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Zygmund Type Estimates and Mapping Properties of Operators with Power-Logarithmic Kernels in Generalized Hölder Spaces

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Abstract

Zygmund type estimates for the integral with power-logarithmic kernel with a variable upper limit and for its inversion are obtained. The results are applied to study mapping properties of operators with power-logarithmic kernels in generalized Hölder spaces $H_0^\omega([a,b])$ with any modulus of continuity ω and to prove an isomorphism between these spaces realized by the above operators.

1. Introduction

Let $H_0^\lambda([a,b])$ be the space $H^\lambda([a,b])$ of Hölderean functions f on a finite interval $[a,b]$ of the real axis such that $f(a)=0$. It is well known by a classical Hardy-Littlewood theorem [3] that if $0<\lambda<1$, $0<\alpha<1$ and $\lambda+\alpha<1$, the Riemann-Liouville fractional integration operator

$$(1) \qquad (I_{a+}^\alpha \phi)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1}\phi(t)dt$$

maps the Hölder space $H_0^\lambda([a,b])$ boundedly into $H_0^{\lambda+\alpha}([a,b])$. This statement was generalized in various directions (see [12, §§3,4,13,17] for historical notes and the review of such results). In particular, in [11] and [9] the Hardy-Littlewood theorem was extended to the weighted Hölder spaces $H_0^\lambda([a,b];\rho) = \{g : \rho g \in H_0^\lambda([a,b])\}$ with the power weights $\rho(x) = (x-a)^\mu(b-x)^\nu$ and

$$(2) \qquad \rho(x) = \prod_{k=1}^{m} |x-x_k|^{\mu_k}, \; a \leqq x_1 \leqq \ldots \leqq x_m \leqq b,$$

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concentrated at the end and inner points of $[a, b]$, respectively. Moreover, in [9] it was shown that I^α_{a+} implements an isomorphism between the spaces $H^\lambda_0([a, b]; \rho)$ and $H^{\lambda+\alpha}_0([a, b]; \rho)$. We also note that such an isomorphism between the spaces $H^\lambda_0([a, b])$ and $H^{\lambda+\alpha}_0([a, b])$ was contained in embryo in [3].

Under certain conditions on the characteristic ω in [7] (see also [12, §13.6] and [14]) it was proved that the operator I^α_{a+} of fractional integration implements an isomorphism between the generalized Hölder spaces $H^\omega_0([a, b])$ and $H^{\omega_\alpha}_0([a, b])$ with $\omega_\alpha(h) = h^\alpha\omega(h)$. This result was extended in [8], [13] and [14] to the weighted generalized Hölder spaces $H^\omega_0([a, b]; \rho)$ with $\rho(x) = (x - a)^\mu(b - x)^\nu$, in [4] to the generalized Hölder spaces H^ω_p with the integral modulus of continuity and in [15] to the convolution operators (see also [12, §§13,17] in this connection). It should be noted that the central point of the investigations in [7], [8], [13]-[15] was the determination of estimates of Zygmund type for the fractional integrals $I^\alpha_{a+}\phi$ and the fractional derivative $D^\alpha_{a+}\phi$.

An analogue of Hardy-Littlewood's theorem for the so-called operators with power-logarithmic kernels

$$(I^{\alpha,\beta}_{a+}\phi)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x - t)^{\alpha-1}\log^\beta\left(\frac{\gamma}{x - t}\right)\phi(t)dt, \tag{3}$$

for $-\infty < a < b < \infty$, $\alpha > 0$, $\gamma > b - a$ and a natural number β, in the spaces $H^\lambda_0([a, b])$ and $H^\lambda_0([a, b]; \rho)$ with the weight (2) were obtained in [5] and [6], respectively. In [6] it was also shown that the operator $I^{\alpha,\beta}_{a+}$ with natural β implements an isomorphism between the spaces $H^\lambda_0([a, b]; \rho)$ and $H^{\lambda+\alpha,\beta}_0([a, b]; \rho)$. In [12, §21] the former results were extended to the operator (3) with a non-negative β.

This paper is devoted to obtain the Zygmund type estimates for the integral (3) and for its inversion, and to apply such results to the investigation of mapping properties of the operator $I^{\alpha,\beta}_{a+}$ in the generalized Hölder spaces $H^\omega_0([a, b])$. Section 2 contains preliminary information. Section 3 deals with the proving the Zygmund type estimate for the integral $I^{\alpha,\beta}_{a+}\phi$ given in (3). Section 4 is devoted to obtain such an estimate for the integral $(I^{\alpha,1}_{0+})^{-1}f$, where $(I^{\alpha,1}_{a+})^{-1}$ is the operator inverse to $I^{\alpha,1}_{a+}$. In Section 5 we apply these results to give conditions for the operator $I^{\alpha,\beta}_{a+}$ to map from the space H^ω_0 into $H^{\omega_{\alpha,\beta}}_0$, $\omega_{\alpha,\beta}(h) = \omega(h)t^\alpha\log^\beta(\gamma/h)$, and to be an isomorphism of these spaces.

