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From Subvarieties Of Abelian Varieties
To Kobayashi Hyperbolic Manifolds:
After P. Vojta and G. Faltings

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The aim of this note is two folds: the first is to understand Faltings' theorem about rational points of subvarieties of abelian varieties, which is conjectured by Lang; the other is to see what we can use from Faltings' proof if we deal with some conjectures of Lang about rational points of Kobayashi hyperbolic manifolds.

For doing this, we first recall the following

Faltings' Theorem. ([F 91b]) Let X be a subvariety of an abelian variety over a field F finitely generated over $\mathbf{Q}$. Then X contains a finite number of translations of abelian subvarieties which contain all but a finite number of points of $X(F)$.

On the other hand, for Kobayashi hyperbolic manifolds, we have the following

Lang's Conjecture. ([L 74]) Let X be a projective variety defined over a number field k. If X is a Kobayashi hyperbolic manifold, then X has only finitely many k-rational points.

An obvious connection between the above Faltings theorem and Lang conjecture is that if X is a subvariety of an abelian vaierty A which does not contain any translation of abelian subvarieties of A, then X is a Kobayashi hyperbolic manifold, hence by the above conjecture, X has only finitely many k-rational points, while that X has only finitely many k-rational points is an immediate consequence of Faltings' theorem. So as one may imagine, the meanning of the title of this paper is not in this sense.

What is the meaning of the title? Roughly speaking, for k-rational points of abelian varieties, there is a natural Néron-Tate pairing among them; while for k-rational points of Kobayashi hyperbolic manifolds, there is a natural Kobayashi distance among them, provided that we can give a good definition for p-adic Kobayashi hyperbolic semi distances. By some classical results, we know that the Néron-Tate pairing is quite rigid, while the Kobayashi distance involves very strong global properties of the space. So philosophically, once we can find some methods to pass concepts from the Néron-Tate pairing to the Kobayashi hyperbolic semi distance, then one should also can verify the Lang conjecture above.

Basically this note comes from several discussions with Lang in the past two years, I would like to thank him warmly.

I. Faltings' Theorem

As we stated above, the following result may be thought of as a special situation of Faltings' theorem.

Theorem. ([F 91a]) Let A be an abelian variety defined over a number field k. If X is an subvariety of A which does not contain any translation of abelian subvarieties of A, then X has only finitely many k-rational points.

On the other hand, we can also deduce Faltings' theorem from this result by applying one of Kawamata's structure theorem about subvarieties of abelian varieties. So, note that only this result has a closed relation with Lang's conjecture above, in this section, we only recall the proof of this theorem.

We begin with some facts and notation from arithmetic geometry [We 92].

a. Ratiopnal Points. From Grothendieck's viewpoint, each point of X defined over a field K is just a morphism from $\mathrm{Spec}(K)$ to X. Thus, arithmetically, if X has its arithmetic model

$$\pi : \mathcal{X} \to \mathrm{Spec}(\mathcal{O}_k),$$

where $\mathcal{O}_k$ denotes the ring of integers of a number field k, then a k-rational point corresponds to a section of π,

$$s(x) : \mathrm{Spec}(\mathcal{O}_k) \to \mathcal{X},$$

so that $\pi \circ s(x) = \mathrm{Id}_{\mathrm{Spec}(\mathcal{O}_k)}$. Suppose $(\mathcal{L}, \rho)$ is a hermitian line sheaf on $\mathcal{X}$, we also call the intersection of $c_1^{\mathrm{Ar}}(\mathcal{L}, \rho)$ with the arithmetic section E_x of $\mathcal{X}$ corresponding to the rational point x as the **degree** of $(\mathcal{L}, \rho)$ at x, which is in fact the degree of the pullback hermitian line sheaf on the arithmetic curve $\mathrm{Spec}(\mathcal{O}_k)$ via $s(x)$.

b. Arithmetic Setup. Suppose m is a positive integer. Later we will consider the arithmetic on the product X^m. So the arithmetic setup now becomes that:

Assume that k is a number field, $\mathcal{O}_k$ its rings of integers, that $\mathcal{A}$ is a normal irreducible projective $\mathcal{O}_k$-scheme whose generic fiber $\mathcal{A}_k$ is an abelian veriety, and that $\mathcal{X}$ is a closed irreducible subscheme such that $\mathcal{X}_k$ does not contain any translate of any abelian subvariety. For a fixed very ample line sheaf $\mathcal{L}$ on $\mathcal{A}$ which is symmetric on the generic fibre, choose a proper normal modification $\mathcal{B} \to \mathcal{A}^m$, trivial over k, such that the Poincaré line sheaves $\mathcal{P}_{ij}$ on B_k extend to line sheaves on $\mathcal{B}$, where $\mathcal{P}_{ij}$ is defined by

$$\mathcal{P}_{ij} := (x_i + x_j)^*(\mathcal{L}) - \mathrm{pr}_i(\mathcal{L}) - \mathrm{pr}_j(\mathcal{L}).$$

Further, for $\mathcal{A}_k$, there is the associated Néron model $\mathbf{A}$. Thus, in addition, we may also assume that $\mathcal{B}$ contains $\mathbf{A}^m$ as an open subset. Let $\mathcal{Y}$ be the closure of $\mathcal{X}^m$ in $\mathcal{B}$, and $\mathcal{Y}^o$ be the open subset by taking the intersection $\mathcal{Y}$ with $\mathbf{A}^m$. Obviously, $\mathcal{Y}^o$ contains of the closure of all k-rational points in X_k^m.

