数理解析研究所講究録
第 819 巻 1993 年 167-187

Diophantine problem of algebraic varieties and Hodge theory

KAZUHISA MAEHARA

Tokyo Institute of Polytechnics

Dedicated to Professor Shoshichi Kobayashi on his sixtieth birthday.

§ 1. INTRODUCTION

In this article, we shall discuss some recent results related to Diophantine problems from point of view of algebraic geometry. Diophantine problems are concerned with polynomials with integral coefficients and most of these problems have their origins in the theory of algebraic curves. For example, P. Samuel's lecture note([S]) provides us a good reference. In the study of curves, the notion of genus is the most fundamental and indicates some complexity of polynomials which defines the curve. For higher dimensional algebraic varieties we have the notion of Kodaira dimension proposed by Iitaka([I]). The variety is said to be of general type if the Kodaira dimension attains the dimension of the variety.

First, note that the birational automorphisms group is finite if the variety is of general type. This was shown by H. Matsumura([Mat]), which is considered as a generalization of a theorem of Hurwitz.

Socond, Kobayashi-Ochiai ([KO])proved finiteness of the set of the generically surjective meromorphic maps from a compact complex manifold onto a variety of general type. In positive characteristic case Deschamps([DM1]) shows finiteness of the separable dominant rational maps from a variety onto a variety of general type. This is de Franchis' theorem in the theory of curves.

Third, fixing an algebraic variety X, we consider the set of separable dominant rational maps from X onto varieties of general type up to birational equivalence. We have the following question;

Iitaka's conjecture based on Severi's theorem.
Is the set finite?

Thanks to Kobayashi-Ochiai's theorem([KO]) or Deschamps'([DM1]), it suffices to show finiteness of birational equivalence classes of the varieties of general type which are images of the given variety X by separable dominant rational maps. In the case of characteristic 0, Deschamps and Mengaud([DM3]) have shown finiteness if X are surfaces of general type with the condition $q > 0$ or $p_g \geq 2$ and the author([M2]) shows it restricting image varieties to such varieties that can be birationally embedded by the m-th multicanonical maps for any given m. To acomplish it, it is enough to show that there exist a minimal model for a variety of general type and an upper bound for the indexes of the canonical divisorial sheaf. K.Ueno suggested to the author that a variety of general type can be replaced by a polarized non uniruled variety in the problems above. We can consider some variation of the conjecture above. The author showed finiteness of isomorphism classes of varieties with ample canonical divisors which are dominated by surjective morphisms from a fixed variety([M1]) and Deschamps and Mengaud proved finiteness of birational classes of surfaces of general type which are dominated by surjective morphisms from a fixed variety([DM2]).

A famous conjecture by Mordell([Mor]) looks like a typical Diophantine problem. It is now Faltings theorem([Fal]). Bombieri and Noguchi conjecture that Faltings theorem should be generalized;

Mordell, Bombieri and Noguchi conjecture(resp. analogue).

Does there exist no variety of general type(resp. with exceptions)over a number field(resp. a function field) which has a dense set of rational points?

Manin([Ma]) and Grauert([Gra]) proved Mordell conjecture analogue over function fields. MBN conjecture analogue has an exception that is a birationally isotrivial variety. On the other hand, Lang([L1]) conjectured that the conjectures above hold if the notion of a variety of general type over a number field be replaced by that of a hyperbolic manifold considered as a complex manifold. Noguchi([N1],[N2],[N3]) proved Lang conjecture analogue over function fields completely.

Shafarevich conjecture which Faltings proved([Pa], [Ar], [Fal]) has classical models in number theory, i.e., Hermite's theorem and Minkowsi's theorem([ZP]);

(1) finiteness of the number of extensions L/K of K with fixed degree and fixed ramification,
(2) there exists no extension unramified over $\mathbb{Q}$.

A generalization of Shafarevich conjecture.

Is the set of birational equivalence classes of varieties of general type with a fixed set of bad reductions and a fixed pluricanonical function ?

Its analogue over function fields has some exceptions, say, isotrivial varieties, isotrivial factors, $\cdots$. We can study this problem replacing varieties of general type by polarized abelian varieties of dimension g or K3 surfaces([S],[SZ]) or polarized non uniruled varieties of a fixed dimension or polarized variations of Hodge structures([pe]), or Hodge Modules.

§ 2. Results

Let X be a given variety of characteristic 0 and m any fixed number. We denote by $\mathcal{E}(X)$ a set of the birational equivalence classes of smooth varieties onto which there exists a dominant rational map from X. We define subsets of $\mathcal{E}(X)$, respectively:

(1) $\mathcal{E}_m(X)$ consists of varieties of general type such that the m-th pluricanonical mappings are birational,

(2) $\mathcal{E}_{pol\geq\ell}(X)$ consists of varieties of general type with the i-th pluri-genus P_{ai+b} polynomials for all $i \geq \ell$ and fixed a, b,
(3) $\mathcal{E}_{nef}(X)$ consists of varieties of general type with the dualizing sheaf ω_X nef,
(4) $\mathcal{E}_{abun}(X)$ consists of varieties with the dualizing sheaf semi-ample.

Theorem 1([M2]).
(1) For all m, $\mathcal{E}_m(X)$ is a finite set,
(2) There exists an m such that $\mathcal{E}_{nef}(X) \subset \mathcal{E}_m(X)$,
(3) There exists an m such that $\mathcal{E}_{pol\geq\ell}(X) \subset \mathcal{E}_m(X)$,
(4) $\mathcal{E}_{abun}(X)$ is at most countable.

Theorem 2([M4],[M5]). *Let K be a function field over $\mathbb{C}$ and X a smooth variety of general type over K. Let $\pi : \mathbb{P}(\Omega_X) \to X$ denote the projective bundle over X and $\mathcal{O}_P(1)$ the fundamental sheaf.*
Suppose that one of equivalent conditions
(1) $\mathcal{O}_P(1)$ is big and semiample,
(2) $\mathcal{O}_P(1)$ is nef and $\mathcal{O}_P(\alpha) \otimes \pi^*\omega_X^{-1}$ is nef and big for some $\alpha > 0$.
Assume that the set of K-rational points is dense in X. Then X is isotrivial.

Proofs of Theorem 1 and 2 are reduced to show local biratinal triviality and boundedness of parametrizing family.

From geometric point of view, we interpret the situation above. Let $\Phi : \mathcal{X} \to \mathcal{S}$ be a surjective morphism between non singular complex varieties with the generic fibre a variety of general type which is isomorphic to X, where the rational function field of $\mathcal{S}$ is K. We may assume that $\mathcal{S}$ is a curve without generality.

Lemma 2.1. *([M4])*
Assume that the set of (rational) sections $(\mathcal{C}_\lambda)_{\lambda\in\Lambda}$ with the intersection number of the section $\mathcal{C}_\lambda$ and a canonical divisor $K_{\mathcal{X}}$ bounded is dense in $\mathcal{X}$. Then $\mathcal{X}$ is isotrivial.

We propose a generalization of the problem above.

Conjecture 2.2. *Let X be a non singular variety and g a fixed number. Consider the set $(C_\lambda)_{\lambda\in\Lambda}$ of genus g embeded in X. We ask if the intersection number of (K_X, C_λ) has an upper bound independent of λ.*

We can replace curves of genus g by higher dimensional subvarieties of a fixed multicanonical function P_m.

REMARK.
(1) If X is a surface, Bogomolov and Miyaoka and Umezu (unpublished) using Miyaoka and Sakai inequality showed Conjecture 2.2.
(2) If Ω_X is ample, Conjecture 2.2 holds.
(3) S. Kobayashi showed if Ω_X is ample, X is hyperbolic([Kob2]).

(4) *Peternell-Campana and Kawamata proved that a hyperbolic manifold of general type has an ample canonical divisor.*

Question 2.3. *Let X be a non singular complex variety with a canonical divisor ample and C a fixed curve with genus ≥ 2. Has the intersection number (K_X, C_λ) with $\jmath_\lambda : C \hookrightarrow C_\lambda$ an embedding an upper bound $\leq$ dim $X \cdot (2g(C) - 2)$?*

Yoga of generalization insists that higher dimensional Shafarevich conjecture should hold. In any case Minkowsi's theorem should be studied to the effect that there exists no smooth complex variety X with some exceptions such that $f : X \to Y$ is smooth with a canonical divisor K_X semiample and Y is complete with var$(f) > 0$. Faltings([Fal2]) gave an example non rigid family of abelian varieties with relative dimension 8 over a curve without isotrivial factors fixing ramifiation locus. M.Saito classifies non rigid families of abelian varieties with no isotrivial factors([S],[SZ]). A check point to a generalization of Shafarevich conjecture over function fields consists in

THE PROBLEM OF LOCAL TRIVIALITY:.
Let $\Phi : X \to S$ be a proper surjective smooth morphism with all fibres of ample canonical divisors between non singular quasi-projective varieties defined ove $\mathbb{C}$. Assume $S = C_1 \times C_2$ where C_1, C_2 are curves. If $\Phi_1 : X|\Phi^{-1}(C_1) \to C_1$ has no isotrivial factors, $\Phi_2 : X|\Phi^{-1}(C_2) \to C_2$ is locally isotrivial.