2. Preliminaries

Let $[a, b]$ be a finite interval of the real axis, a function f be given on $[a, b]$, and

$$\omega(f, h) = \sup_{0<t<h}\ \sup_{x,x+t\in[a,b]} |f(x + t) - f(x)| \tag{4}$$

be the modulus of continuity of f. Let $\omega(h)$ be a continuous and almost increasing function on $[0, b - a]$ such that $\omega(0) = 0$. We denote by $H^\omega = H^\omega([a, b])$ the space of functions $f(x)$ with the finite norm

$$\|f\|_{H^\omega} = \|f\|_{C([a,b])} + \sup_{h>0}\frac{\omega(f, h)}{\omega(h)}. \tag{5}$$

We also denote by $H_0^\omega = H_0^\omega([a,b])$ a subspace of $H^\omega = H^\omega([a,b])$:

$$H_0^\omega = H_0^\omega([a,b]) = \{f \in H^\omega : \ f(a) = 0\}, \tag{6}$$

and define the norm by

$$\|f\|_{H_0^\omega} = \|f\|_{H^\omega}.$$

In particular, if $\omega(h) = h^\lambda$, then $H^\omega = H^\lambda$ and $H_0^\omega = H_0^\lambda$ are the spaces of usual Hölderian functions (see, e.g. [12, §1.1]).

For $\delta \geqq 0, \nu \geqq 0$, we say that

$$\omega \in \Phi_\nu^\delta, \tag{7}$$

if the function $\omega(t)$ satisfy the conditions

$$\int_0^t \left(\frac{t}{\xi}\right)^\delta \omega(\xi)\,\frac{d\xi}{\xi} \leqq c\omega(t)\ . \tag{8}$$

and

$$\int_t^{b-a} \left(\frac{t}{\xi}\right)^\nu \omega(\xi)\,\frac{d\xi}{\xi} \leqq c\omega(t) \tag{9}$$

with a constant $c > 0$. Φ_ν^δ is the subspace of the Bari-Stechkin class Φ_ν (see [2]). Note that the class Φ_ν^δ is empty if $\delta \geqq \nu$. Therefore we assume that $0 < \delta < \nu$.

Let $D_{a+}^\alpha\phi$ be the Riemann-Liouville fractional derivative of order α with $0 < \alpha < 1$:

$$(D_{a+}^\alpha\phi)(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x (x-t)^{-\alpha}\phi(t)dt, \quad 0 < \alpha < 1. \tag{10}$$

The following assertions are true:

Theorem A. [12, Theorem 13.15] *Let $\phi(x)$ be a continuous function on $[a,b]$ and $\phi(a) = 0$. If $0 < \alpha < 1$, then the Zygmund type estimate*

$$\omega(I_{a+}^\alpha\phi, h) \leqq ch\int_h^{b-a} \frac{\omega(\phi,t)}{t^{2-\alpha}}dt, \quad c > 0, \tag{11}$$

holds for the fractional integral $I_{a+}^\alpha\phi$.

Theorem B. [12, Theorem 13.16] *Let $\phi(x)$ be a continuous function on $[a,b]$ and $\phi(a) = 0$. If $0 < \alpha < 1$, then the Zygmund type estimate*

$$\omega(D_{a+}^\alpha\phi, h) \leqq c\int_0^h \frac{\omega(\phi,t)}{t^{1+\alpha}}dt, \quad c > 0, \tag{12}$$

holds for the fractional derivative $D_{a+}^\alpha\phi$.

Theorem C. [12, Theorem 13.17] *Let* $0 < \alpha < 1$ *and* $\omega(t) \in \Phi^0_{1-\alpha}$. *Then the operator* I^α_{a+} *maps* H^ω_0 *isomorphically onto* $H^{\omega_\alpha}_0$, $\omega_\alpha(t) = t^\alpha \omega(t)$.

It is known [10] (see also [12, §34.2]) that the following characterization and inversion of the operator $I^{\alpha;\beta}_{0+}$ given in (3) for $\beta = 1$ hold valid in terms of the different construction of Marchaud type via the special Volterra function

$$\mu_\alpha(x) = -\int_0^\infty \frac{x^{t-\alpha}}{\Gamma(t-\alpha+1)} e^{t\psi(t)} dt, \tag{13}$$

where α is any complex number and $\psi(z) = \Gamma'(z)/\Gamma(z)$.