c. The Index. If f is a homogenous polynomial with variables x_{ij}, $i = 1, \dots, m$, $j = 1, \dots, n_i$, such that for a fixed i, the degree of f on x_{ij} is d_i, then for any point $x \in P := \mathbf{P}^{n_1} \times \dots \times \mathbf{P}^{n_m}$, the **index** of f at x with respect to $d_1, \dots, d_m$, $i(x, f; d_1, \dots, d_m)$ or $i(x, f)$, is defined as the maximal rational number σ such that for any set of integers $j_1, \dots, j_m$ with

$$\frac{j_1}{d_1} + \frac{j_2}{d_2} + \dots + \frac{j_m}{d_m} < \sigma,$$

and any choice of differential operators D_i of degree $\leq j_i$, on the i-th factor $\mathbf{P}^{n_i}$,

$$D_1 \circ D_2 \circ \ldots \circ D_m(f)$$

vanishes in x.

d. Arithmetic Height. Let X be an algebraic cycle of pure codimension p on the projective space $P := \mathbf{P}^n \times \mathrm{Spec}(\mathcal{O}_k)$. Then the **height** of X with respect to a hermitian line sheaf $(\mathcal{L}, \rho)$, $h_{(\mathcal{L},\rho)}(X)$, is defined by

$$h_{(\mathcal{L},\rho)}(X) := \deg(X_{\mathrm{Ar}}\, c_1^{\mathrm{Ar}}(\mathcal{L}, \rho)^{\dim X+1}) \in \mathbf{R},$$

where X_{Ar} is the associated arithmetic cycle of X, which may be defined as follows:

Suppose X is an irreducible subvariety of codimension p in P. If X lies in a fiber over a finite place v of k, then $X_{\mathrm{Ar}} := (X, 0)$. Otherwise, let $h_X := \deg(X_k) \cdot h^p$ denote the harmonic (p,p)-form on P_∞ representing X. Then there exists a uniuqe Green's current $g_X \in \tilde{\mathcal{D}}(X_{\mathbf{R}})$ so that

$$dd^c g_X - \delta_X = -h_X.$$

Thus, $(X, g_X) \in \mathrm{CH}^p_{\mathrm{Ar}}(P)$. We then let $X_{\mathrm{Ar}} := (X, g_X)$.

As an immediately consequence, we have the following

Proposition. For any effective algebraic cycle X,

$$h_{(\mathcal{O}(1),\rho)}(X) \geq 0,$$

where ρ denotes the canonical metric induced from the Fubini-Study metric on P.

e. Norm. Let $(\mathcal{E}, \rho)$ be a hermitian vector sheaf on an arithmetic variety $\mathcal{X}$. Then the metric ρ naturally induces a metric on the vector space of global sections s of the pullback vector sheaf at infinity. Usually, we denote this norm as $\|s\|$. Note that this norm depends also on the metric of the manifold at infinity.

Next we explain the basic strategy for proving Faltings' theorem. The method Faltings used is Diophantine Approximations, which was recently re-emphasized by Vojta in his proof of Mordell's conjecture. To explain it, we recall the following

Roth's Theorem. Let α be a fixed algebraic number. Given $\varepsilon > 0$, one has the inequality

$$|\alpha - \frac{p}{q}| \geq \frac{1}{q^{2+\varepsilon}}$$

for all but a finite number of fractions p/q in lowest form with $q > 0$.

In order to prove Roth's theorem, we first find a polynomial $P(X)$, such that

1. The coefficients are not too large;
2. $P(X)$ vanishes with higher order at $(\alpha, \ldots, \alpha)$;

Thus if there are too many approximation rational points a_i/b_i, we may choose $\beta_i := a_i/b_i$ such that

a. $h(\beta_i) >> 0$;
b. $h(\beta_{i+1})/h(\beta_i) >> 0$.

Hence, by Dyson's lemma, we have

3. $P(X)$ cannot vanish with higher order at $(\beta_1, \ldots, \beta_m)$. That is, there exist $j_1, \ldots, j_m$ so that

$$\zeta := D^{j_1} \ldots D^{j_m} P(\beta_1, \ldots, \beta_m) \neq 0.$$

Therefore, by consider the height of $\zeta \in \mathbf{Q}$, we get a contradiction: On one hand, because many derivatives vanish at $(\alpha, \ldots, \alpha)$, hence the height should have an upper bound; while on the other hand, by 3, the height should be bounded below.

Faltings' proof actually has the same pattern. First, note that polynomials are just the global sections of very ample line sheaves on products of projective spaces, we may use sections of ample line sheaves. At this point, Vojta's makes his first essential contribution: There are more ample line sheaves on products of varieties than on products of projective spaces. In order to get 1 and 2, Vojta uses an arithmetic Riemann-Roch theorem, while Faltings uses a generalization of Siegel's lemma about the control of the size of solutions for a system of linear equations by the size of the integer coefficients. Finally, to give 3, Vojta uses a generalization of Dyson's lemma, while Faltings uses his Product Theorem.

I.1. Several Intermediate Results

I.1.1. Find Ample Line Sheaves

At this point, we have Vojta's remarkable discovery. That is, on the product of varieties, there may exist more ample line sheaves than on the product of projective spaces. The importance here is that, classically, we only use homogeneous polynomials to do diophantine approximations, but homogeneous polynomials are just global sections of (very ample) line sheaves on products of projective spaces. So if, instead of working only with homogeneous polynomials, we consider sections of ample line sheaves on products of varieties in question, we may have more choices.

With this advantage of using global sections of ample line sheaves, which do not just come from the pull-back of ample line sheaves on products of projective spaces, we actually need to pay a little bit: Note that the index is defined for homogeneous polynomials, if we choose a global section of any ample line sheaf, we first need to embed this ample sheaf to a pull-back line sheaf from a product of projective spaces; second, when we use the embedding I to study the index for s at any point, we need to control the norm of $I(s)$ well in order to use the classical approach. On the other hand, as we need to pass from the complex situation to the arithmetic situation, we must consider the denominators of the coeffients. Basicall, in Faltings' proof, Faltings uses his Theorem 4.4 and Prop. 5.2 to deal with the problems above.