§ 3. Methods

We restrict ourselves to analogues over function fields over the complex number field. In these problems the following tools are essential, which come from Hodge theory to prove local birational triviality and boundedness of degrees.

We recall Viehweg's Definitions([V2],[V3],[V4],[M3]).

Definition 3. *Let Y/S be a scheme and T a functor of the category of coherent sheaves over Y/S to that of coherent sheaves over X/S. Let $f : X \to S$ be a morphism, $\mathcal{F}$ a coherent $\mathcal{O}_Y$-Module, L an invertible sheaf over X and U an open subset of X with depth$_{X\setminus U}(\mathcal{O}_X) \geq 2$. $\mathcal{F}$ is said to be f-weakly positive (with respect to T, L and U), if for any $\alpha > 0$ there exists $\beta_0 > 0$ such that for any $\beta \geq \beta_0$ the following canonical homomorphism over U are generically surjective*

$$f^* f_* T(\mathcal{F}^{\otimes \alpha\beta} \otimes L^{\otimes \beta}) \longrightarrow T(\mathcal{F}^{\otimes \alpha\beta} \otimes L^{\otimes \beta}).$$

If $U = X, T = id$, $\mathcal{F}$ is said to be f-pseudo-effective. If $U = X, T = id$ and the canonical homomorphisms are surjective, $\mathcal{F}$ is said to be f-numerically effective(f-nef) or f-semipositive.

Remark

Let $f : M \to C$ be a surjective morphism with connected fibres from a Kähler manifold M onto a curve C. Hinted from Grifiths's work, Fujita([F1]) found the semipositivity of $f_* \omega_{M/C}$. Kawamata([Kaw2]) and Viehweg([V2]) generalized it. Kollár([Kol1],[Kol2]) found another approach([EV1],[M6]).

Theorem 4. *Let $f : X \to S$ be a surjection with connected fibres between non singular varieties over $\mathbb{C}$. Then*

(1) $f_*(\omega_{X/S}^{\otimes m})$ *is weakly positive for all* $m > 0$.

(2) $\omega_{X/S}$ *is pseudo-effective with respect to* f_*.

Definition 5. *Let $(Varieties)/k$ be the category of geometrically irreducible, reduced proper algebraic schemes over the ground field k restricting the morphisms to surjections. Let $X/S, Y/T$ be surjective morphisms of varieties. We consider X/S and X'/S to be equivalent if there exists a biratinal map over S' between $X_{S'}/S'$ and $X'_{S'}/S'$ where S' is generically finite extention of S. Moreover X/S and Y/T are looked to be equivalent if main parts of $X_T/S \times T$ and $Y_S/S \times T$ are equivalent. We denote by var(X/S) the minimal dimension of a base variety T such that Y/T is equivalent to X/S.*

Iitaka-Viehweg Conjecture 6.

(1)

$$\max_{m>0} \kappa(det\ f_*(\omega_{X/S}^{\otimes m})) \geq \mathrm{var}(X/S)$$

(2) *for a general point* $s \in S$

$$\kappa(\omega_{X/S}) - \kappa(\omega_{X_s}) \geq \mathrm{var}(X/S)$$

Remark.

(1) *If the generic fibre of X/S is birationally equivalent to a variety with a canonical divisor semiample, Kawamata([Kaw3]) proves it.*

(2) *If X/S has a variety of general type as the generic fibre, the conjecture above is essentially shown by Kollár and Viehweg([Kol3],[V4],[V5],[V6]).*

§ 4. Hodge Theory

Definition. *Let X be a normal variety. Let $f : X \to S$ be a smooth morphism of varieties and D a divisor on X. Assume D has only simple normal crossings. If for each component C of D, the restriction $f : C \to S$ is smooth, then D is said to have only f-simple normal crossings.*

Letting X be a variety, we denote by $[D]$ the integral part of $D \in \mathbf{Div}(X) \otimes \mathbb{Q}$, $\{D\} = D - [D]$ and $\lceil D \rceil = -[-D]$.

Theorem 0([M6]). *Let X be a non singular variety and $f : X \to S$ a projective smooth morphism of analytic varieties. Let D be a $\mathbf{Q}$-divisor on X and D f-numerically equivalent to zero. Let A, B be divisors on X. Let D decompose itself into $D' + D''$ without common component. Assume $\{D\}_{red} + A + B$ has only f-simple normal crossings. Then Hodge-Deligne spectral sequence degenerates at E_1:*

$$E_1^{a,b} = R^b f_* \Omega^a_{X/S}(A + \{D'\}_{red}, B + \{D''\}_{red})(\lceil D' \rceil + [D'']) \Rightarrow$$

$$R^{a+b}f_*\Omega^*_{X/S}(A+\{D'\}_{red}, B+\{D''\}_{red})(\lceil D'\rceil+[D'']).$$

We denote by $\mathbf{Q}_{(p)}$ the subring of $\mathbf{Q}$ consisting of the irreducible fractionals without the denominator multiple of p.

THEOREM A(CF. DELIGNE-ILLUSIE([DI])). *Let k be a perfect field of characteristic $p>0$, $S=Spec\ k$, $\tilde{S}=Spec(W_2(k))$. Let X be a proper smooth scheme over k of pure dimension d. Let D be a $\mathbf{Q}_{(p)}$-divisor on X. Let $A+B$ be a divisor on X. Suppose $\{D\}_{red}+A+B$ has only normal crossings. Assume there exist liftings $\tilde{X}$ and $\tilde{D}$, $\tilde{A}$, $\tilde{B}$ such that $\tilde{X}\to\tilde{S}$ is flat and $\{\tilde{D}\}_{red}+\tilde{A}+\tilde{B}$ has only normal crossings on $\tilde{X}$ relative to $\tilde{S}$. Then*

(1) *Let C be a component of $\{D\}$. If D is numerically equivalent zero, the next sequence is exact for all a,b*

$$H^{b-1}(C,\Omega^a_C(A,B+\{D\}_{red}-C)([D]))\to$$

$$H^b(X,\Omega^a_X(A,B+\{D\}_{red})(-C+[D]))\to$$

$$H^b(X,\Omega^a_X(A,B+\{D\}_{red}-C)([D]))\to 0.$$

(2) *If D is ample, the next cohomologies vanish for $a+b>\max(d_X,2d_X-p)$*

$$H^b(X,\Omega^a_X(A,B+\{D\}_{red})([D]))=0.$$

COROLLARY B.

Let k be a perfect field of characteristic $p>0$, $S=Spec\ k$, $\tilde{S}=Spec(W_2(k))$. Let X,Y be a projective smooth schemes over k of pure dimension. Let $f:X\to Y$ be a surjective morphism whose restriction is smooth over Y^o with $Y\backslash Y^o$ a normal crossing divisor. Assume there exists a lifting $\tilde{X}$ such that $\tilde{X}\to\tilde{S}$ is flat. Let L be an ample invertible sheaf over Y. Then

(1) *$(f_*\omega_{X/Y})^{\otimes a}\otimes L^{\otimes b}$ is generically generated by global sections for arbitrally a and $b>dimY$.*
(2) *If $\omega_{X/Y}$ is f-semiample, $f_*\omega^{\otimes\ell}_{X/Y}$ is weakly positive for $\ell>0$.*

COROLLARY C.

Let k be a perfect field of characteristic $p>0$, $S=Speck$, $\tilde{S}=Spec(W_2(k))$. Let X be a variety over k. Let $\mathcal{E}^p_{am}(X)$ be the set of smooth k-varieties $\{V\}$ onto which there exists a dominant separable rational map from X such that V have flat liftings over $Spec(W_2(k))$ and ample canonical divisors with a fixed pluricanonical polynomial P_m.

Then $\mathcal{E}^p_{am}(X)$ is finite.

Definition of Arithmetic variety.

An arithmetic variety $\mathcal{X}$ is a projective flat regular scheme X over Spec $\mathbb{Z}$ the base change to Spec $\mathbb{C}$ of whose generic fibre has a Kähler metric compatible with complex conjugate action.

Problem D

For an arithmetic variety, we want some vanishing theorems.

These Hodge theoretical cosequenses through covering trick give us tools to attack Diophantine problems, say, vanishings and weak positivity of direct image sheaves of powers of dualizing sheaves.

Next we explain a covering technique to apply Hodge theory to algebraic geometry.

Definition 1([GM]).

Let X be a scheme and M an abelian group. We denote Spec $\mathcal{O}_X[M]$ by $D(M)$. An X-group $D(M)$ is called a diagonalizable group.