Theorem D. [12, Theorem 34.1] *For a function* $f \in L_p(a,b)$ $(-\infty < a < b < \infty)$, *to be representable in the form* $f = I^{\alpha,1}_{a+}\phi$ $(0 < \alpha < 1)$ *with* $\phi \in L_p(a,b)$, *it is necessary when* $1 < p < \infty$ *and sufficient when* $1 \leqq p < \infty$ *that the limit*

$$(Bf)(x) = \lim_{\epsilon \to 0} \int_a^{x-\epsilon} [f(x) - f(t)]\mu'_\alpha(x-t)dt \tag{14}$$

exists in $L_p(a,b)$ (*we suppose that* $f(x) = 0$ *outside of the interval* $[a,b]$). *If this condition is satisfied, then the function* $\phi(x)$ *is given by*

$$\phi(x) = \mu_\alpha(x-a)f(x) - (Bf)(x). \tag{15}$$

From the properties of the Volterra function (13) we obtain the behaviour of $\mu_\alpha(x)$ and its derivative $\mu'_\alpha(x)$, as $|x| \to 0$ (see [1, §18.3] and [12, §32.1]),

$$\mu_\alpha(x) = \frac{x^{-\alpha}}{\Gamma(1-\alpha)\log x}[1 + O(1)], \tag{16}$$

$$\mu'_\alpha(x) = -\frac{\alpha x^{-\alpha-1}}{\Gamma(1-\alpha)\log x}[1 + O(1)]. \tag{17}$$

In what follows, we shall denote by c, c_1, c_2, etc. the different positive constants, which do not depend on x, and suppose that all integrals will be convergent.

3. Zygmund type estimate for the integral with power-logarithmic kernel

Let a function ϕ be given on a finite interval $[a,b]$, $\omega(\phi,h)$ be the modulus of continuity of ϕ defined in (4) and $I^{\alpha;\beta}_{a+}\phi$ be the integral (3). The following analogy of Theorem A is true:

Theorem 1. *Let* $\phi(x)$ *be a continuous function on* $[a,b]$ *with* $\phi(a) = 0$ *and* $\gamma > b - a$. *Then the Zygmund type estimate*

$$\omega(I^{\alpha,\beta}_{a+}\phi, h) \leqq c \log^\beta\left(\frac{\gamma}{h}\right)\left[h^\alpha \omega(\phi,h) + h\int_h^{b-a} \frac{\omega(\phi,t)}{t^{2-\alpha}}dt\right] \tag{18}$$

holds with $0 < \alpha < 1$ *and* $\beta > 0$ *for the integral* $I_{a+}^{\alpha,\beta}\phi$.

Proof. By (3) and the hypothesis of the theorem we have

$$(I_{a+}^{\alpha,\beta}\phi)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\log^\beta\left(\frac{\gamma}{x-t}\right)[\phi(t)-\phi(a)]dt.$$

We denote

$$g(x) = \phi(x) - \phi(a),\ \psi(x) = \int_a^x (x-t)^{\alpha-1}\log^\beta\left(\frac{\gamma}{x-t}\right)g(t)dt \tag{19}$$

and note that

$$|g(x) - g(y)| \leqq \omega(\phi, |x-y|). \tag{20}$$

Let $h > 0$. For any $x, x+h \in [a,b]$ we have

$$\psi(x+h) - \psi(x) = \int_{-h}^{x-a} \frac{g(x-t)}{(t+h)^{1-\alpha}}\log^\beta\left(\frac{\gamma}{t+h}\right)dt - \int_0^{x-a}\frac{g(x-t)}{t^{1-\alpha}}\log^\beta\left(\frac{\gamma}{t}\right)dt \tag{21}$$

$$\begin{aligned}
&= \int_{-h}^{0} \frac{g(x-t)-g(x)}{(t+h)^{1-\alpha}}\log^\beta\left(\frac{\gamma}{t+h}\right)dt \\
&+ \int_0^{x-a}\left[\frac{\log^\beta(\gamma/(t+h))}{(t+h)^{1-\alpha}} - \frac{\log^\beta(\gamma/t)}{t^{1-\alpha}}\right][g(x-t)-g(x)]dt \\
&+ g(x)\left[\int_{-h}^{x-a}\frac{\log^\beta(\gamma/(t+h))}{(t+h)^{1-\alpha}}dt - \int_0^{x-a}\frac{\log^\beta(\gamma/t)}{t^{1-\alpha}}\right]dt \\
&\equiv I_1 + I_2 + I_3.
\end{aligned}$$