Theorem 1 (=Theorem 4.4 [F 91a]) Suppose m is big enough. For any very ample symmetric line sheaf $\mathcal{L}$ on $\mathcal{A}_k$, there exists a positive number ε_0, such that, for any $\varepsilon < \varepsilon_0$, there exists a real number s which makes the Faltings line sheaf, $\mathcal{L}(-\varepsilon, s_1, s_2, \ldots, s_m)$, defined by

$$\mathcal{L}(-\varepsilon, s_1, s_2, \ldots, s_m) := -\varepsilon \sum_i s_i^2 \operatorname{pr}_i^*(\mathcal{L}) + \sum_i (s_i x_i - s_{i+1} x_{i+1})^*(\mathcal{L}),$$

ample on $\mathcal{X}_k^m$, whenever

$$\frac{s_1}{s_2} \geq s, \; \frac{s_2}{s_3} \geq s, \; \ldots, \frac{s_{m-1}}{s_m} \geq s.$$

Remark 1. In fact, it suffices to choose m large enough so that the map

$$\alpha_m : X^m \to A^{m-1}$$

defined by

$$\alpha_m(x_1, x_2, \dots, x_m) := (2x_1 - x_2, 2x_2 - x_3, \dots, 2x_{m-1} - x_m)$$

is finite. On the other hand, by the condition for X, i.e. X is a subvariety of A which does not contain any translate of abelian subvariety, we see that such an m exists.

Remark 2. For ε_0, we may determine it by the fact that there exists a positive ε_0 satisfing the following condition: For any $\varepsilon \leq \varepsilon_0$ and any product subvariety $Y \subset X^m$, the intersection number

$$\mathcal{L}(-\varepsilon, s_1, s_2, \dots, s_m)^{\dim(Y)} Y$$

is positive. The existence of such an ε_0 is guaranteed by Remark 1 and the fact that $\mathcal{L}(0, s_1, s_2, \dots, s_m)$ "is" the pullback by α_m of an ample line sheaf. ∎

Now by the fact that

$$(s_i x_i - s_{i+1} x_{i+1})^*(\mathcal{L}) + (s_i x_i + s_{i+1} x_{i+1})^*(\mathcal{L}) = 2s_i^2 \mathcal{L}_i + 2s_{i+1}^2 \mathcal{L}_{i+1},$$

where $\mathcal{L}_i$ denotes $\mathrm{pr}_i(\mathcal{L})$, and note that the terms on the left hand side are generated by their global sections, we see that, for d big enough, on A, there are natural injections I_d of $\mathcal{L}(-\varepsilon, s_1, s_2, \dots, s_m)^d$ into $4d \sum_i s_i^2 \mathcal{L}_i := \sum_i d_i \mathcal{L}_i$ without a common zero, where $d_i = 4s_i^2 d$. In particular, we get some sub-line sheaves of the pullback of an ample line sheaf $\pi^* \mathcal{O}(d_1, d_2, \dots, d_m)$. Thus by a twisted Koszul complex a standard discussion, we may have the following

Proposition 2 (=Proposition 5.2 [F 91a]). There exist some effective bounded positive integers a, b, a suitable constant c, and an exact sequence

$$0 \to \Gamma(\mathcal{X}_k^m, \mathcal{L}(-\varepsilon, s_1, s_2, \dots, s_m)^d) \to \Gamma(\mathcal{X}_k^m, \otimes_i \mathcal{L}_i^{d_i})^a \to \Gamma(\mathcal{X}_k^m, \otimes_i \mathcal{L}_i^{3d_i})^b,$$

such that the follows hold:

1. The norms of the maps are bounded by $\exp(c \sum_i d_i)$.
2. The difference of the natural norm on

$$\Gamma(\mathcal{Y}, \mathcal{L}(-\varepsilon, s_1, s_2, \dots, s_m)^d)$$

and that induced on it by the restriction from $\Gamma(\mathcal{Y}, \otimes_i \mathcal{L}_i^{d_i})^a$ is bounded by $\exp(c \sum_i d_i)$.
3. If a section s of $\mathcal{L}(-\varepsilon, s_1, s_2, \dots, s_m)^d$ over $\mathcal{X}_k^m$ maps to $\Gamma(\mathcal{Y}, \otimes_i \mathcal{L}_i^{d_i})^a$, then on the open subset $\mathcal{Y}^o$ of $\mathcal{Y}$, the denominator of s is bounded by $\exp(c \sum_i d_i)$.

I.1.2. An Upper Bound For The Index

Once we have Step 1, it is very easy for us to deduce an upper bound for the index of a global section with a suitable restiction on the norm by certain standard methods with the help of Siegel's lemma.

Theorem 3.(=Theorem 5.3 [F 91a]). For any point $x = (x_1, x_2, \dots, x_m)$ in the smooth locus of $\mathcal{X}_k^m$, and a positive number σ so that $0 < \sigma < \varepsilon < \varepsilon_0$, if all s_i / s_{i+1} are large enough, then there exists for big d a section $s \in \Gamma(\mathcal{Y}^o, \mathcal{L}(\sigma - \varepsilon, s_1, s_2, \dots, s_m)^d)$ such that

1. $i(x,s) < \sigma$.
2. $||s|| \leq \exp(c\sum_i d_i)$, for a suitable constant c which depends only on σ and ε.

I.1.3. A Lower Bound For The Index

With above, in this step, we need to show that if $\mathcal{X}_k$ contains infinitely many k-rational points, then we may get a lower bound for the global sections at a suitable point. Then finally, the finite statement comes by getting a contradiction from comparing the lower bound and the upper bound for the index. So at first, we need to choose a suitable point.