Note that for any variable X-scheme T one has

$$D(M)(T) = Hom_{\mathbf{groups}}(M, \Gamma(T, \mathcal{O}_T^*)).$$

Let $\mathbb{Z}_{((n_i)_{i\in I})}$ denote $\bigoplus_{i\in I} \mathbb{Z}/n_i$. We denote an X-group $D(\mathbb{Z}_{((n_i)_{i\in I})})$ by $\mu_{((n_i)_{i\in I})}$. The μ_n is the usual group of n-th root of unity.

We define a Kummer covering and a generalized Kummer covering according to Grothendieck- Murre ([GM]). Let X be a scheme and $\mathbf{a} = (a_i)_{i\in I}$ a finite set of sections of $\mathcal{O}_X$. Let $\mathbf{n} = (n_i)_{i\in I}$ be a set of positive integers. We denote $\mathcal{A}_{\mathbf{n}}^{\mathbf{a}} = \mathcal{O}_X[(T_i)_{i\in I}]/((T_i^{n_i} - a_i)_{i\in I})$ and $Z_{\mathbf{n}}^{\mathbf{a}} = Spec\ \mathcal{A}_{\mathbf{n}}^{\mathbf{a}}$. We give the action of μ_n on $Z_{\mathbf{n}}^{\mathbf{a}}$ with $\phi : \mathcal{A}_{\mathbf{n}}^{\mathbf{a}} \to \mathcal{A}(\mu_n) \otimes \mathcal{A}_{\mathbf{n}}^{\mathbf{a}}$ given $\phi(t_i) = u_i \otimes t_i (i \in I)$.

Definition 2.

Assuming the a_i are regular, a couple (Y, G) consisting of an X-scheme and an X-group with the action on Y is said to be a Kummer covering of X relative to the sections $\mathbf{a}$, if (Y, G) is isomorphic to a couple $(Z_{\mathbf{n}}^{\mathbf{a}}, \mu_n)$ for a suitable set of integers $\mathbf{n}$, with every n_i prime to the residue characteristics of X.

Lemma ([GM], lemma 1.2.4).

Let $\mathbf{a} = (a_i)_{i\in I}$ and $\mathbf{b} = (b_i)_{i\in I}$ be sets of regular sections on X. Assume that there exist a set of $g_i \in \Gamma(X, \mathcal{O}_X^*)$ for $i \in I$ such that $a_i = g_i b_i$. Then there exists an etale surjective morphism $e : U \to X$ such that one has a $\mu_{\mathbf{n}}$-isomorphism

$$(Z_{\mathbf{n}}^{\mathbf{b}})_U \cong (Z_{\mathbf{n}}^{\mathbf{a}})_U.$$

Here $U = Spec\ \mathcal{O}_X[(V_i)_{i\in I}]/((V_i^{n_i} - g_i)_{i\in I})$. The isomorphism $\mathcal{A}_{\mathbf{n}}^{\mathbf{b}} \to \mathcal{A}_{\mathbf{n}}^{\mathbf{a}}$ is given by

$$t_i' \mapsto v_i t_i.$$

We will define the generalized Kummer covering due to Grothendieck-Murre ([GM]). Let L be a factor group of $\mu_{\mathbf{n}}$:

$$\mathbb{Z}_{\mathbf{n}} \to L \to 0.$$

$D(L) \subset D(\mathbf{Z_n}) = \mu_\mathbf{n}$ is a diagonal subgroup. The kernel N of $\mathbf{Z_n} \to L$ determines the quotient $\mu_\mathbf{n}/K = D(N)$.

We construct the quotient space $Z^\mathbf{a}_\mathbf{n}/K$. Identifiying $\mathcal{A}(\mu_\mathbf{n}) \cong \mathcal{O}_X[\mathbf{Z_n}]$, one has

$$\mathbf{u}^\alpha = \Pi_{i\in I} u_i^{\alpha_i}.$$

By the homomorphism $\phi : \mathcal{O}_X[Z_\mathbf{n}] \to \mathcal{O}_X[L]$, putting $\mathbf{u}^\alpha = \Pi_{i\in I}\, u_i^{\alpha_i}$ and $u'_i = \phi(u_i)$, we have $N = \{\alpha \in \mathbf{Z_n} : \Pi_{i\in I}(u')_i^{\alpha_i} = 1\}$. The N is said to be the orthogonal subgroup of $\mu_\mathbf{n}$ to $K = D(L)$. Let $\mathcal{B}$ denote the $\mathcal{O}_X$- subalgebra $\bigoplus_{\alpha\in N} t^\alpha \mathcal{O}_X$ of $\mathcal{A}^\mathbf{a}_\mathbf{n}$. Note that the $\mathcal{O}_X$-subalgebra B is identified with the kernel of

$$p_1 : \mathcal{A}^\mathbf{a}_\mathbf{n} \to \mathcal{O}_X[L] \bigotimes_{\mathcal{O}_X} \mathcal{A}^\mathbf{a}_\mathbf{n}$$

and

$$p_2 : \mathcal{A}^\mathbf{a}_\mathbf{n} \to \mathcal{O}_X[L] \bigotimes_{\mathcal{O}_X} \mathcal{A}^\mathbf{a}_\mathbf{n}$$

with $p_1(\mathbf{t}^\alpha) = \mathbf{u}'^\alpha \otimes \mathbf{t}^\alpha, p_2(\mathbf{t}^\alpha) = 1 \otimes \mathbf{t}^\alpha$.

Lemma 3([GM], Prop.1.3.2).

(1) $\mu_\mathbf{n}/K \cong D(N)$

(2) $Z^\mathbf{a}_\mathbf{n}/K = Spec\ \mathcal{B}$.

Definition 4.

Given the regular sections $\mathbf{a} = (a_i)_{i\in I}$, *a couple* (Y, G) *is called a generalized Kummer covering of* X *relative to a set of* $\mathbf{a}$ *if there exists a set of positive integers* $\mathbf{n} = (n_i)_{i\in I}$ *each* n_i *prime to the residue characteristics of* X *and a diagonalizable subgroup* K *of* X *such that the couple* (Y, G) *is isomorphic to the couple* $(Z^\mathbf{a}_\mathbf{n}/K, D(N))$.

We remark that there exists the canonical morphism

$$(u, \phi) : (Z^\mathbf{a}_\mathbf{n}, \mu_\mathbf{n}) \to (Z^\mathbf{a}_\mathbf{n}/K, D(N)).$$

Let L_1 be a quotient of $\mathbf{Z_n}$ and L a quotient of L_1. Let $K = D(L), K_1 = D(L_1)$ and $N = \ker(\mathbf{Z_n} \to L), N_1 = \ker(\mathbf{Z_n} \to L_1)$. Then the canonical morphism

$$(Z^\mathbf{a}_\mathbf{n}/K, \mu_\mathbf{n}/K = D(N)) \to (Z^\mathbf{a}_\mathbf{n}/K_1, \mu_\mathbf{n}/K_1 = D(N_1))$$

induces the isomorphism

$$((Z^\mathbf{a}_\mathbf{n}/K)/(K_1/K), (\mu_\mathbf{n}/K)/(K_1/K) = D(N_1))$$

$$\cong (Z^\mathbf{a}_\mathbf{n}/K_1, \mu_\mathbf{n}/K_1 = D(N_1)).$$

Lemma 5([GM], Prop.1.3.5). *Let* $\mathbf{a} = (a_i)_{i\in I}$, $\mathbf{b} = (b_i)_{i\in I}$ *be sets of sections and* $\mathbf{n} = (n_i)_{i\in I}$, $\mathbf{m} = (m_i)_{i\in I}$ *sets of positive integers with* $m_i = n_i q_i$ *where* $q_i \in \mathbb{Z}$, *and* $a_i = g_i^{n_i} b_i$ *with* $g_i \in \Gamma(X, \mathcal{O}_X^*)$ *for* $i \in I$. *Let* L *be a quotient of* $\mathbb{Z}_{\mathbf{n}}$ *and* $K = D(L)$. *Then the canonical morphism*

$$(Z_{\mathbf{m}}^{\mathbf{b}}, \mu_{\mathbf{m}}) \to (Z_{\mathbf{n}}^{\mathbf{a}}, \mu_{\mathbf{n}})$$

induces the isomorphism

$$(Z_{\mathbf{m}}^{\mathbf{b}}/K', D(N')) \to (Z_{\mathbf{n}}^{\mathbf{a}}/K, D(N)).$$

Here $\phi_{\mathbf{n},\mathbf{m}} : \mu_{\mathbf{m}} \to \mu_{\mathbf{n}}$ *is given by*

$$\mathbb{Z}_{\mathbf{n}} \to \mathbb{Z}_{\mathbf{m}}$$

$$(\alpha_i) \mapsto (q_i\alpha_i).$$

$N' = \{(q_i\ \alpha_i) : (\alpha_i) \in \mathbb{Z}_{\mathbf{n}}\}, K' = D(\mathbb{Z}_{\mathbf{m}}/N')$.

Note that the construction and the morphisms above are compatible with arbitrary base change.

