over B. By the superstability there is a finite sequences $\bar{b} \in B$ such that $ab \downarrow_{\bar{b}} B$ and $\bar{b}_1 \subset \bar{b}$. Then we obtain that b and a are not independent over $\bar{b}$. So we can get a formula $\psi(x,\bar{b},a)$ such that $\models \psi(b,\bar{b},a)$, and if $\models \psi(b',\bar{b},a)$ then $b' \not\downarrow_{\bar{b}} a$. Let $\Gamma(\bar{b},a)$ denote $(\exists x)(\varphi(x,\bar{b_0}) \wedge \psi(x,\bar{b},a) \wedge \theta(x,\bar{b_1}))$. On the other hand there is a finite subset I' of I such that $I - I'$ is the infinite Morley sequence of $p|\bar{b}$ since $\kappa(T)$ is finite. Moreover we can assume that $a \in I - I'$, since $\bar{b}$ and a are independent. So we can pick some $a' \in J(\subset B)$ such that $\Gamma(\bar{b},a')$ holds. Therefore there is an element $b' \in \varphi^M$ such that $R^\infty(b'/\bar{b}) = R^\infty(b/B)$ and $b' \not\downarrow_{\bar{b}} a'$. But $R^\infty(b'/B) = R^\infty(b'/\bar{b}a') < R^\infty(b'/\bar{b}) \leq R^\infty(b/B)$. This contradicts the minimality of $R^\infty(b/B)$. Hence b and a are independent over B.

So we have $b \in B$ by the maximality of B and the above claim. Hence φ is realised by the element b of B. This completes the proof of the claim. ∎

Our theorem follows directly from the above lemma:

Theorem. *Let T be superstable and let A be any set. Then there is no minimal model over A which has an infinite set of indiscernibles over A.*

Proof: Suppose that M is a model containing a set A and an infinite set I of indiscernibles over A. We can assume that I is an infinite Morley sequence over A because $\kappa(T)$ is finite. By the lemma we get a proper elementary submodel of M. So M is not minimal over A. ∎

3. Example

The following example shows that our theorem can not be extended to a stable theory. It is a slightly improvement of Marcus' one (see [1]).

Example: We construct a countable structure M with the following conditions: i) M is minimal, ii) M has an infinite indiscernible set and iii) $Th(M)$ is stable but non-superstable. Let L_0 be a language with an equality only. For $i < \omega$, let $L_{i+1} = \{P_{i+1}\} \cup \{R^n_{i+1} : n < \omega\} \cup L_i$, where P_{i+1} is a unary predicate symbol and R^n_{i+1}'s are binary predicate symbols. For each $i < \omega$ we define inductively countable L_i-structures M_i and countable subgroups H_i of $Aut(M_i)$ satisfying the following properties:

(1) $P_{i+1}^{M_{i+1}} = M_{i+1} - M_i$.

(2) $R_{i+1}^n \subset P_i^{M_{i+1}} \times P_{i+1}^{M_{i+1}}$. For any $a \in P_i^{M_i}$ and $b \in P_{i+1}^{M_{i+1}}$ there is a predicate $R_{i+1}^n \in L_{i+1}$ such that $\models R_{i+1}^n(x,b)$ if and only if $x = a$.

(3) M_0 is a countable set. H_0 is a countable subgroup of permutation of M_0 which move only a finite number of elements.

(4) For all $f \in H_0$ and $i < \omega$ there is a unique extension of f to an automorphism $f^* \in H_i$.

Now assume that M_i and H_i are defined as required. Let $M_{i+1} = \{b_f : f \in H_i\} \cup M_i$. Then M_{i+1} is countable (because H_i is so). Define a predicate $P_{i+1}^{M_{i+1}} = M_{i+1} - M_i$. Let $\{a_n : n < \omega\}$ be an enumeration of $P_i^{M_i}$. For every $n < \omega$ define a predicate ${R_{i+1}^n}^{M_{i+1}} = \{(f(a_n), b_f) : f \in H_i\}$. Clearly R_{i+1}^n's satisfy the condition (2). For $g \in H_i$ define a g^* as follows:

$$\begin{cases} g^*(b_f) = b_{g \cdot f} & \text{for each } b_f \in M_{i+1} - M_i, \\ g^*(a) = g(a) & \text{for each } a \in M_i. \end{cases}$$

Then g^* is an automorphism of M_{i+1}. In fact we can see that $(f(a), b_f) \in R_{i+1}^n$ iff $((g \cdot f)(a), b_{g \cdot f}) \in R_{i+1}^n$ iff $g^*((f(a), b_f)) \in R_{i+1}^n$. Let $H_{i+1} = \{g^* : g \in H_i\}$. Then H_{i+1} is a countable subgroup of $Aut(M_{i+1})$ since H_i is so. Hence we can construct M_i's and H_i's.

Let $L = \bigcup L_i$. Let M be an L-structure with $M = \bigcup M_i$.

(i) *M is a minimal model* : Let N be any submodel of M. Take any element a of M. Since M is the union of P_i^M's there is minimum $i < \omega$ such that $a \in P_i^M$. Pick an arbitrary element b of P_{i+1}^N. By the condition (2) there is some predicate $R \in L_{i+1}$ such that $R(x,b)$ holds if and only if $x = a$. Hence $a \in dcl(b) \subset N$, so $N = M$. Therefore M is minimal.

(ii) *M_0 is an indiscernible set* : Let $\bar{a}$, $\bar{b}$ be any elements of M_0 with the same length. By the condition (3) there is an $f \in H_0$ such that $f(\bar{a}) = \bar{b}$. Moreover by (4) f can be extended to an automorphism of M. So $tp(\bar{a}) = tp(\bar{b})$.

(iii) *$Th(M)$ is not superstable* : Let $\{a_n : n < \omega\}$ be an enumeration of M_0. For all $n < \omega$ let $\bar{a}_n = a_0{}^\frown a_1{}^\frown ...{}^\frown a_n$. For all $n < \omega$ let $\varphi_n(x, \bar{a}_n)$ denote $R_1^0(a_0, x) \wedge ... \wedge R_1^n(a_n, x)$. Then $(\varphi_n)_{n<\omega}$ is a infinite chain of forking formulas. In fact, for each $n < \omega$, $\{\varphi_n(x, \bar{a}_{n-1}^\frown a)^{\mathfrak{C}} : a \in M_0 - \{a_0, ..., a_{n-1}\}\}$ is a pairwise disjoint set. Hence $Th(M)$ is not superstable.

References

1. Marcus, L., *A minimal prime model with an infinite set of indiscernibles*, Israel J. Math. **11** (1972), 180–183.

2. Shelah S., "Classification Theory," North-Holland, Amsterdam, 1990.