Using (20) and making the change of variable $t = h\tau$, we estimate I_1:

$$\begin{aligned}
|I_1| &\leqq \int_0^h \frac{\omega(\phi,t)}{(h-t)^{1-\alpha}}\log^\beta\left(\frac{\gamma}{h-t}\right)dt \\
&= h^\alpha\int_0^1 \frac{\omega(\phi,h\tau)}{(1-\tau)^{1-\alpha}}\left(\log\left(\frac{\gamma}{h}\right)+\log\left(\frac{1}{1-\tau}\right)\right)^\beta d\tau \\
&\leqq ch^\alpha\log^\beta\left(\frac{\gamma}{h}\right)\int_0^1\frac{\omega(\phi,h\tau)}{(1-\tau)^{1-\alpha}}d\tau + ch^\alpha\int_0^1\frac{\omega(\phi,h\tau)}{(1-\tau)^{1-\alpha}}\log^\beta\left(\frac{1}{1-\tau}\right)d\tau \\
&\leqq c_1h^\alpha\log^\beta\left(\frac{\gamma}{h}\right)\omega(\phi,h) + c_2h^\alpha\omega(\phi,h) \\
&\leqq c_3h^\alpha\log^\beta\left(\frac{\gamma}{h}\right)\omega(\phi,h).
\end{aligned} \tag{22}$$

For I_2 we have

$$|I_2| \leqq \int_0^{x-a} \omega(\phi,t)\left|(t+h)^{\alpha-1}\log^\beta\left(\frac{\gamma}{t+h}\right) - t^{\alpha-1}\log^\beta\left(\frac{\gamma}{t}\right)\right| dt$$

$$= h^\alpha \int_0^{(x-a)/h} \omega(\phi,h\tau)\left|(\tau+1)^{\alpha-1}\log^\beta\left(\frac{\gamma}{(\tau+1)h}\right) - \tau^{\alpha-1}\log^\beta\left(\frac{\gamma}{h\tau}\right)\right| d\tau.$$

If $x-a \leqq h$, then

$$|I_2| \leqq h^\alpha \int_0^1 \omega(\phi,h\tau)\left[(\tau+1)^{\alpha-1}\left|\log^\beta\left(\frac{\gamma}{(\tau+1)h}\right)\right| + \tau^{\alpha-1}\left|\log^\beta\left(\frac{\gamma}{h\tau}\right)\right|\right] d\tau \tag{23}$$

$$\leqq c_4 h^\alpha \log^\beta\left(\frac{\gamma}{h}\right)\omega(\phi,h) + c_5 h^\alpha \omega(\phi,h)$$

$$\leqq c_6 h^\alpha \log^\beta\left(\frac{\gamma}{h}\right)\omega(\phi,h).$$

If $x-a \geqq h$, then by applying the mean value theorem, we obtain

$$|I_2| \leqq h^\alpha\left(\int_0^1 + \int_1^{(x-a)/h}\right)\omega(\phi,h\tau)\left|(\tau+1)^{\alpha-1}\log^\beta\left(\frac{\gamma}{(\tau+1)h}\right) - \tau^{\alpha-1}\log^\beta\left(\frac{\gamma}{h\tau}\right)\right| d\tau \tag{24}$$

$$\leqq h^\alpha\left[c_7\log^\beta\left(\frac{\gamma}{h}\right)\omega(\phi,h) + c_8 h\int_1^{(x-a)/h}\omega(\phi,h\tau)\tau^{\alpha-2}\log^\beta\left(\frac{\gamma}{h\tau}\right)d\tau\right]$$

$$\leqq h^\alpha\left[c_7\log^\beta\left(\frac{\gamma}{h}\right)\omega(\phi,h) + c_8 h^{2-\alpha}\int_h^{b-a}\omega(\phi,t)t^{\alpha-2}\log^\beta\left(\frac{\gamma}{t}\right)dt\right]$$

$$\leqq h^\alpha\left[c_7\log^\beta\left(\frac{\gamma}{h}\right)\omega(\phi,h) + c_8\log^\beta\left(\frac{\gamma}{h}\right)\omega(\phi,h)(b-a-h)\right]$$

$$\leqq c_9 h^\alpha\log^\beta\left(\frac{\gamma}{h}\right)\omega(\phi,h).$$

Finally we estimate I_3:

$$|I_3| \leqq \omega(\phi,x-a)\left|\int_0^{x-a+h} t^{\alpha-1}\log^\beta\left(\frac{\gamma}{t}\right)dt - \int_0^{x-a} t^{\alpha-1}\log^\beta\left(\frac{\gamma}{t}\right)dt\right|.$$