Lemma 6([GM], Lemma 1.3.10).
Let $\mathbf{a} = (a_i)$ *be a set of regular sections on* X *and let* a_i *be decomposed into* $a_i = \Pi_{\lambda\in J_i} a_{i\lambda}$. *Putting* $\mathbf{b} = (a_{i\lambda}), \mathbf{n}' = (n_{i\lambda})$ *with* $n'_{i,\lambda} = n_i$, *one has the canonical* X*-morphism:*

$$(Z_{\mathbf{n}'}^{\mathbf{b}}, \mu_{\mathbf{n}'}) \to (Z_{\mathbf{n}}^{\mathbf{a}}, \mu_{\mathbf{n}})$$

which is given by the $\mathcal{O}_X$*-Algebra homomorphism and the group homomorphism*

$$\mathcal{A}_{\mathbf{n}}^{\mathbf{a}} \to \mathcal{A}_{\mathbf{n}'}^{\mathbf{b}}$$

such that

$$t_i \mapsto \Pi_{\lambda\in J_i} t_{i\lambda}$$

$$\psi_{\mathbf{n},\mathbf{n}'} : \mathbb{Z}_{\mathbf{n}} \to \mathbb{Z}_{\mathbf{n}'}$$

$$(\alpha_i) \mapsto (\beta_{i\lambda})$$

such that $\beta_{i\lambda} = \alpha_i$. *Let* $\mathbb{Z}_{\mathbf{n}} = \{(\alpha_i) : 0 \le \alpha_i < n_i\}$. *Putting* $M = \psi_{\mathbf{n},\mathbf{n}'}(\mathbb{Z}_{\mathbf{n}})$, *one has the induced isomorphism*

$$(Z_{\mathbf{n}'}^{\mathbf{b}}/D(\mathbb{Z}_{\mathbf{n}'}/M), \mu_{\mathbf{n}'}/D(\mathbb{Z}_{\mathbf{n}'}/M) = D(M)) \to (Z_{\mathbf{n}}^{\mathbf{a}}, \mu_{\mathbf{n}}).$$

Given each subgroup N *of* $\mathbb{Z}_{\mathbf{n}}$, *putting* $N' = \psi_{\mathbf{n},\mathbf{n}'}(N), K = D(\mathbb{Z}_{\mathbf{n}}/N), K' = D(\mathbb{Z}_{\mathbf{n}'}/N')$, *one has, moreover, the canonical morphism*

$$(Z_{\mathbf{n}'}^{\mathbf{b}}, \mu_{\mathbf{n}'}) \to (Z_{\mathbf{n}}^{\mathbf{a}}/K, \mu_{\mathbf{n}}/K)$$

which induces the isomorphism

$$(Z^{\mathbf{b}}_{\mathbf{n}'}/K', \mu_{\mathbf{n}'}/K' = D(N')) \to (Z^{\mathbf{a}}_{\mathbf{n}}/K, \mu_{\mathbf{n}}/K = D(N)).$$

Lemma 7([GM], Prop.1.8.5).

Let $\mathbf{D} = (D_i)$ *be a set of divisors on a locally noetherian normal scheme* X *and* (Y, G) *a generalized Kummer covering of* X *relative to* $\mathbf{D}$. *Then*

(1) *if* (D_i) *has normal crossings,* Y *is normal,*

(2) *if* (D_i) *are regular divisors with normal crossings, a Kummer covering* $(Z^{\mathbf{D}}_{\mathbf{n}}, \mu_{\mathbf{n}})$ *is regular over the points of* $\bigcup_{i \in I} \mathbf{supp} D_i$.

(3) *if* (D_i) *are regular divisors with normal crossings a generalized Kummer covering* (Y, G) *is regular over the regular points of* $\bigcup_{i \in I} \mathbf{supp} D_i$.

Example 1. *Let* X *be a locally noetherian normal scheme and* a *a section of* $\mathcal{O}_X$ *such that* **div** a *has normal crossings. The Kummer covering* $Z^{a^2}_3 =$ *Spec* $\mathcal{O}_X[T]/(T^3 - a^2)$ *is not normal. The map*

$$\mathcal{O}_X[T]/(T^3 - a^2) \to \mathcal{O}_X[U]/(U^3 - a)$$

$$T \mapsto U^2$$

and the isomorphism

$$\mathbb{Z}_3 \to \mathbb{Z}_3$$

$$\alpha \mapsto 2\alpha$$

give a birational morphism

$$(Z^a_3, \mu_3) \to (Z^{a^2}_3, \mu_3).$$

The normalization of Spec $\mathcal{O}_X[T]/(T^3 - a^2)$ *is Spec* $\mathcal{O}_X[U]/(U^3 - a)$.

Example 2. *Let* X *be a locally noetherian normal scheme and* a *a section of* $\mathcal{O}_X$ *such that* **div** a *has normal crossings. The Kummer covering* $Z^{a^n}_n =$ *Spec* $\mathcal{O}_X[T]/(T^n - a^n)$ *is not irreducible. A morphism*

$$(Z^a_n = \mathit{Spec}\ \mathcal{O}_X[U]/(U^n - a), \mu_n) \to (\mathit{Spec}\ \mathcal{O}_X[T]/(T^n - a^n), \mu_n)$$

defined by

$$\mathcal{O}_X[T]/(T^n - a^n) \to \mathcal{O}_X[U]/(U^n - a)$$

$$T \mapsto U^n$$

and

$$\mathbb{Z}_n \to \mathbb{Z}_n$$

$$1 \mapsto n.$$

This induces an isomorphism

$$(Spec\ \mathcal{O}_X[U^n]/(U^n - a) = X, 0) \cong (Spec\ \mathcal{O}_X[T]/(T^n - a^n)/\mu_n, 0).$$

Proposition 1. *Let X be a locally noetherian normal scheme and $a = \Pi_{i\in I}\, a_i^{k_i}$ a section of $\mathcal{O}_X$. Assume that $\Sigma_{i\in I}$ $\mathbf{div}(a_i)$ has normal crossings. Let n be a positive integer such that the greatest common number $(n, k_i)_{i\in I} = 1$. Consider a Kummer covering $Z_n^a = Spec\ \mathcal{O}_X[T]/(T^n - a)$ and take a map*

$$\mathcal{O}_X[T]/(T^n - a) \to \mathcal{O}_X[(U_i)_{i\in I}]/((U_i^n - a_i)_{i\in I})$$

$$T \mapsto \Pi_{i\in I}\, U_i^{k_i}$$

and a homomorphism

$$\mathbb{Z}_n \to \Pi_{i\in I}\, \mathbb{Z}_n$$

$$\alpha \mapsto (\alpha k_i),$$

whose image we denote by M. This gives a birational morphism

$$(Spec\ \mathcal{O}_X[U_i]/((U_i^n - a_i))/D(\Pi_{i\in I}\, \mathbb{Z}_n/M), D(M)) \to (Spec\ \mathcal{O}_X[T]/(T^n - a), \mu_n).$$

Hence the normalization of $Spec\ \mathcal{O}_X[T]/(T^n - a)$ is idenified with

$$Spec\ \mathcal{O}_X[U_i]/((U_i^n - a_i))/D(\Pi_{i\in I}\, \mathbb{Z}_n/M) = Spec \bigoplus_{\alpha\in M} \mathbf{u}^\alpha \mathcal{O}_X$$

proof: One has an isomorphism

$$\bigoplus_\alpha t^\alpha \mathcal{O}_X \to \bigoplus_\alpha \Pi_{i\in I} u_i^{\alpha k_i} \mathcal{O}_X$$

$$t \mapsto \Pi\, u_i^{k_i}.$$

The integral closure of $\bigoplus_\alpha\ \Pi_{i\in I} u_i^{\alpha\ k_i} \mathcal{O}_X$ is $\bigoplus_{\alpha\in M}\ \mathbf{u}^\alpha \mathcal{O}_X$. Hence this induces a birational morphism

$$(Spec \bigoplus_{\alpha\in M} \mathbf{u}^\alpha \mathcal{O}_X, D(M)) \to (Spec\ \mathcal{O}_X[T]/(T^n - a), \mu_n).$$

Corollary 8. *Assume $e = (n, k_i)_{i\in I} > 1$. One has the isomrphism*

$$(Spec\ \mathcal{O}_X[T]/(T^n - a)/D(\mathbb{Z}_n/e\mathbb{Z}_n), D(e\mathbb{Z}_n)) \cong$$

$$(Spec\ \mathcal{O}_X[U]/(U^{n/e} - \Pi_{i\in I}\ a_i^{k_i/e}), D(\mathbb{Z}_{n/e}))$$

