Sufficient Conditions for Univalent Functions

By Mamoru NUNOKAWA

Department of Mathematics, University of Gunma

(布川 護, 群馬大学)

Let A(p) denote the class of functions of the form

$$f(z) = z^{p} + \sum_{n=p+1}^{\infty} a_{n}z^{n}$$
  $(p \in N = \{1,2,3,\cdots\})$ 

which are analytic in the unit disk  $U = \{z : |z| < 1\}$ .

Ozaki, Ono and Umezawa [5] proved the following result.

Theorem A. If  $f(z) \in A(1)$  and |f''(z)| < 1 in U, then f(z) is univalent in U.

Recently, Nunokawa, Kwon and Cho [2] have proved the following result.

Theorem B. Let  $p \ge 2$ . If  $f(z) \in A(p)$  and suppose that

$$|f^{(p+1)}(z)| < 2(p!)$$
 in U,

then f(z) is p-valent in U.

Applying Schwarz's lemma, it is easily confirmed that Theorem A and the following Theorem A' are equivalent.

Theorem A'. If  $f(z) \in A(1)$  and |zf''(z)| < 1 in U, then f(z) is univalent in U.

We here obtain the following result concerned with Theorem A and Theorem A'. Theorem 1. Let  $f(z) \in A(1)$  and suppose that

(1) 
$$\left| \arg \left( zf''(z) + \frac{1}{2} \right) \right| < \pi \text{ in } U.$$

Then f(z) is univalent in U.

Proof. Let us put p(z) = f'(z) and

$$\phi(z) = \frac{1 - p(z)}{1 + p(z)}.$$

Then, if there exists a point  $z_0 \in U$  such that

Re 
$$p(z) > 0$$
 for  $|z| < |z_0|$   
Re  $p(z_0) = 0$ ,

then we have that  $\phi(0) = 0$ ,  $|\phi(z)| < 1$  for  $|z| < |z_0|$  and  $|\phi(z_0)| = 1$ .

Applying the same method as the proof of [ 3, Theorem 1 ], we have that  $z_0 \dot{p}(z_0) \ \ \text{is a real number and}$ 

$$z_0 p'(z_0) \le -\frac{1}{2} (1 + |p(z_0)|^2) \le -\frac{1}{2}$$
.

This contradicts (1). Therefore we have

Re 
$$f'(z) > 0$$
 in U.

From Noshiro's theorem [1], f(z) is univalent in U. This completes our proof.

Concerning a sufficient conditions for the univalency of f(z), it is trivial that Theorem 1 is better than Theorem A and Theorem A'.

Theorem 2. Let  $f(z) \in A(p)$  and suppose that

$$| arg (zf^{(p+1)}(z) + \frac{p!}{2}) | < \pi \text{ in } U.$$

Then f(z) is p-valent in U.

Proof. Applying the same method as the proof of Theorem 1, we have

Re 
$$f^{(p)}(z) > 0$$
 in U.

Then, from Ozaki's theorem [4, Theorem 2], f(z) is p-valent in U. This completes our proof.

Theorem 2 is also better than Theorem B.

## References

- [1] K. Noshiro: On the theory of schlicht functions. J. Fac. Sci. Hokkaido Univ., (1)2, 129-155 (1934-1935).
- [2] M. Nunokaw, O. Kwon and N. K. Cho: On the multivalent functions, Tsukuba J. Math., (1)15, 141-143 (1991).
- [3] M. Nunokawa: On properties of non-Carathéodory functions, Proc. Japan Acad., 68A(6), 152-153 (1992).
- [4] S. Ozaki: On the theory of multivalent functions, Sci. Rep. Tokyo Bunrika
  Daigaku A, 2, 167-188 (1935).
- [5] S. Ozaki, I. Ono and T. Umezawa: On a general second order derivative, Sci.

  Rep. Tokyo Bunrika Daigaku, A, 5, 111-114 (1956).