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RESEARCH ON THE THEORY OF FINITE MODELS WITHOUT EQUALITIES

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(REPORT SCRIPT)

1) LANGUAGE

$L = < c_1, \ldots, c_m, R_1, \ldots, R_n >$

$c_1, \ldots, c_m$, are constants.

$R_1, \ldots, R_n$ are relations

with arities $r_1, \ldots, r_n$.

m, n are finite positive integers.

No function symbols are in L.

No equal signs are used.

2) FORMULAS

We use the following formulas:

First order formulas.

Infinite conjunction of first order formulas.

Infinite disjunction of first order formulas.

3) MODELS

$\mathfrak{A} =< A, c_1, \ldots, c_m, R_1, \ldots, R_n >$

A is the universe of the model.

c_i is the interpretation of constant c_i in L.

R_i is the interpretation of relation R_i in L.

$|A|$ is finite.

4) EQUIVALENCES ARE DIFFERENT

$\mathfrak{A} \models T$: all sentences of T are satisfied in $\mathfrak{A}$

$T \models T'$: for all model $\mathfrak{A}$ $\mathfrak{A} \models T \Rightarrow \mathfrak{A} \models T'$.

T is equivalent to T': For all finite models $T \models T'$ and $T' \models T$.

T is logically equivalent to T': For all models (finite or infinite) $T \models T'$ and $T' \models T$.

Comparing equivalence and logical equivalence:

Let T be the theory of Abelian groups.

$\varphi : T \wedge \forall x \exists y (y + y = x)$.

$\psi : T \wedge \forall z (z \neq 0 \rightarrow z + z \neq 0)$.

φ and ψ are equivalent but not logically equivalent to each other.

5) THEOREMS IN GENERAL MODEL THEORY BUT NOT TRUE IN FINITE MODEL THEORY

(Numbers are in Chang and Keisler's 'Model Theory'.)

Theorem 1. 3. 20. Gödel's Completeness Theorem.

Theorem 2. 2, 20. Craig's Interpolation Theorem.

Corollary 1. 2. 12. Compactness Theorem.

There was a *general conjecture* saying that: All theorems proved by using the above 3 theorems are not true in the theory of finite models. Of course it cannot be proved.

6) THEOREMS IN GENERAL MODEL THEORY WHICH ARE ALSO TRUE IN FINITE MODEL THEORY

Theorem. Ehrenfeucht game theorem.

7) PRESERVATION THEOREMS IN GENERAL MODEL THEORY

Theorem 3. 2. 2 (Taski). A theory T is preserved under submodels if and only if T has a set of universal axioms.

Theorem 5. 2. 3 (Taski). T is preserved under model extensions if and only if T is equivalent to a countable conjunction of first order existential sentences.

Theorem 3. 2. 4 (Lyndon). T is preserved under onto homomorphisms if and only if T is equivalent to a countable conjunction of first order positive sentences.

Exercise 5. 2. 6. (Keisler). T is preserved under into homomorphisms if and only if T is equivalent to a countable conjunction of first order positive existential sentences.

8) NEGATIVE RESULTS IN FINITE MODEL THEORY

The following theorems G1 and G2 are given by Gurevich but G3 and G4 are still unconfirmed.

Theorem G. 1. There is a first order sentence φ such that φ is preserved under submodels but φ is not equivalent to a first order universal sentence.

Theorem G. 2. There is a first order sentence φ preserved under model extensions but it is not equivalent a first order existential sentence.

Theorem G. 3. There is a first order sentence φ preserved under onto homomorphisms but it is not equivalent to a first order positive sentence?

Theorem G. 4. There is a first order sentence φ preserved under into homomorphisms but it is not equivalent to a first order positive existential sentence?

9) POSITIVE RESULTS IN FINITE MODEL THEORY

We prove the following theorems.

Theorem L. 1. A theory T is preserved under submodels if and only if T has a set of universal axioms.

Theorem L. 2. T is preserved under model extensions if and only if T is equivalent to a countable disjunction of first order existential sentences.

Theorem L. 3. T is preserved under onto homomorphisms if and only if T is equivalent to a countable disjunction of countable conjunctions of first order positive sentences.

Theorem L. 4. T is preserved under into homomorphisms if and only if T is equivalent to a countable disjunction of first order positive existential sentences.

10) A COUNTER EXAMPLE

Example G. The language L has two relations $P(x, y, z)$, $Q(x, y)$ expressing the relations $x + y = z$, $x \leq y$ respectively and two constants 0 and 1 representing the zero element and a unit of a group.

(1) $\forall xyzuvw(P(x, y, u) \wedge P(u, z, v) \wedge P(y, z, w) \rightarrow P(x, w, v))$,

(2) $\forall x P(x, 0, x)$,

(3) $\forall xyz(P(x, y, z) \rightarrow P(y, x, z))$.