Theorem 9 ([EV], [K2]). *Let X is a locally noetherian regular scheme. Let $\mathcal{L}$ be an invertible sheaf over X. Assume that $\mathcal{L}^{\otimes n}$ is represented by a divisor $D = \Sigma\ \nu_i C_i$ where (C_i) are regular divisors with normal crossings. The local Kummer covering (Z_n^D, μ_n) is well defined globally. It has quotient singularieties over the non regular points on* **supp**D. *Let $f : Z_n^D \to X$ be a structure morphism and $\delta : Y \to Z_n^D$ an arbitrary resolution of singularities. Let Δ be a divisor such that $\Delta + \Sigma_i\ C_i$ has normal crossings. Let $J \subset I$ and $\ell \in \mathbb{Z}$. One has, then,*

$$(f\circ\delta)_*(\Omega_Y^a\langle(f\circ\delta)^*(\Sigma_{i\in J}\ C_i + \Delta)\rangle((f\circ\delta)^*\frac{\ell}{n}\ D)) =$$

$$\bigoplus_{0\leq k<n} \Omega_X^a\langle\{\frac{k}{n}D\} + \{\frac{\ell}{n}\ D\} + \Sigma_{i\in J}\ C_i + \Delta\rangle([\frac{k}{n}D] - \mathcal{L}^{\otimes k} + [\frac{\ell}{n}D] + (-[\frac{\ell}{n}D] + \ell\,\mathbf{div}\ t))$$

. *The Galois group $Gal(R(Y)/R(X)) = Gal(R(Z_D^n)/R(X)) = \mu_n$ acts naturally on*

$$(f\circ\delta)_*(\Omega_Y^a\langle(f\circ\delta)^*(\Sigma_{i\in J}\ C_i + \Delta)\rangle((f\circ\delta)^*\frac{\ell}{n}D))$$

and its invariant part is a direct factor, i.e.,

$$H^0(\mu_n, (f\circ\delta)_*(\Omega_Y^a\langle(f\circ\delta)^*(\Sigma_{i\in J}\ C_j + \Delta)\rangle((f\circ\delta)^*\frac{\ell}{n}\ D)) =$$

$$\Omega_X^a\langle\{\frac{\ell}{n}\ D\} + \Sigma_{i\in J}\ C_i + \Delta\rangle([\frac{\ell}{n}\ D]).$$

proof: see the proof of Theorem 17.

Corollary10(Kawamata-Esnault-Viehweg covering).

Let X is a locally noetherian regular scheme. Let $(\mathcal{L}_j)_{j\in K}$ be invertible sheaves over X. Assume that $\mathcal{L}_j^{\otimes n}$ are represented by divisors $D_j = \Sigma_i\ \nu_{ji}C_i$ with $(C_i)_{i\in I}$ regular divisors with normal crossings. Let $\mathbf{D} = (C_i)_{i\in I}, \mathbf{n} = (n)_{i\in I}$. The local Kummer covering $(Z_{(n)_{j\in K}}^{(D_j)_{j\in K}}, \mu_{(n)_{j\in K}})$ is well defined globally. It has quotient singularieties over the non regular points on **supp** *D_j for each j. Let $f : Z_{(n)_{j\in K}}^{(D_j)_{j\in K}} \to X$ be a structure morphism and $\delta : Y \to Z_{(n)_{j\in K}}^{(D_j)_{j\in K}}$ an arbitrary resolution of singularities. Let $g = f\circ\delta$. Let Δ be a divisor such that $\Delta + \Sigma_j\ D_j$ has normal crossings. Let $J \subset I$.*

Then

$$g_*(\Omega^a_Y\langle g^*(\Sigma_{i\in J}\ C_i+\Delta)\rangle(g^*\Sigma_j\ \frac{\ell_j}{n}\ D_j))=$$

$$\bigoplus_{j\in K}\bigoplus_{0\leq k_j<n}\ \Omega^a_X\langle\{\frac{\Sigma_{j\in K}k_jD_j}{n}\}+\Sigma_{i\in J}C_i+\Delta\rangle($$

$$[\frac{\Sigma_{j\in K}k_jD}{n}]-\otimes^{j\in K}\ \mathcal{L}_j^{\otimes k_j}+[\frac{\Sigma_j\ell_jD_j}{n}]+(-[\frac{\Sigma_j\ell_jD_j}{n}]+\Sigma_j\ell_j\mathbf{div}\ t_j)).$$

The Galois group $Gal(R(Y)/R(X))=\mu_{(n)_{j\in K}}$ *acts naturally on*

$$g_*\Omega^a_Y\langle g^*(\Sigma_{i\in J}\ C_i+\Delta)\rangle(g^*\frac{\Sigma_j\ell_jD_j}{n})$$

and its invariant part is a direct factor, i.e.,

$$H^0(\mu_{(n)_j},g_*(\Omega^a_Y\langle g^*(\Sigma_{i\in J}\ C_i+\Delta)\rangle(g^*\frac{\Sigma_j\ell_jD_j}{n})=$$

$$\Omega^a_X\langle\{\frac{\sum_j\ell_jD_j}{n}\}+\Sigma_{i\in J}\ C_i+\Delta\rangle([\frac{\sum_j\ell_jD_j}{n}]).$$

Let $X_{\mathbf{et}}$ denote the site of the category of etale schemes over X endowed with the etale topology.

Intuitive Definition 11. *Let* $(X_i)_{i\in I}$ *be a set of schemes and the* $e_{ij}: X_{ij}\to X_i$ *etale morphims such that* $\phi_{ij}: X_{ij}\cong X_{ji}$ *are isomorphisms. We identify* X_{ij} *of* $X_{i\mathbf{et}}$ *and* X_{ji} *of* $X_{j\mathbf{et}}$ *for all* $i,j\in I$ *and obtain a new site. This site is said to be a scheme in etale topology.*

Remark. *One can replace the etale topology by arbitrary Grothendieck topology. This process of enlarging the notion of shemes enables us to take polynomial roots of divisors*

This forms in fact a **Gerbe**([GM]). We denote by X_{ijk} the scheme defined by the universal property such that $T\to X_{ij}, T\to X_{ik}$ are X_i-morphisms, $T\to X_{jk}, T\to X_{ji}$ are X_j-morphisms and $T\to X_{ki}, T\to X_{kj}$ are X_k-morphisms respectively.

Definition 12. *Let* X *be a scheme in etale topology. A sheaf* $\mathcal{F}$ *over* X *is defined to be a functor satisfying an exact sequence*

$$\mathcal{F}(X)\longrightarrow\Pi_i\mathcal{F}(X_i)\rightrightarrows\Pi_{ij}\mathcal{F}(X_{ij}).$$

The cohomology groups are calculated by Čeck cohomologies.

We can say a scheme in etale topology is regular or normal and so on if it is of local property as you can easily imagine.

Theorem 13(Hodge-Kodaira-Deligne). *Let X be a complete non singular variety in etale topology over the complex number field and D a divisor which is numerically equivalent to zero. The natural maps*

$$H^b(X, \Omega_X^a(D)) \to H^b(X, \Omega_X^{a+1}(D))$$

are killed and the Hodge spectral sequence degenerates at E_1. Furthermore, let (Δ_i) be a set of regular divisors with normal crossings on X with $\Delta = \Sigma\Delta_i$. Then the natural maps

$$H^b(X, \Omega_X^a\langle\Delta\rangle(D)) \to H^b(X, \Omega_X^{a+1}\langle\Delta\rangle(D))$$

are killed and the Deligne spectral sequence degenerates at E_1

$$E_1^{ab} = H^b(X, \Omega_X^a\langle\Delta\rangle(D)) \Longrightarrow H^{a+b}(X, \Omega_X^\bullet\langle\Delta\rangle(D)).$$

Definition-Proposition 14. *Let X be a scheme in etale topology, $\mathbf{n} = (n_i)_{i\in I}$ and $\mathbf{D} = (D_i)$ a set of divisors on X satisfied the same conditions as in local Kummer coverings. Then $(Z_{\mathbf{n}}^{\mathbf{D}}, \mu_{\mathbf{n}})$ is defined globally as a scheme in etale topology. $(Z_{\mathbf{n}}^{\mathbf{D}}, \mu_{\mathbf{n}})$ is said to be a Kawamata covering if the (D_i) are a set of regular divisors with normal crossings. It is said to be an Esnault-Viehweg covering if the $(\mathbf{supp}D_i)$ are a set of divisors with normal crossings and $I = \{1\}$. It is said to be a Kawamata-Esnault-Viehweg covering if the $(\mathbf{supp}D_i)$ are a set of divisors with normal crossings.*

Theorem 15. *Let X be a locally noetherian regular scheme in etale topology and $\mathbf{D} = (D_i)$ a set of regular divisors with normal crossings. Then a Kawamata covering $(Z_{\mathbf{n}}^{\mathbf{D}}, \mu_{\mathbf{n}})$ is a regular scheme in etale topology.*