By the Mordell-Weil theorem, $A(k) \otimes \mathbf{R}$ is a finite dimensional vector space with an inner product given by the Néron-Tate pairing, so by a sphere packing, we can find k-rational points $x_1, x_2, \ldots, x_m$ with heights $h_1, h_2, \ldots, h_m$, such that

1. h_1 is big enough;
2. h_i/h_{i-1} are all bounded below by s^2;
3. $< x_i, x_{i+1} > \geq (1-\varepsilon/2)\,||x_i||\cdot||x_{i+1}||$, which means that all x_i are almost in the same direction.

But for the purpose here, these conditions are not enough. In fact, as we need to have an induction on the dimension of X to deduce the assertion, so we may put a certain condition on x_i with respect to projections, which comes from the idea of the following

Product theorem (=Theorem 3.1 [F 91a]). Suppose P, $P := \mathbf{P}^{n_1} \times \mathbf{P}^{n_2} \times \ldots \times \mathbf{P}^{n_m}$, is a product of projective spaces over a field k of characteristic zero, $\mathcal{L} := \mathcal{O}(d_1, d_2, \ldots, d_m)$ a line sheaf on P with positive integers d_i, and s a non-zero global section of $\mathcal{L}$ over P. Let $Z_\sigma \subset P$ denote the subset consisting of the points p with the index of s at p at least σ. Then for any positive number $\varepsilon > 0$, there exists an r, depending on ε, such that the following holds:

Suppose Z is an irreducible component of $Z_{\sigma+\varepsilon}$, which is also an irreducble component of Z_σ. Then if $d_1/d_2 \geq r, d_2/d_3 \geq r, \ldots, d_{m-1}/d_m \geq r$, we have
1. $Z = Z_1 \times Z_2 \times \ldots \times Z_m$ is a product of closed subvarieties $Z_i \subset \mathbf{P}^{n_i}$;
2. The degree $\deg(Z_i)$ are bounded by some constant only depending on ε.

With this, we define an **essential projection** in the following sense:

Suppose X, $X \subset \mathbf{P}^n$, is a projective variety. There exist a projection $\pi : X \longrightarrow \mathbf{P}^{\dim(X)}$ and a hypersurface $Y \subset \mathbf{P}^n$ not containing X, with $\deg(Y) \leq (n-d)\deg(X)$, such that the ideal of Y annihilates $\Omega_{X/P}$. In particular, π is étale outside $X \cap Y$. Furthermore $Z = \pi(X \cap Y) \subset P$ is a hypersurface of degree $\leq (n-d)\deg(X)^2$ whose ideal annihilates $\Omega_{X/P}$.

Proposition 4 (=Proposition 2.2 [F 91a]). The essential projection exists for a setup $(X, \mathbf{P}^n)$.

With this proposition, for each i choose an essential projection $\pi_i : X \longrightarrow P_i = \mathbf{P}^{\dim(X)}$, and a hypersurface $Z_i \subset P_i$, defined by a homogeneous polynomial G_i with integral coefficients, whose ideal annihilates $\Omega_{X/P}$. We also let $\pi : X^m \longrightarrow P = P_1 \times P_2 \times \ldots \times P_m$ denote the product of π_i. Thus by induction on $\dim(X)$, we may further assume that there are infinitely many k-rational points not in any $\pi_i^{-1}(Z_i)$. In this way, we put an additional condition for the rational points x_i above:

4. $\pi_i(x_i)$ is not in Z_i.

On the other hand, when we do arithmetic, the above essential projection may not work well. We need a process by removing the denominators for the local equations. Thus, for example, when we discuss the essential projection, we find that a **good projection** in arithmetic in the following sense is useful:

Assume $\mathcal{X}_k \subset \mathbf{P}_k^n$ is irreducible. Choose a k-rational point x in $\mathbf{P}_k^n$ so that

1. x is not in $\mathcal{X}$.
2. The homogeneous coordinates of x are all integers of absolute value at most $\deg(\mathcal{X}_k)[k:\mathbf{Q}]$.
3. For each finite place v, the distance $d_v(x, \mathcal{X})$ is bounded below by a positive constant only depending on $\deg(\mathcal{X}_k)$.

Hence, the projection π with x as the center satisfies the following conditions:

a. $\pi : \mathbf{P}_k^n \to \mathbf{P}_k^{n-1}$ makes $\pi(\mathcal{X}_k) \subset \mathbf{P}^{n-1}$.
b. The projection of $\mathcal{X}_k$ to $\mathbf{P}_k^{n-1}$ has degree l at most $\deg(\mathcal{X}_k)[k:\mathbf{Q}]$.
c. There exists a nontrivial homogeneous polynomial F of degree l with coefficients in $\mathbf{Q}$ such that F vanishes on the projection.

Therefore, for a good projection, we may give a necessary estimate at infinity. As a corollary, combining with the facts about essential projections, we have the following

Proposition 5. There exists a composition of good projections $\pi : \mathcal{X}_k \to P := \mathbf{P}_k^{\dim(\mathcal{X}_k)}$ and a homogeneous polynomial F of degree at most $(n-d)\deg(X_k)\,[k:\mathbf{Q}]$, whose coefficients are rational integers bounded in size by $\exp(c_1 h(\mathcal{X}) + c_2)$, such that F does not vanish identically in $\mathcal{X}_k$, but annihilates $\Omega_{\mathcal{X}/P}$. There also exists a hypersurface $Z \subset P$, of degree less than $(n-d)\deg(\mathcal{X}_k)^2\,[k:\mathbf{Q}]^2$, defined by a polynomial G with coefficients in $\mathbf{Z}$ and bounded in size by $\exp(c_1 h(\mathcal{X}) + c_2)$, such that G annihilates $\Omega_{\mathcal{X}/P}$.