If $x-a \leqq h$, then we have

$$|I_3| \leqq \omega(\phi,x-a)\left[(x-a+h)^\alpha\int_0^1 \tau^{\alpha-1}\log^\beta\left(\frac{\gamma}{(x-a+h)\tau}\right)d\tau\right. \tag{25}$$

$$\left. +(x-a)^\alpha\int_0^1 \tau^{\alpha-1}\log^\beta\left(\frac{\gamma}{(x-a)\tau}\right)d\tau\right]$$

$$\leqq \omega(\phi, x-a)\left[c_{10}(x-a+h)^\alpha \log^\beta\left(\frac{\gamma}{x-a+h}\right) + c_{11}(x-a+h)^\alpha \right.$$

$$\left. + c_{12}(x-a)^\alpha \log^\beta\left(\frac{\gamma}{x-a}\right) + c_{13}(x-a)^\alpha\right]$$

$$\leqq \omega(\phi, h)[c_{14}h^\alpha \log^\beta\left(\frac{\gamma}{h}\right) + c_{15}h^\alpha]$$

$$\leqq c_{16}h^\alpha \log^\beta\left(\frac{\gamma}{h}\right)\omega(\phi, h).$$

If $x-a \geqq h$, then

$$\begin{aligned} |I_3| &\leqq \omega(\phi, x-a)\left|\int_{x-a}^{x-a+h} t^{\alpha-1}\log^\beta\left(\frac{\gamma}{t}\right)dt\right| \\ &\leqq \omega(\phi, x-a)\log^\beta\left(\frac{\gamma}{x-a}\right)|(x-a+h)^\alpha - (x-a)^\alpha| \\ &\leqq ch(x-a)^{\alpha-1}\log^\beta\left(\frac{\gamma}{h}\right)\omega(\phi, x-a). \end{aligned}$$

From this, applying the estimate

$$(x-a)^{\alpha-1}\omega(\phi, x-a) \leqq c\int_h^{b-a}\omega(\phi, t)t^{\alpha-2}dt,$$

(see [7] and [12, §13.6]), we obtain

$$|I_3| \leqq ch\log^\beta\left(\frac{\gamma}{h}\right)\int_h^{b-a}\omega(\phi, t)t^{\alpha-2}dt. \tag{26}$$

Substituting these estimates (22)-(26) into (21) and taking (19) and (4) into account, we arrive at the estimate (18) which completes the proof of the theorem.

Remark 1. In [15] for the convolution integral

$$(K\phi)(x) = \int_0^x k(x-t)\phi(t)dt$$

with a positive kernel $k(u)$ the estimate

$$\omega(\rho K\phi, h) \leqq c\int_h^{b-a}\frac{\omega(\phi, t)}{t}dt$$

was proved under the assumption that $t^{-\alpha}\rho(t)$ $(0 < \alpha < 1)$ is a non-decreasing function almost everywhere on $[0, b-a]$.

4. Zygmund type estimate for the integral inverse to the integral with power-logarithmic kernel

Let $(I_{a+}^{\alpha,\beta})^{-1}$ be the operator inverse to the operator $I_{a+}^{\alpha,\beta}$ given in (3). It is known (see [10], [12, §34.2] and Theorem 2.4) that, when $\beta = 1$, $(I_{a+}^{\alpha,\beta})^{-1}f$ has the form

$$(I_{a+}^{\alpha,1})^{-1}f(x) = \mu_\alpha(x-a)f(x) - \int_a^x [f(x)-f(t)]\mu'_\alpha(x-t)dt, \tag{27}$$

where $\mu_\alpha(x)$ is the special Volterra function given in (13) and $\mu'_\alpha(x)$ is its derivative. The following analogy of Theorem B is true.

Theorem 2. *Let $f(x)$ be a continuous function on $[a,b]$ and $f(a) = 0$. Then the Zygmund type estimate*

$$\omega((I_{a+}^{\alpha,1})^{-1}f,h) \leqq c_1 \int_0^h \omega(f,t)|\mu'_\alpha(t)|dt + \omega(f,h)\left[c_2|\mu_\alpha(h)| + c_3\int_h^{b-a} |\mu'_\alpha(t)|dt\right] \tag{28}$$

holds for the function $(I_{a+}^{\alpha,1})^{-1}f(x)$ given in (27).

Proof. Let $h > 0$, x, $x + h \in [a,b]$,

$$\phi(x) \equiv (I_{a+}^{\alpha,1})^{-1}f(x) = \mu_\alpha(x-a)f(x) - \int_a^x [f(x)-f(t)]\mu'_\alpha(x-t)dt \equiv F(x) - B(x). \tag{29}$$

At first we estimate $\omega(B,h)$. We have

$$\begin{aligned} B(x+h) - B(x) &= \int_0^{x+h-a} [f(x+h) - f(x+h-t)]\mu'_\alpha(t)dt \\ &\quad - \int_0^{x-a} [f(x) - f(x-t)]\mu'_\alpha(t)dt \\ &= \int_0^{x-a} [f(x+h) - f(x+h-t) - f(x) + f(x-t)]\mu'_\alpha(t)dt \\ &\quad + \int_{x-a}^{x+h-a} [f(x+h) - f(x+h-t)]\mu'_\alpha(t)dt \\ &\equiv B_1 + B_2. \end{aligned} \tag{30}$$