Theorem 16. *Let X be a locally noetherian regular scheme in etale topology, $\mathbf{n} = (n_i)_{i\in I}$ and $\mathbf{D} = (D_i)_{i\in I}$ a set of regular divisors with normal crossings on X. Let $\pi : Z = Z_{\mathbf{n}}^{\mathbf{D}} \to X$ be the structure morphism of a Kawamata covering $(Z_{\mathbf{n}}^{\mathbf{D}}, \mu_{\mathbf{n}})$. Let Δ be a divisor such that $\Delta + \Sigma D_i$ has normal crossings. Let $\ell_i \in \mathbf{Z}$ and $J \subset I$. Then*

$$\pi_*(\Omega_Z^a\langle\pi^*(\Sigma_{i\in J}\, D_i + \Delta)\rangle(\pi^*(\Sigma_{i\in I}\frac{\ell_i}{n_i}\, D_i))) =$$

$$\bigoplus_{(k_i)\in\mathbf{Z}_{\mathbf{n}}} \Omega_X^a\langle\Sigma_{i\in I}\{\frac{k_i}{n_i}\}\, D_i + \Sigma_{i\in I}\,\{\frac{\ell_i}{n_i}\}\, D_i + \Sigma_{i\in J}\, D_i + \Delta\rangle(\Sigma_{i\in I}\, - k_i\, \mathbf{div}\, t_i +$$

$$\Sigma_{i\in I}\,[\frac{\ell_i}{n_i}]\, D_i + \{\frac{\ell_i}{n_i}\}\, n_i\, \mathbf{div}\, t_i).$$

The Galois group $Gal(R(Z_{\mathbf{n}}^{\mathbf{D}})/R(X)) = \mu_{\mathbf{n}}$ acts naturally on

$$\pi_*(\Omega_Z^a\langle\pi^*(\Sigma_{i\in J}\, D_i + \Delta)\rangle(\pi^*(\Sigma_{i\in I}\frac{\ell_i}{n_i}\, D_i)))$$

, *the invariant part of which is a direct factor*

$$H^0(\mu_{\mathbf{n}}, \pi_*(\Omega^a_Z\langle\pi^*(\Sigma_{i\in J}\, D_i + \Delta)\rangle(\pi^*(\Sigma_{i\in I}\frac{\ell_i}{n_i}\, D_i)))) =$$

$$\Omega^a_X\langle\Sigma_{i\in J}\, D_i + \Sigma_{i\in I}\, \{\frac{\ell_i}{n_i}\}\, D_i + \Delta\rangle(\Sigma_{i\in I}\, [\frac{\ell_i}{n_i}]\, D_i).$$

proof: The problem is a local question and so X can be seen an affine regular scheme in etale topology. Let $D_i = \mathbf{div}\ z_i$ such that the z_i are a part of regular parameters for X. Let $t_i^{n_i} = z_i$. One has as $\mathcal{O}_X$-modules

$$\mathcal{O}_Z = \bigoplus_{(k_i)\in\mathbb{Z}_{\mathbf{n}}} \Pi_{i\in I}\, t_i^{k_i}\mathcal{O}_X$$

and

(1) for $i \in J$

$$\frac{dt_i}{t_i} = \frac{dz_i}{z_i}$$

(2) for $i \in I \setminus J$

$$dt_i = t_i\, \frac{dz_i}{z_i}$$

(3) as $\mathcal{O}_X$-modules

$$\mathcal{O}_Z(\pi^*\Sigma_{i\in I}\frac{\ell_i}{n_i}D_i) = \bigoplus_{(k_i)\in\mathbb{Z}_{\mathbf{n}}} \Pi_{i\in I}t_i^{k_i} \cdot \Pi_{i\in I}t_i^{-\ell_i}\mathcal{O}_X$$

(4)

$$t_i^{-\ell_i} = z_i^{-[\frac{\ell_i}{n_i}]}t_i^{-\{\frac{\ell_i}{n_i}\}n_i}.$$

Corollary 17. *Let X be a locally noetherian regular scheme in etale topology and $\mathbf{D} = (C_i)_{i\in I}$ a set of regular divisors with normal crossings and $D = \Sigma_{i\in I}\ \nu_i C_i$, , $\mathbf{n} = (n)_{i\in I}$. Let Y be the normalization of an Esnault-Viehweg covering Z_n^D and $\eta : Y \to X$ the structure morphism. Let Δ be a divisor such that $\Delta + \Sigma_{i\in I}\ C_i$ has normal crossings. Let $J \subset I$ and $\ell_i \in \mathbb{Z}$. Let $M = \mathbf{im}(\mathbb{Z}_n \to \mathbb{Z}_{\mathbf{n}}\ (1 \mapsto (\nu_i))$. Then one has*

$$\mathcal{O}_Y = \bigoplus_{(k_i)\in M} \Pi_{i\in I}u_i^{k_i}\mathcal{O}_X$$

and

$$\eta_*\Omega^a_Y\langle\eta^*(\Sigma_{i\in J}\, C_i + \Delta)\rangle(\eta^*(\frac{\ell}{n}\, D)) =$$

$$\bigoplus_{0\leq k<n} \Omega_X^a\langle\{\frac{k}{n}\ D\}+\{\frac{\ell}{n}\ D\}+\Sigma_{i\in J}\ C_i+\Delta\rangle((\,[\frac{k}{n}\ D]-k\ \mathbf{div}\ t)+$$

$$[\frac{\ell}{n}\ D]+(\ell\ \mathbf{div}\ t-[\frac{\ell}{n}\ D]))=$$

$$\bigoplus_{(k_i)\in M} \Omega_X^a\langle\Sigma_{i\in I}\{\frac{k_i}{n}\}\ C_i+\Sigma_i\{\frac{\ell_i}{n}\}C_i+\Sigma_{i\in J}\ C_i+\Delta\rangle(\Sigma_{i\in I}\ ([\frac{k_i}{n}]C_i-k_i\mathbf{div}\ u_i)+$$

$$[\frac{\ell}{n}\ D]+\Sigma_{i\in I}(-[\frac{\ell\nu_i}{n}]\ C_i+\ell\nu_i\mathbf{div}\ u_i)).$$

The Galois group $Gal(R(Y)/R(X))=\mu_n$ acts naturally on

$$\eta_*(\Omega_Y^a\langle\eta^*(\Sigma_{i\in J}\ C_i+\Delta)\rangle(\eta^*(\frac{\ell}{n}\ D)))$$

, the invariant part of which is a direct factor

$$H^0(\mu_n,\eta_*(\Omega_Y^a\langle\eta^*(\Sigma_{i\in J}\ C_i+\Delta)\rangle(\eta^*(\frac{\ell}{n}\ D))))=$$

$$\Omega_X^a\langle\Sigma_{i\in J}\ C_i+\{\frac{\ell}{n}\ D\}+\Delta\rangle([\frac{\ell}{n}\ D]).$$

proof: Let Z be a Kawamata covering $Z_{\mathbf{n}}^{\mathbf{D}}$ and $\pi: Z\to X$ the structure morphism. Then letting $D=\mathbf{div}\ d, C_i=\mathbf{div}\ c_i$,

$$\mathcal{A}(Z_D^n)=\mathcal{O}_X[T]/(T^n-d)\to\mathcal{A}(Z)=\mathcal{O}_X[(U_i)_{i\in I}]/((U_i^n-c_i)_{i\in I})$$

$$T\to\Pi_{i\in I}\ U_i^{\nu_i},$$

one has the canonical morphism $f: Z\to Z_D^n$ which factors $\pi: Z\to X$ and Let $M=\mathbf{im}(\mathbb{Z}_n\to\mathbb{Z}_\mathbf{n}(1\mapsto(\nu_i))$. Then the structure sheaf of the normalization of the Esnault-Viehweg covering relative to D is

$$\mathcal{O}_Y=\bigoplus_{(k_i)\in M}\Pi_{i\in I}u_i^{k_i}\mathcal{O}_X.$$

Making account of

(1)

$$\mathcal{O}_X(\frac{\ell}{n}D)=\mathcal{O}_X(\{\frac{\ell}{n}\}\ \mathbf{div}(t^n)+[\frac{\ell}{n}D])$$

$$=t^{-\{\frac{\ell}{n}\}n}\mathcal{O}_X([\frac{\ell}{n}D])$$

(2)

$$\frac{k\nu}{n}=[\frac{k\nu}{n}]+\{\frac{k\nu}{n}\}$$

(3)

$$M = (\{\frac{k\nu_i}{n}\}n)_{i\in I}, k_i = k\nu_i$$

(4)

$$u_i^{\{\frac{k\nu_i}{n}\}n}\mathcal{O}_X = \mathcal{O}_X(-\{\frac{k\nu_i}{n}\}\ \mathbf{div}(u_i^n)) =$$

$$\mathcal{O}_X([\frac{k\nu_i}{n}]C_i - \frac{k\nu_i}{n}\ \mathbf{div}(u_i^n)).$$

Thus

$$\mathcal{O}_X(\Sigma_{i\in I}([\frac{k\nu_i}{n}]C_i - \frac{k\nu_i}{n}\ \mathbf{div}(u_i^n))) = \mathcal{O}_X([\frac{k}{n}\ D] - k\ \mathbf{div}(t)).$$