With these conditions for rational points, by a standard discussion in the sense of arithmetic intersection theory, and note that there is a difference between the arithmetic height and the Néron-Tate height, we may easily have the following

Proposition 6. With the same notation as above, the arithmetic degree of the hermitian line sheaf $\mathcal{L}(\sigma - \varepsilon, s_1, s_2, \ldots, s_m)^d$ at $x = (x_1, x_2, \ldots, x_m)$ is bounded above by

$$d\,(\sigma - \frac{\varepsilon}{2})\,m + c\sum_i d_i.$$

Next, we give a lower bound for the index $i(x,s)$. We hope that $i(x,s) \geq \sigma$. In practice, we will give a local discussion from the above conditions, and hence show the following

Proposition 7. With the same notation as above, choose G_i for each π_i. Suppose $i(x,s) < \sigma$. If e_i are positive integers so that

$$\sum_i \frac{e_i}{d_i} \leq i(x,s),$$

then the arithmetic degree of the hermitian line sheaf

$$\mathcal{L}(\sigma - \varepsilon, s_1, s_2, \ldots, s_m)^d \otimes \otimes_i \mathcal{L}_i^{e_i \deg(G_i)}$$

at $x = (x_1, x_2, \ldots, x_m)$ is bounded below by

$$-c\sum_i d_i.$$

So, by comparing Proposition 6 and Proposition 7, once σ is small enough and h_1 is big enough, we may get a contradiction by the upper bound norm condition for G_i from Proposition 5 and the fact that the second term of

$$d\,(\sigma - \frac{\varepsilon}{2})\,m + c\sum_i d_i$$

if of size d/h_1 so that the whole expression becomes negative if h_1 is sufficiently large. This completes the proof.

I.2. The Proof Of Intermediate Results: A Sketch

Due to the lack of space, we omit this section.

II. From Subvarieties of Abelian Varieties To Kobayashi Hyperbolic Manifolds: A Speculation

In this section, we look at the present possibility of using diophantine approximations to prove the Lang conjecture stated in the introduction about rational points of Kobayashi hyperbolic manifolds.

II.1. Kobayashi Hyperbolic Manifolds

Let $\mathbf{D}$ denote the (open) unit disc in the complex plane $\mathbf{C}$. We introduce the **Poincaré hyperbolic norm** on $\mathbf{D}$ as follows:

If $z \in \mathbf{D}$ and $v \in T_z(\mathbf{D})$ is a tangent vector at z, which in this case can be identified with a complex number, then

$$|v|_{\mathrm{hyp},z} := \frac{|v|_{\mathrm{ecu}}}{(1 - |z|^2)},$$

where $|v|_{\mathrm{euc}}$ denotes the euclidean norm on $\mathbf{C}$. Similarly, for any positive number r, we let $\mathbf{D}(0,r)$ be the open disc of radius r with center 0. The **Poincaré hyperbolic metric** on $\mathbf{D}(0,r)$ is defined by

$$|v|_{\mathrm{hyp},r,z} := \frac{r|v|_{\mathrm{ecu}}}{(r^2 - |z|^2)}.$$

Thus multiplication by r

$$m_r : \mathbf{D} \to \mathbf{D}(0,r)$$

gives an analytic isometry between $\mathbf{D}$ and $\mathbf{D}(0,r)$.

On the other hand, for the unit disc, we have the following

Schwarz Lemma. Every analytic map $f : \mathbf{D} \to \mathbf{D}$ with $f(0) = 0$ satisfies $|f(z)| \le |z|$ for all $z \in \mathbf{D}$ and $|f'(0)| \le 1$. Furthermore, if there is at least one point $c \in \mathbf{D}^*$, the punctured disc, with $|f(c)| = |c|$, or $|f'(0)| = 1$, then f is a rotation around 0.

The proof of this lemma can be obtained by the maximal modules principle and the fact that there exists a sequence of point z_k in $\mathbf{D}$ such that $|z_k| \to 1$ for $k \to \infty$.

As direct consequences of this result, we have the following

Theorem. The analytic automorphic group of $\mathbf{D}$ is given by

$$\begin{aligned}\operatorname{Aut}\mathbf{D} =& \{\frac{az+b}{\bar{b}z+\bar{a}} : a, b \in \mathbf{C}, |a|^2 - |b|^2 = 1\}\\ =& \{e^{i\theta}\frac{z-w}{\bar{w}z-1} : w \in \mathbf{D}, 0 \le \theta < 2\pi\}.\end{aligned}$$

Schwarz-Pick Lemma. Let $f : \mathbf{D} \to \mathbf{D}$ be an analytic morphism of the disc onto itself. Then

$$\frac{|f'(z)|}{1-|f(z)|^2} \le \frac{1}{1-|z|^2}.$$

Therefore, we know that analytic automorphisms of $\mathbf{D}$ are also isometry with respect to the hyperbolic metric. So, by the double transitive of $\operatorname{Aut}(\mathbf{D})$, we can cauculate the hyperbolic distance for any two point a, $b \in \mathbf{D}$ as follows:

First, if s is a positive real number in $\mathbf{D}$, then

$$d_{\text{hyp}}(0, s) = \int_0^s \frac{1}{1-t^2}dt = \frac{1}{2}\log\frac{1+s}{1-s}.$$

So, in general,

$$\begin{aligned}d_{\text{hyp}}(a, b) &= d_{\text{hyp}}(0, |\frac{b-a}{1-\bar{a}b}|)\\ &= \frac{1}{2}\log\frac{1+\frac{|b-a|}{|1-\bar{a}b|}}{1-\frac{|b-a|}{|1-\bar{a}b|}}.\end{aligned}$$

Also we know that, for $r > 1$,

$$d_{\text{hyp},\mathbf{D}}(0, \frac{1}{r}) = d_{\text{hyp},\mathbf{D}(0,\text{r})}(0, 1).$$

Starting from this, for any connected complex space X, we may introduce the Kobayashi hyperbolic semi distance, $d_{\text{hyp}}(x, y)$, as follows:

Let x, $y \in X$. We consider a sequence of holomorphic maps

$$f_i : \mathbf{D} \to X, \quad i = 1, 2, \ldots, m$$

and points p_i, $q_i \in \mathbf{D}$ such that

$$f_1(p_1) = x, \quad f_i(q_i) = f_{i+1}(p_{i+1}), \quad f_m(q_m) = y.$$

In other words, we join x and y by a Kobayashi chain of discs. Add the hyperbolic distances between p_i and q_i, and take the inf over all such choices of f_i, p_i, q_i to define the **Kobayashi hyperbolic semi distance**

$$d_{\mathrm{Kob},X}(x,y) = d_X(x,y) = \inf \sum_{i=1}^{m} d_{\mathrm{hyp}}(p_i, q_i).$$

Obviously, d_X satisfies the properties of a distance, except that $d_X(x,y)$ may be 0 if $x \neq y$.