We first estimate B_1. If $x - a \leqq h$, then

$$|B_1| \leqq 2\int_0^{x-a} \omega(f,t)|\mu'_\alpha(t)|dt \leqq 2\int_0^h \omega(f,t)|\mu'_\alpha(t)|dt. \tag{31}$$

When $x - a \geqq h$, we have

$$|B_2| \leqq 2\int_0^h \omega(f,t)|\mu'_\alpha(t)|dt + 2\omega(f,h)\int_h^{x-a} |\mu'_\alpha(t)|dt \tag{32}$$

$$\leqq 2\int_0^h \omega(f,t)|\mu_\alpha'(t)|dt + 2\omega(f,h)\int_h^{b-a} |\mu_\alpha'(t)|dt.$$

As far as B_2 is concerned, for $x-a \leqq h$ by using the properties of the moduli of continuity (see, e.g. [2, Chapter II, §1]), we obtain

$$\begin{aligned} |B_2| &\leqq \int_{x-a}^{x-a+h} \omega(f,t)|\mu_\alpha'(t)|dt \\ &\leqq \int_0^{2h} \omega(f,t)|\mu_\alpha'(t)|dt \leqq c\int_0^h \omega(f,t)|\mu_\alpha'(t)|dt. \end{aligned} \tag{33}$$

If $x-a \geqq h$, then making the change of variable $t = \tau + x - a$ and applying the properties of the moduli of continuity again, we find

$$\begin{aligned} |B_2| &\leqq \int_0^h \omega(f, x-a+\tau)|\mu_\alpha'(x-a+\tau)|d\tau \\ &\leqq c_1\int_0^h \omega(f,t)|\mu_\alpha'(t)|dt. \end{aligned} \tag{34}$$

Substituting (31)-(34) into (30) and taking (4) into account we obtain the estimate

$$\omega(B,h) \leqq c_2\int_0^h \omega(f,t)|\mu_\alpha'(t)|dt + c_3\omega(f,h)\int_h^{b-a} |\mu_\alpha'(t)|dt. \tag{35}$$

Now we estimate $\omega(F,h)$. We have

$$F(x+h)-F(x) = f(x)[\mu_\alpha(x+h-a)-\mu_\alpha(x-a)]+\mu_\alpha(x+h-a)[f(x+h)-f(x)] \equiv F_1+F_2. \tag{36}$$

For F_1 we have

$$|F_1| = \left|f(x)\int_{x-a}^{x-a+h} \mu_\alpha'(t)dt\right| \leqq \omega(f,x-a)\int_{x-a}^{x-a+h} |\mu_\alpha'(t)|dt \leqq \int_{x-a}^{x-a+h} \omega(f,t)|\mu_\alpha'(t)|dt.$$

From this by arguments similar to the above for (33), we obtain

$$|F_1| \leqq c_4\int_0^h \omega(f,t)|\mu_\alpha'(t)|dt. \tag{37}$$

Finally we estimate F_2:

$$\begin{aligned} |F_2| &\leqq \omega(f,h)[|\mu_\alpha(x+h-a)-\mu_\alpha(h)| + |\mu_\alpha(h)|] \\ &\leqq \omega(f,h)\left[\int_h^{x-a+h} |\mu_\alpha'(t)|dt + |\mu_\alpha(h)|\right] \\ &\leqq \omega(f,h)\left[\int_h^{b-a} |\mu_\alpha'(t)|dt + |\mu_\alpha(h)|\right]. \end{aligned} \tag{38}$$

Substituting (37), (38) into (36) and taking (4) into account we arrive at the estimate

$$\omega(F,h) \leqq c_4 \int_0^h \omega(f,t)|\mu_\alpha'(t)|dt + \omega(f,h)\left[\int_h^{b-a} |\mu_\alpha'(t)|dt + |\mu_\alpha(h)|\right]. \tag{39}$$

According to (29) from (35) and (39) (after re-denoting the constants) we obtain the estimate (28), which completes the proof of the theorem.

5. Mapping properties and an isomorphism implemented by operators with power-logarithmic kernels

Let $I_{a+}^{\alpha,\beta}$ be the operator (3) and H_0^ω be the generalized Hölder space (6). Mapping property of $I_{a+}^{\alpha,\beta}$ in H_0^ω is characterized by the following statement.

Theorem 3. *Let* $0<\alpha<1$, $\beta \geqq 0$, *a function* $\omega(t)$ *be continuous and almost increasing on* $[0,b-a]$ *with* $\omega(0)=0$ *and*

$$\int_h^{b-a} \frac{\omega(t)}{t^{2-\alpha}}dt \leqq c\,\frac{\omega(h)}{h^{1-\alpha}}. \tag{40}$$

Then the operator $I_{a+}^{\alpha,\beta}$ *maps the generalized Hölder space* H_0^ω *boundedly into the space* $H_0^{\omega_{\alpha,\beta}}$ *with the characteristic* $\omega_{\alpha,\beta}(t)=\omega(t)t^\alpha\log^\beta(\gamma/t)$.