Hence one has

$$\eta_*\Omega_Y^a\langle\eta^*(\Sigma_{i\in J}\ C_i + \Delta)\rangle(\eta^*(\frac{\ell}{n}\ D)) =$$

$$\bigoplus_{(k_i)\in M}\ \Omega_X^a\langle\Sigma_{i\in I}\{\frac{k_i}{n}\}\ C_i + \{\frac{\ell}{n}D\} + \Sigma_{i\in J}\ C_i + \Delta\rangle(\Sigma_{i\in I}\ ([\frac{k_i}{n}]C_i - k_i\mathbf{div}\ u_i) +$$

$$([\frac{\ell}{n}\ D] + \Sigma_i(-[\frac{\ell\nu_i}{n}]\ C_i + \ell\nu_i\ \mathbf{div}\ u_i)).$$

Corollary 18. *Let X be a locally noetherian regular scheme in etale topology, $\mathbf{D} = (C_i)_i = (\mathbf{div}\ c_i)_i$ a set of regular divisors with normal crossings on X, $\mathbf{n} = (n_i)_{i\in I}$ and $D_j = \mathbf{div}\ d_j = \Sigma_i\nu_{ji}\ C_i$. Let $\pi : Z_{(n)_j}^{(D_j)_j} \to X$ be the structure morphism of a Kawamata-Esnault-Viehweg covering $(Z_{(n)_j}^{(D)_j}, \mu_{(n)_j})$. Let Δ be a divisor such that $\Delta + \Sigma_i\ C_i$ has normal crossings. Let Y be the normalization of Kawamata-Esnault-Viehweg covering which factors $Z_{\mathbf{D}}^{\mathbf{n}} \to X$ and $\eta : Y \to X$ the structure morphism. Let $Z = Z_{\mathbf{D}}^{\mathbf{n}}$ be a Kawamata covering. Let*

$$M = \mathbf{im}(\mathbb{Z}_{(n)_j} \to \mathbb{Z}_{(n)_i}\ (k_j \mapsto (m_i = \Sigma_j\ k_j\nu_{ji}))_i.$$

Let ϕ

$$\mathcal{A}(Z_{(D_j)_j}^{(n)_j}) = \mathcal{O}_X[(T_j)_j]/((T_j^n - d_j)_j) \to \mathcal{A}(Z) = \mathcal{O}_X[(U_i)_i]/((U_i^n - c_i)_i)$$

be a $\mathcal{O}_X$-homomorphism defined by $T_j \mapsto \Pi_i\ U_i^{\nu ji}$. Then

$$\mathcal{O}_Y = \oplus_{(m_i)\in M}\ \Pi_i\ u_i^{m_i}\mathcal{O}_X$$

and

$$\eta_*(\Omega_Y^a\langle\eta^*(\Sigma_{i\in Q}\ C_i + \Delta)\rangle(\eta^*(\Sigma_j\frac{\ell_j}{n}\ D_j))) =$$

$$\bigoplus_j \bigoplus_{0\leq k_j<n} \Omega^a\langle\{\frac{\Sigma_j k_j D_j}{n}\}+\{\frac{\Sigma_j \ell_j D_j}{n}\}+\Sigma_{i\in Q}C_i+\Delta\rangle(([\frac{\Sigma_j k_j D_j}{n}] -\Sigma_j k_j \mathbf{div}\ t_j)+$$

$$[\frac{\Sigma_j \ell_j D_j}{n}]+(-[\frac{\Sigma_j \ell_j D_j}{n}]+\Sigma_j \ell_j \mathbf{div}\ t_j)) =$$

$$\bigoplus_{(m_i)\in M} \Omega^a_X\langle\Sigma_i\{\frac{m_i}{n}\}\ C_i+\Sigma_i\{\frac{\Sigma_j \ell_j \nu_{ji}}{n}\}C_i+\Sigma_{i\in Q}\ C_i+\Delta\rangle($$

$$\Sigma_i\ ([\frac{m_i}{n}]\ C_i\ -m_i\ \mathbf{div}\ u_i)+([\frac{\Sigma_j\ \ell_j D_j}{n}]+\Sigma_i(-[\frac{\Sigma_j \ell_j \nu_{ji}}{n}]C_i+(\Sigma_j \ell_j \nu_{ji})\ \mathbf{div}\ u_i)).$$

The Galois group $Gal(R(Z^{(D_j)_j}_{(n)_j})/R(X))=\mu_{(n)_j}$ acts naturally on

$$\eta_*(\Omega^a_Y\langle\eta^*(\Sigma_{i\in Q}\ C_i+\Delta)\rangle(\eta^*(\frac{\Sigma_j \ell_j D_j}{n})))$$

, the invariant part of which is a direct factor

$$H^0(\mu_{(n)_j},\eta_*(\Omega^a_Y\langle\eta^*(\Sigma_{i\in Q}\ C_i+\Delta)\rangle(\eta^*(\frac{\Sigma_j \ell_j D_j}{n})))) =$$

$$\Omega^a_X\langle\Sigma_{i\in Q}\ C_i+\{\frac{\Sigma_j\ \ell_j D_j}{n}\}+\Delta\rangle([\frac{\Sigma_j \ell_j D_j}{n_i}]).$$

Theorem 19(Deligne-Illusie, [DI]). *Let k be a field of characteristic $p>0$, $S=Spec\ k$, $\tilde{S}=Spec\ W_2(k)$ and X an S-scheme in etale topology. Let D and $\tilde{D}$ be divisors which are numerically equivalent to zero on X and $\tilde{X}$, respectively. Associated to any flat $\tilde{S}$-scheme in etale topology $\tilde{X}$ lifting X an isomorphism is determined canonically:*

$$\phi_{\tilde{X}}:\bigoplus_{a<p}\Omega^a_{X'/S}(D')[-a]\cong\tau_{<p}F_*\Omega^\bullet_{X/S}(D)$$

in $D(X')$ such that $\mathcal{H}^a\phi_{\tilde{X}}=C^{-1}$ for $a<p$.

Corollary 20. *Let k be a field of characteristic $p>0$, $S=Spec\ k$, $\tilde{S}=SpecW_2(k)$ and X an S-scheme. Let $\mathbf{D}=(D_i)_{i\in I}$ be a set of smooth divisors with normal crossings on X. Let Δ be a divisor on X such that $\Delta+\Sigma_i D_i$ has normal crossings. Let (n_i) be a set of positive integer prime to p and the k_i integers.*

(1) *Assume that $\Sigma_{i\in I}\frac{k_i}{n_i}D_i$ is numerically equivalent to zero.*

Associated to any flat $\tilde{S}$-couple of scheme and relative divisor $(\tilde{X},\tilde{D}+\tilde{\Delta})$ lifting $(X,D+\Delta)$, an isomorphism is determined canonically:

$$\phi_{(\tilde{X},\tilde{D}+\tilde{\Delta})}:\bigoplus_{a<p}\Omega^a_{X'/S}\langle\Delta'+\Sigma_{i\in J}D'_i+\{\Sigma_{i\in I}\frac{k_i}{n_i}D'_i\}\rangle([\Sigma_{i\in I}\frac{k_i}{n_i}D'_i])[-a]\cong$$

$$\tau_{<p}F_*\Omega^\bullet_{X/S}\langle\Delta + \Sigma_{i\in J}D_i + \{\Sigma_{i\in I}\frac{k_i}{n_i}D_i\}\rangle([\Sigma_{i\in I}\frac{k_i}{n_i}D_i])$$

in $D(X')$ *such that* $\mathcal{H}^a\phi_{\tilde{X}} = C^{-1}$ *for* $a < p$.

Corollary 21. *Let* $(E_j)_{j\in J}$ *be a set of reduced divisors different from each other and any sum of* D_i*'s. Instead of (i), we assume that* $\Sigma_{i\in I}\frac{k_i}{n_i}D_i + \Sigma_{j\in J}\frac{\ell_j}{n_j}E_j$ *is numerically equivalent to zero and that* $[\Sigma_{j\in J}\frac{\ell_i}{n_j}E_j] = 0$. *Assume moreover that* (E_j) *has a lifting property. Then*

$$\phi_{(\tilde{X},\tilde{D}+\tilde{\Delta})} : \bigoplus_{a<p} \Omega^a_{X'/S}\langle\Delta' + \Sigma_{i\in J}D'_i + \{\Sigma_{i\in I}\frac{k_i}{n_i}D'_i\}\rangle([\Sigma_{i\in I}\frac{k_i}{n_i}D'_i])[-a] \cong$$

$$\tau_{<p}F_*\Omega^\bullet_{X/S}\langle\Delta + \Sigma_{i\in J}D_i + \{\Sigma_{i\in I}\frac{k_i}{n_i}D_i\}\rangle([\Sigma_{i\in I}\frac{k_i}{n_i}D_i]).$$

Further the Hodge-Deligne spectral sequence degenerates at E_1 *for* $a + b < p$

$$E_1^{ab} = H^b(X, \Omega^a_X\langle\Delta + \Sigma_{i\in J}D_i + \{\Sigma_{i\in I}\frac{k_i}{n_i}D_i\}\rangle([\Sigma_{i\in I}\frac{k_i}{n_i}D_i]) \Rightarrow$$

$$H^{a+b}(X, \Omega^\bullet_X\langle\Delta + \Sigma_{i\in J}D_i + \{\Sigma_{i\in I}\frac{k_i}{n_i}D_i\}\rangle([\Sigma_{i\in I}\frac{k_i}{n_i}D_i]).$$

Remark 22. *We can take the coefficients of divisors in real numbers or adic numbers.*

REFERENCES

[Ar] Arakelov, *Families of algebraic curves with fixed degeneracies*, Izv.Akad.Nauk SSSR **Ser.Mat. 35** (1971), 1277–1302.