(4) $\forall xy(Q(x, y) \vee Q(y, x))$,

(5) $\forall x Q(x, x)$,

(6) $\forall xyz(Q(x, y) \wedge Q(y, z) \rightarrow Q(x, z))$,

(7) $\forall x Q(0, x)$,

(8) $\forall xy(P(x, 1, y) \wedge \neg P(y, 0, 0) \rightarrow Q(x, y) \wedge \neg P(x, 0, y))$,

(9) $\forall xy \exists z P(x, y, z)$,

(10) $\forall xz \exists y P(x, y, z)$,

Sentences (1,2,3,9,10) are the axioms of Abelian groups. (4,5,6,7) are the axioms of a linear order with 0. (8) says that if $x + 1 \neq 0$, then $x + 1 > x$. (1) - (8) are universal sentences but (9), (10) are not. Now we define our main sentence.

φ : (1) $\wedge$ (2) $\wedge$ (3) $\wedge$ (4) $\wedge$ (5) $\wedge$ (6) $\wedge$ (7) $\wedge$ (8) $\rightarrow$ (9) $\wedge$ (10).

Theorem G. 2. 1. φ is preserved by extensions of models.

Proof. Let $\mathfrak{G}$ be a model of φ, $\mathfrak{H}$ be an extension of the model $\mathfrak{G}$. $\mathfrak{H} \models$ (1) - (8) $\longrightarrow$ $\mathfrak{G} \models$ (1) - (8) $\mathfrak{G} \models \varphi$. Hence (9) and (10) are true in $\mathfrak{G}$ From (1), (2), (3), (9), (10) $\mathfrak{G}$ is an Abelian group From (4) - (8) $\mathfrak{G}$ has a linear order with smallest 0 and $x + 1$ is always greater than x except for $x+1=0$ We give symbols to elements of $\mathfrak{G}$, $\{0,1,2,\ldots,n-1\}$ Now (1) - (8) are true in $\mathfrak{H}$. Let $a \in H$. List $(a, a+1, a+1+1, \ldots)$ has $n-1$. Hence $\mathfrak{H}$ has exactly $0,1,\ldots,n-1$. Therefore $\mathfrak{H} \models$ (9), (10) and $\mathfrak{H} \models \varphi$. $\square$

Theorem G. 2. 2. φ is not equivalent to an existential sentence.

Proof. Suppose that φ is equivalent to an existential sentence $\exists x_1,\ldots,x_k\ \varphi_1(x_1,\ldots,x_k)$ where φ_1 is a formula without quantifiers. Cyclic group $\mathfrak{Z}_{k+4}$ in order $0,1,\ldots,k+3$ $\mathfrak{Z}_{k+4} \models \exists x_1,\ldots,x_k \varphi_1[x_1,\ \ldots,x_k]$. There are k elements $a_1,\ldots,a_k \in Z_{k+4}$ such that $\mathfrak{Z}_{z+4} \models \varphi_1[a_1,\ldots,a_k]$ Find $a \in Z_{k+4} \neq 0,1,a_1,\ldots,a_k,n-1$ Let $\mathfrak{Z}'$ be a model with elements $Z_{k+4} - \{a\}$ and the same P and Q except for the element a $\mathfrak{Z}' \models$ (1-8) but not (9,10). Therefore φ is not true in $\mathfrak{Z}'$. $0,1,a_1,\ldots,a_k,n-1$ are having the same relations P, Q in $\mathfrak{Z}_{k+4}$ as in $\mathfrak{Z}'$ $\varphi_1(a_1,\ldots,a_k)$ then φ is true in $\mathfrak{Z}'$ Conclusion $\varphi \not\equiv$ existential sentence. $\square$

11) A POSITIVE RESULT

Theorem L. 2. T is preserved under model extensions if and only if T is equivalent to a countable disjunction of existential sentences $\bigvee \exists \varphi_i$.

Proof. For every finite model $\mathfrak{A}$ with n elements we write an existential sentence φ_i which describes all positive and negative relations among all elements. Let Σ be the disjunction of the above sentence φ_i over all models of the theory T. Every model $\mathfrak{A}$ of the theory T is a model of Σ because $\mathfrak{A}$ satisfies a disjunct.

Conversely, every model $\mathfrak{A}$ of the sentence Σ satisfies a disjunct of the sentence Σ which is written according to a model $\mathfrak{B}$. $\mathfrak{A}$ is an extension of the model $\mathfrak{B}$. $\mathfrak{B}$ is a model of the theory T. Therefore $\mathfrak{A}$ is a model of the theory T. $\square$

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