There are several important properties for the Kobayashi hyperbolic semi distance.

1. Every analytic morphism is distance decreasing for the Kobayashi hyperbolic semi distances.
2. d_X is the largest semi distance on X such that every analytic morphism $f : \mathbf{D} \to X$ is distance decreasing.
3. $d_{\mathbf{D}} = d_{\mathrm{hyp}}$.

Actually, the above properties may character the Kobayashi hyperbolic semi distance.

With above, we define **Kobayashi hyperbolic manifolds** as these complex manifolds on which the Kobayashi hyperbolic semi distance is a distance, that is, $d_X(x,y) = 0$ if and only if $x = y$ for all points x, $y \in X$.

II.2. A Possible Definition of p-adic Kobayashi Semi Distance

For many reasons, p-adic Kobayashi semi distances are quite important. In particular, in order to do arithmetic for Kobayashi hyperbolic manifolds, we need to give the corresponding definition for the Kobayashi hyperbolic semi distance in p-adic situations. In this subsection, we give a possible definition for p-adic Kobayashi hyperbolic semi distance.

In more details, this definition has its foundation on the theory of rigid analytic spaces. We know that for p-adic situation, since usually the corresonding objects are discrete, it is very hard to do calculus: There are too many strang phenomenons if we use a very simple translation from ∞-adic cauculus to a p-adic one. In this direction, Tate, in his remarkable paper "Rigid Analytic Spaces", motivated by the question how to characterize elliptic curves with bad reduction, discovered a new category of analytic-algebraic objects with a structure rich enough to make do algebraic geometry in the sense of Grothendieck possible. More precisely, we may use the affinoids to build rigid analytic spaces, and hence analytic morphisms in this category should come from the associated morphisms for certain affinoid algebras. We also have Tits-Bruhat buildings and their geometric realization. For these, see [BGR 84] and [Be 90].

On the other hand, calculus for this theory is still not well studied. The point is that now we do not have a solid foundation for calculus, say, the concept for distances and so on.

The first breaking point in this direction, according to my knowledge, is Drinfel'd's works about the upper half plane, which may be thought as the moduli space of elliptic modules (Drinfel'd modules of rank 2). Parallel to the complex situation about the moduli space of elliptic curves, we may introduce calculus to study the problems at hands. For more details, see the survey paper [DH 84].

Recently, p-adic calculus becomes more and more important: We have arithmetic geometry in the sense of Arakelov, in which the arithmetic can be realized as a global version of all v-

adic calculus, where v are infinity places and finite places. Nevertheless, we also have the p-adic superrigidity theorem, which has its roots from the classical rigidity theorem for ∞ geometry.

As a more precise example, Rumely also studies p-adic calculus in order to deal with a certain arithmetic problem, which has its root from some results of Fekete, Szegö, and Cantor. In [Ru 89], we may find a definition of capaticity for certain subsets over all places. He actually goes quite far, even through with the restriction on dimension one objects. Among others, we may find that p-adic Green's functions, the canonical distances very useful, even through a canonical distance is in fact not a distance at all.

Now what we can do for p-adic Kobayashi hyperbolic semi distance. First, for p-adic geometry, we have the corresponding concepts about analytic spaces, analytic morphisms and so on, which have their names rigid analytic spaces, analytic morphisms (affinoid morphisms), etc.. Instead of recall them all, next we only mention the following for the purpose here:

In the complex situation, a complex space can be built up by patching complex discs. Similarly, we build up a rigid analytic space X by the following affionid domain:

$$\mathbf{D}^n(k) := \{(x_1, \ldots, x_n) \in k^n : \max_{1 \le i \le n} |x_i| \le 1\},$$

where k is a p-adic field which comes from the completion of a certain *algebraic closed field.* Let T_n be the subalgebra of the k-algebra $k[[x_1, \ldots, x_n]]$ of formal power series in n-indeterminates over k, defined by

$$\begin{aligned} T_n(k) :=& k < x_1, \ldots, x_n > \\ :=& \{ \sum_{i_1,\ldots,i_n \ge 0} a_{i_1,\ldots i_n} x_1^{i_1} \ldots x_n^{i_n} : a_{i_1 \ldots i_n} \in k, |a_{i_1,\ldots,i_n}| \to 0, \text{ if } i_1 + \ldots + i_n \to +\infty \}. \end{aligned}$$

Usually, we call T_n the **free Tate algebra** in n indeterminates over k. Obviously, there exists a natural bijection between $\mathbf{D}^n(k)$ and the maximal spectrum of T_n. (If we use the language in [Be 90], we may do even better.)

So, in general, we can use the Grothendieck language to define the rigid analytic space in the sense that, locally, it is a spectrum of T_n for a certain n, and these local patches may be glued by affinoid morphisms, which come from algebraic morphisms among T_n's. Hence we also get the definition for analytic morphisms in the rigid analytic space category.

As direct consequences of the definition, we have the following

Maximal Modules Principle. The maximum of the values taken a strictly convergent power series f is assumed on the subset

$$\{(x_1, \ldots, x_n) \in \mathbf{D}^n(k) : |x_1| = \ldots = |x_n| = 1\}$$

of the unit ball $\mathbf{D}^n(k)$.