[DM1] Deschamps, M.M., Menegaud, R.L, *Applications rationelles separables dominantes sur une variété de type général*, Bull.Soc. Math. France **106** (1978), 279–287.

[DM2] *Surfaces de type général dominées par une variété fixe*, C.R.Acd.Sci.Paris Ser.A **765** (1979), 288.

[DM3] *Surfaces de type général dominées par une variété fixe. II*, C.R.Acd.Sci.Paris Ser.A **342** (1980), 7.

[EV1] Esnauld, H and Viehweg, E, *Logarithmic de Rham complexes and vanishing theorems*, Invent. Math. **86** (1986), 161–194.

[Fal1] Faltings, G, *Endlichkeitssätze für abelshe Varietäten über Zählkörpern*, Invent. Math. **75** (1984), 381.

[Fal2] *Arakelov's Theorem for Abelian Varieties*, Invent.math. **73** (1983), 337–347.

[F1] Fujita, T, *On Kähler fiber spaces over curves*, J. Math. Soc. Japan **30** (1978), 779–794.

[F2] Fujita, T, *On semi-positive line bundles*, Proc Japan Acad **56 Ser. A** (1980), 393-396.

[Gra] Grauert, H, *Mordells Vermutung über rationale Punkte auf Algebraischen Kurven und Funktionenkörper*, Inst. Hautes Études Sci. Publ. Math. **25** (1965), 131–149.

[Gr] Grothendieck, A, *Technique de descente et théorèmes d'existence en géométrie algébrique; IV. Les schémas de Hilbert*, "Fondements de la géometrie algébrique, Extraits du Séminaire Bourbaki 1957-1962," Secrétariat math., Paris, 1962.

[GM] Grothendieck, A., Murre, P., "The tame fundamental group of a formal neighbourhood of a divisor with normal crossings on a scheme," Lect. Notes Math.208, 1971.

[I] Iitaka, S, "Algebraic Geometry - an introduction to birational geometry of algebraic varieties.," G.T.M.Springer-Verlag, Berlin, Heidelberg, New York, 1981.

[Kaw 1] Kawamata, Y, *A generalization of Kadaira-Ramanujam's vanishing theorem*, Math. Ann. **261** (1982), 43–46.

[Kaw 2] *Characterization of abelian varieties*, Comp. Math. **43** (1985), 1–46.

[Kaw 3] *Minimal models and the Kodaira dimension of algebraic fibre spaces*, J. reine angew. Math. **363** (1985), 1–46.

[Kob 1] "Hypebolic manifolds and holomorphic mappings," Dekker, New York, 1970.

[Kob2] *Negative vector bundles and complex Finsler structures*, Nagoya Math. J. **57** (1975), 153–166.

[Kob3] *The first Chern class and holomorphic symmetric tensor fields*, J. Math. Soc. Japan **32** (1980), 325–329.

[Kob4] "Differential Geometry of Complex Vector Bundles," Publ. Math. Soc. Japan, 1987.

[KO] Kobayashi, S., Ochiai, T, *Meromorphic mappings into complex spaces of general type*, Invent. Math. **31** (1975), 7-16.

[Kol1] Kollár, J, *Higher direct images of dualizing sheaves I*, Ann. Math. **123** (1986), 11–42.

[Kol2] *Higher direct images of dualizing sheaves II*, Ann. Math. **124** (1986), 171–202.

[Kol3] *Subadditivity of the Kodaira dimension: Fibres of eneral type*, "Sympos. Alg. Geom. Sendai," Adv. Stud. Pure Math. North-Holland 1987, 1985, pp. 361–398.

[L1] Lang, S, *Higher dimensional Diophantine problems*, Bull. Amer. Math. Soc. **80** (1974), 779–787.

[L2] "Fundametals of Diophantine Geometry," Springer-Verlag, 1983.

[L3] *Hyperbolic and Diophantine analysis*, Bull. Amer. Math. Soc. (N.S.) **14** (1986), 159–205.

[L4] *Introduction to complex hyperbolic spaces*, Springer-Verlag, New York, Berlin and Heiderberg.

[M1] Maehara,K, *Family of varieties dominated by a variety*, Proc. J. Acad. **55(A)** (1979), 146-151.

[M1-bis] *Flat deformation theory of a family dominated by another family.*

[M2] *Finiteness property of varieties of general type*, Math. Ann. **265** (1983), 101–123.
[M3] *The weak 1-positivity of direct image sheaves*, J. reine angew Math. **364** (1986), 112–129.
[M4] *The Mordell-Bombieri-Noguchi conjecture over function fields*, Kodai Math. J. **11** (1988), 1–4.
[M5] *On the higher dimensional Mordell conjecture over function fields*, Osaka J. Math. **28** (1991), 255–261.
[M6] *Kawamata covering and logarithmic de Rham complex*, "Alg. Geom., Mejiro-Hongo Seminar," Math.Sci.Gakusyuin uni., 1990, pp. 71–90.
[Ma] Manin Yu.I, *Rational points of algebraic curves over function fields*, Izv. Akad. Nauk SSSR **50** (1966), 1395–1440; Amer. Math. Soc. Transl.(2).
[Mat] Matsumura, H, *Birational automorphism groups of algebraic varieties*, Acad. Natz. Lincei **Ser.7 Vol.34** (1963), 151–155.
[Mor] Mordell,L.J., "Diophantine Equations," Pure and Applied Math. Columbia uni, New York, 1969.
[N1] Noguchi, J., *A higher dimensional analogue of Mordell's conjecture over function fields*, Math. Ann. **258** (1981), 207–212.
[N2] *Hyperbolic fibre spaces and Mordell's conjecture over function fields*, Publ. Res. Inst. Math. Sci. **21** (1985), 27–46.
[N3] *Hyperbolic Manifolds and Diophantine geometry*, Sugaku Expositions **4 No.1** (1991), 63–81.
[N4] *Logarithmic Jet Spaces and extensions of de Franchis' Theorem*, "Contributions to several complex variables, In honour of Wilhelm Stoll," Friedr. Vieweg and Sohn, Braunschweig/Wiesbaden, pp. 227–249.
[N5] *Moduli spaces of holmorphic mappings into hypebolically imbedded complex spaces and locally symmetric spaces*, Invent. math. **93** (1988), 15–34.
[NO] Noguchi, J., Ochiai, T., *Geometric function theory in several complex variables*, Transl. Math. Monographs **no.80** (1990), Amer. Math. Soc., Providence.
[NS] Noguchi, J., Sunada, T, *Finiteness of the family of rational and meromorphic mappings into algebraic varieties*, Amer. J. Math. **104** (1982), 887-900.
[Pa] Parshin, A.N., *Algebraic curves over function fields I*, Izv. Akad. Nauk SSSR **Ser. Mat. 32** (1968), 1145–1170.
[Pe] Peters C.A.M, *Rigidity for variations of Hodge structure and Arakelov-type finiteness theorems*, Compostio Math. **75** (1990), 113–126.
[SZ] Saito, M.H., Zucker, S, *Classification of non-rigid families of K3 surfaces and a finiteness theorem of Arakelov-type*, Math. Ann. **289** (1991), 1–31.
[V1] Viehweg, E, *Vanishing theorems*, J. reine angew. Math. **335** (1982), 1–8.
[V2] *Weak positivity and the additivity of the Kodaira dimension for certain fibre spaces*, Adv. Studies Pure Math. **1** (1983), 329–353.
[V3] *Weak positivity and the additivity of the Kodaira dimension for certain fibre spaces II: The local Torelli map*, "Classification of algebraic and analytic varieties, Katata," Birkhäuser, 1982, pp. 567–589.
[V4] *Weak positivity and the stability of certain Hilbert points I*, Invent. math. **96**, 639–667.
[V5] *Weak positivity and the stability of certain Hilbert points II.*
[V6] *Weak positivity and the stability of certain Hilbert points III.*
[ZP] Zarkhin, Yu.G., Parshin, A.N., *Finiteness problems in Diophantine geometry*, Amer. Math. Soc. Transl.(2) **Vol. 143** (1989).

Tokyo Institute of Polytechnics, 1583 Iiyama, Atsugi, Kanagawa 243-02 Japan