Proof. When $\beta=0$, this theorem was proved in [7] (see also [12, §13.6]). We consider the case $\beta>0$. Let

$$\psi(x) = (I_{a+}^{\alpha,\beta}\phi)(x), \tag{41}$$

where $\phi(x)\in H_0^\omega = H_0^\omega([a,b])$. Then according to Theorem 1 the Zygmund type estimate (18) holds for the integral (41). Applying this estimate and the condition (40) we have

$$\sup_{0<h\leq b-a} \frac{\omega(\psi,h)}{\omega(h)h^\alpha\log^\beta(\gamma/h)} \leqq c\left[\frac{\omega(\phi,h)}{\omega(h)} + h^{1-\alpha}\int_h^{b-a}\frac{\omega(\phi,t)}{t^{2-\alpha}}dt\right] \leqq c\,\|\phi\|_{H_0^\omega}. \tag{42}$$

The equality $\psi(a)=0$ follows from the definition (3) of the operator $I_{a+}^{\alpha,\beta}$ with power-logarithmic kernel. Further, we have

$$\|\psi\|_{C([a,b])} = \max_{a\leq x\leq b}\left|\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\log^\beta\left(\frac{\gamma}{x-t}\right)\phi(t)dt\right| \leqq c_1\,\|\phi\|_{C([a,b])}. \tag{43}$$

From (41)-(43) and the definition (6) of the space H_0^ω we obtain

$$\left\|I_{a+}^{\alpha,\beta}\phi\right\|_{H_0^{\omega_{\alpha,\beta}}} = \|\psi\|_{H_0^{\omega_{\alpha,\beta}}} \leqq \|\phi\|_{H_0^\omega}.$$

The theorem is proved.

Corollary 1. *Let $0<\alpha<1$, $\beta \geqq 0$, $\lambda>0$ and $\lambda+\alpha<1$, then the operator $I_{a+}^{\alpha,\beta}$ maps H_0^{λ} boundedly into $H_0^{\lambda+\alpha,\beta}$.*

Remark 2. Corollary 1 was obtained by direct estimates in [5] (see also [12, §21]).

Now we consider the mapping property of the operator $(I_{a+}^{\alpha,1})^{-1}$ given in (27) on $H_0^{\omega_{\alpha,1}}$ with $\omega_{\alpha,1}(t)=\omega(t)t^{\alpha}|\log(t)|$.

Theorem 4. *Let $0<\alpha<1$, a function $\omega(t)$ be continuous and almost increasing on $[0,b-a]$ with $\omega(0)=0$ and*

$$\int_0^h \frac{\omega(t)}{t}dt \leqq c\omega(h). \tag{44}$$

Then the operator $(I_{a+}^{\alpha,1})^{-1}$ maps the generalized Hölder space $H_0^{\omega_{\alpha,1}}$ with the characteristic $\omega_{\alpha,1}(t)=\omega(t)t^{\alpha}|\log(t)|$ boundedly into the space H_0^{ω}.

Proof. Let $f(x)\in H_0^{\omega_{\alpha,1}}=H_0^{\omega_{\alpha,1}}([a,b])$, then in view of (27) we have

$$g(x)\equiv (I_{a+}^{\alpha,1})^{-1}f(x)=\mu_\alpha(x-a)f(x)-\int_a^x [f(x)-f(t)]\mu_\alpha'(x-t)dt. \tag{45}$$

We show that

$$\sup_{0<h\leq b-a}\frac{\omega(g,h)}{\omega(h)}\leqq c<\infty. \tag{46}$$

Applying the Zygmund type estimate (28), the relations (16) and (17) for the special Volterra function (13) and its derivative, and also the condition (44), we have

$$\begin{aligned}
\frac{\omega(g,h)}{\omega(h)} &\leqq \frac{1}{\omega(h)}\left[c_1\int_0^h \omega(f,t)|\mu_\alpha'(t)|dt+\omega(f,h)\left(c_2|\mu_\alpha(h)|+c_3\int_h^{b-a}|\mu_\alpha'(t)|dt\right)\right]\\
&\leqq \|f\|_{H_0^{\omega_{\alpha,1}}}\left[\frac{c_1}{\omega(h)}\int_0^h \omega(t)t^{\alpha}|\log(t)||\mu_\alpha'(t)|dt\right.\\
&\qquad\left.+h^{\alpha}|\log(h)|\left(c_2|\mu_\alpha(h)|+c_3\int_h^{b-a}|\mu_\alpha'(t)|dt\right)\right]\\
&\leqq \|f\|_{H_0^{\omega_{\alpha,1}}}\left[\frac{c_4}{\omega(h)}\int_0^h\frac{\omega(t)}{t}+c_5+c_6h^{\alpha}|\log(h)|\int_h^{b-a}\frac{dt}{t^{\alpha+1}|\log(t)|}\right]\\
&\leqq \|f\|_{H_0^{\omega_{\alpha,1}}}\left[c_7+c_8|\log(h)|\int_1^{(b-a)/h}\frac{d\tau}{\tau^{\alpha+1}|\log(h\tau)|}\right]\\
&\leqq c_9\,\|f\|_{H_0^{\omega_{\alpha,1}}}.
\end{aligned}$$