Identity Theorem. If $f \in T_n$ vanishes for all $x \in \mathbf{D}^n(k)$, then $f = 0$. In particular, the map associating to a series $f \in T_n$ its corresponding function from $\mathbf{D}^n(k)$ to k is an injection.

Furthermore, for automorphisms of the unit disc, we have the following

Proposition. The series $f = \sum_{i=0}^{\infty} a_i x^i \in T_1$ defines a bi-affinoid map of the unit disc into itself if and only if

$$|a_0| \leq 1, \ |a_1| = 1, \ \text{and } |a_i| < 1 \text{ for all } i > 1.$$

With the above preparation, we define the p-adic **Poincaré hyperbolic distance** on the uint disc $\mathbf{D}^1(k)$ by

$$d_{\mathrm{hyp,v}}(a,b) := \log_v \frac{1+|b-a|_v}{1-|b-a|_v}$$

where if q_v denotes the number of elements of the primitive residus field of k, $\log_v$ is defined by

$$\log_v := \log_{q_v}.$$

Next let us "justify" this definition. First, for simplicity, we define

$$\log_v = \begin{cases} \log_{q_v}, & \text{if } v \text{ is finite;} \\ \log_e, & \text{if } v \text{ is a real value;} \\ \log_{e^2}, & \text{if } v \text{ is a complex value.} \end{cases}$$

Thus by the definition for the complex situation, we know that

$$\begin{aligned} d_{\mathrm{hyp}}(a,b) &= d_{\mathrm{hyp}}(0, |\frac{b-a}{1-\bar{a}b}|) \\ &= \frac{1}{2}\log\frac{1+\frac{|b-a|}{|1-\bar{a}b|}}{1-\frac{|b-a|}{|1-\bar{a}b|}} \\ &= \log_v \frac{1+\frac{|b-a|_v}{|1-\bar{a}b|_v}}{1-\frac{|b-a|_v}{|1-\bar{a}b|_v}}, \end{aligned}$$

where v denotes the complex place. Note that in this expression, $\bar{a}b$ is also in $\mathbf{D}$, therefore when we take a finite place, by the ultrametric inequality, we see that $|1-x|_v = 1$ for any point in $\mathbf{D}$.

Now suppose that this definition makes sense, then in a similar manner, we may introduce the p-adic Kobayashi hyperbolic semi distance d_X for a general space X. Thus we need to show that the p-adic Kobayashi hyperbolic semi distance has similar properties as these for the Kobayashi hyperbolic semi distance over $\mathbf{C}$. In particular, we need to have a corresponding result in the p-adic category for the following:

Every analytic morphism is distance decreasing for the Kobayashi hyperbolic semi distances.

For this, we first need to show that an automorphism f of the unit disc in rigid analytic space theory actually keeps the distance above. This is rather obvious by the Proposition above: We may first assume that $f(0) = 0$, i.e. $f \in T_1$ and $a_0 = 0$. Thus, we find that, in this case, f should be an element of T_1 and so the $a_1 x$ term dominates others.

Thus, we need prove that the corresponding p-adic Schwarz Lemma holds.

p-adic Schwarz Lemma. Let $f : \mathbf{D} \to \mathbf{D}$ be an analytic morphism in the sense of rigid analytic spaces. If $f(0) = 0$, then $|f(x)|_v \leq |x|_v$ for all $x \in \mathbf{D}$.

Proof. Here k is a completion of an algebraic closed field is very important. In fact, in this case, we know that for each point $a \in |k|$, there exists a sequence of points x_j of k such that

1. $|x_k|_v \neq a$.
2. $\lim_j |x_j|_v = a$.

Form here, it is not difficult to show that the closure of $|k|_v$ is $\mathbf{R}_{\geq 0}$. Then we may have the assertion by the fact that the maximal modules principle holds.

II.3. Classical Relations of Arithmetic Heights And Distances

Due of the lack of space, we omit this section.

II.4. The Situation For Kobayashi Hyperbolic Manifolds: A Strategy

In this section we propose certain steps towards Lang's conjecture for projective Kobayashi hyperbolic manifolds.

By the discussion in the last subsection, we know that we may first need the following steps, which are quite possible:

1. For any two k-rational points x, y, find a direct relation between *Kobayashi hyperbolic distance* $d(x, y)$ and the *arithmetic intersection* $E_x E_y$. And this relation should come from the local contributions.
2. To give a relation between *Kobayashi hyperbolic distance* and *arithmetic heights*.

Now let us discuss the above items in more details.

1. We need to define Kobayashi hyperbolic semi distance for all places. Once we have such a definition, with the discussion from the previous subsection, we need it to be good in the following sense:

Let X be a projective Kobayashi hyperbolic manifold defined over a number field k, then a definition for p-adic Kobayashi hyperbolic semi distances is good if the following is satisfied:

Suppose X has its arithmetic model $\mathcal{X}$ over Spec$(\mathcal{O}_k)$. For any two rational point x, y of X, the arithmetic intersection of the corresponding sections E_x, E_y of $\mathcal{X}$ over Spec$(\mathcal{O}_k)$ has a natural relation with

$$d_{\mathrm{HYP}}(x, y) := \sum_v d_{\mathrm{hyp,v}}(x_v, y_v),$$

where x_v and y_v denotes the reduction of x and y at the place v, and only finite terms on the right hand side is non-zero. Also this global relation should come from the local contributions.

Even through in II.2, we proposed a defintion for p-adic Kobayashi hyperbolic semi distances, but it is very hard to show that this definition is good in the sense above. Since we may expect that any relation should come from the local contributions, so we can go a little bit further: We give a more precise statement about the word "good".

We start with one dimensional case and with a fixed place, e.g. with Riemann surface C. By the well-known uniformization theorem, we know that the upper half plane $\mathcal{H}$ is the universal covering of any Riemann surface with genus at least 2, outside of the cusps. Thus, we can choose a uniformization

$$\pi : \mathcal{H}/\Gamma \to C - S,$$

where Γ is a discrete group of $\mathrm{PSL}_2(\mathbf{R})$ and S is a finite set of cusps on X. So we may use a standard process to offer the Green functions of C by certain twisting process. (For more details, see [La 75] or [Gr 86].) We will not go further in this direction. Instead, we look at the situation over $\mathcal{H}$, which may give us a basic idea to general problems, as corresponding objects for Riemann surfaces may be obtained from an average process (modulo divergence).

First, we know that on $\mathcal{H} := \{z = x + iy : y > 0\}$, the natural metric

$$d\mu = \frac{dz\, d\bar{z}}{y^2}$$

gives a hyperbolic mesaure on $\mathcal{H}$. By an easy calculation, we know that the hyperbolic distance of $z, w \in \mathcal{H}$ is given by

$$d_{\mathrm{hyp}}(z, w) = \cosh^{-1}(\frac{|z-w|^2}{2\mathrm{Im}z\,\mathrm{Im}w} + 1).$$

So note that the corresponding "Green function" may be given by

$$g(z, w) = -2\log|\frac{z-w}{\bar{z}-w}|,$$

we see that

$$e^{\pm d_{\mathrm{hyp}}(z,w)} = \frac{1 + e^{-g(z,w)/2}}{1 - e^{-g(z,w)/2}}.$$

Thus by passing a similar relation to Riemann surfaces, say, by averaging over Γ modulo the divergence, note that the Green function has its contribution to the arithmetic intersection, so over C, we may "get" the following relation

$$(e^{c_1 d_{\mathrm{hyp}}(x,y)} - 1)(e^{c_2[E_x, E_Y]_\infty} - 1) = O(1),$$

for certain constants c_1, c_2. In particular, we see that $[E_x, E_y] \to \infty$ "iff" $d_{\mathrm{hyp}} \to 0$. Then we may see that if there are infinitely many k-rational points with the arithmetic heights decay rapidly, then the Kobayashi hyperbolic distances among them should be small enough. Hence we may have a kind of special sphere packing:

X is compact, so we always can choose an equal-dimension sphere in X, which contains infinitely many k-rational points. But on the other hand, for a unit sphere, the Kobayashi hyperbolic distance of the center and the point nearby the boundary should be arbitary large, so by the discussion above, k-rational points cannot scatter in this way.

Suppose we now have a good definiton so that the above assertion holds, then to apply Diophantine Approximations to show the finiteness theorem, we may go a step further:

By the Product Theorem, we may choose the k-rational points $x_1, \ldots, x_m$ so that

a. x_i in the same unit ball of dimension $\dim X$ with mutually small hyperbolic distance.
b. $h(x_i)$ go to infinite rapidly;
c. x_i are not too twisted. That is, they close to a certain fixed proper subvariety.

d. Once we take the good projection in the sense of Faltings, they are not in the corresponding hypersurface Z_i.

2. Now let us look at how one could have a natural connection beteween Kobayashi hyperbolic distance and the arithmetic height function. We make this connection by introducing a definition proposed by Philippon [Ph 90]:

Suppose $\phi : X \to \mathbf{P}^n$ is a closed embedding of an m-dimansional smooth variety X in $\mathbf{P}^n$ defined over a number field k. Then we may first define the **elimination form** of X, denoted as $E(X)$ by the following process: Let $\check{\mathbf{P}}^n$ be the dual projective space of $\mathbf{P}^n$: A point ζ of $\check{\mathbf{P}}^n$ corresponds a hyperplane H_ζ of $\mathbf{P}^n$ defined by the equation $\zeta \mathbf{z} = 0$ for $\mathbf{z} \in \mathbf{P}^n$. Let Y, $Y \subset (\check{\mathbf{P}}^n)^{m+1}$ be the subvariety consisting of the points $(\zeta_0, \ldots, \zeta_m)$ so that

$$(\cap_i H_{\zeta_i}) \cap X \neq \emptyset.$$

Then Y is a hypersurface defined over k, which is defined by a multihomogeneous polynomial E with degree $d = \deg(X)$. With E, we may define the **Philippon height** of X as

$$h_{\mathrm{PH}}(X) := \sum_{v \text{ finite}} \log |E|_v + \sum_{v,\text{ infinite}} \int_{(S^n)^{m+1}} \log|E(v,X)|d\mu,$$

where as usual $|E|_v$ denotes the maximum v-adic norm of the coefficients of E, $E(v, X)$ denotes the v-conjugation of $E(X)$, and $d\mu$ denotes the natural $(U(n+1))^{m+1}$ metric on copys of the unit sphere S.

Theorem. ([So 91]) The natural relation between the Philippon height and the arithmetic height is given by

$$h(X) = h_{\mathrm{PH}}(X) + \frac{1}{2}(n+1)\sum_{j=1}^{n} \frac{1}{j}[k : \mathbf{Q}]\deg X.$$

With above, note that the essential fact behind the proof of the following theorem is that, on one hand, the arithmetic height is naturally associated with a certain distance, which is essentially a quadratic form, while on the other hand the height should decay rapidly, say, exponentially, we may conclude that now the situation for Kobayashi hyperbolic manifold is somehow as the one for the following

Theorem. (Faltings [Fa 91a]) Let A be an abelian variety over a number field k, and $E \subset A$ a closed subvariety. Suppose that for any place v of k and any positive κ, the number of k-rational points $x \in A - E$, for which the v-local distance $d_v(x, E)$ from x to E is less than $H(x)^{-\kappa}$, is finite.

Thus, with enough choices of hermitian ample vector sheaves on products of varieties, it is hopeful to prove the Lang conjecture stated in the introduction. We hope that later this idea would be applied to reprove the special case of Faltings' theorem, the Mordell conjecture.

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