From here we obtain the estimate of the form (46):

$$\sup_{0<h\leq b-a}\frac{\omega(g,h)}{\omega(h)}\leqq c_9\,\|f\|_{H_0^{\omega_{\alpha,1}}}. \tag{47}$$

Now we estimate $\|g\|_{C([a,b])}$. We have

$$\|g\|_{C([a,b])} \leqq \sup_{a<x\leq b} \left[|\mu_\alpha(x-a)|\omega(f,x-a) + \int_0^{x-a} |\mu'_\alpha(t)|\omega(f,t)dt\right]$$

$$\leqq c\,\|f\|_{H_0^{\omega_{\alpha,1}}} \Big[(x-a)^\alpha |\log(x-a)|\omega(x-a)|\mu_\alpha(x-a)|$$

$$+ \int_0^{x-a} t^\alpha |\log(t)|\omega(t)|\mu'_\alpha(t)|dt\Big]$$

$$\leqq \|f\|_{H_0^{\omega_{\alpha,1}}} \left[c_{10}\omega(x-a) + c_{11}\int_0^{x-a} \frac{\omega(t)}{t}dt\right]$$

$$\leqq c_{12}\,\|f\|_{H_0^{\omega_{\alpha,1}}}\,\omega(x-a).$$

From here we have

$$\|g\|_{C([a,b])} \leqq c_{12}\,\|f\|_{H_0^{\omega_{\alpha,1}}}\,,$$

and taking (46) and (47) into account, we finally arrive at the estimate

$$\|g\|_{H_0^\omega} \leqq c\,\|f\|_{H_0^{\omega_{\alpha,1}}}\,.$$

The condition $g(a) = 0$ follows directly from (45) if we take the relations (16), (17) and (44) into account. This completes the proof of this theorem.

If X and Y are Banach spaces and T is an operator, we denote by $T: X \longmapsto Y$ the imbedding with the properties

(i) if $f \in X$, then $Tf \in Y$;

(ii) $\|Tf\|_Y \leqq c\,\|f\|_X$.

Thus, in Theorems 3 and 4 we have proved the following imbeddings:

$$I_{a+}^{\alpha,\beta}: H_0^\omega \longmapsto H_0^{\omega_{\alpha,\beta}},\ 0<\alpha<1,\ \beta \geqq 0, \tag{48}$$

and

$$(I_{a+}^{\alpha,1})^{-1}: H_0^{\omega_{\alpha,1}} \longmapsto H_0^\omega,\ 0<\alpha<1. \tag{49}$$

Thus we obtain the analogy of Theorem C about an isomorphism of the generalized Hölder spaces H_0^ω and $H_0^{\omega_{\alpha,1}}$ implemented by the operator $I_{a+}^{\alpha,1}$ with the power-logarithmic kernel.

Theorem 5. *Let $0<\alpha<1$, $\beta \geqq 0$ and $\omega(t) \in \Phi_{1-\alpha}^0$, where $\Phi_{1-\alpha}^0$ is the space defined in (7). Then the operator $I_{a+}^{\alpha,1}$ maps the space H_0^ω isomorphically onto the space $H_0^{\omega_{\alpha,1}}$ with the characteristic $\omega_{\alpha,1}(t) = \omega(t)t^\alpha|\log(t)|$.*

Proof. To show that the assertion of this theorem follows from (48) and (49) we have to prove that any function $f \in H_0^{\omega_\alpha,1}$ is representable by the integral (3) $f = I_{a+}^{\alpha,1}\phi$ with a function $\phi \in H_0^\omega$. For this we use the criterion of representability of a function f via the power-logarithmic integral $f = I_{a+}^{\alpha,1}\phi$ of a function $\phi \in L_p(a,b)$ given in Theorem D. We verify that the conditions of Theorem D hold for a function $f \in H_0^{\omega_\alpha,1}$. The condition $f \in L_p(a,b)$ is valid because

$$|f(x)| \leqq \omega(x-a)(x-a)^\alpha |\log(x-a)| \, \|f\|_{H_0^{\omega_\alpha,1}} \leqq c.$$

We verify the convergence in $L_p(a,b)$ of the functions as $\epsilon \to 0$

$$\psi_\epsilon(x) = \begin{cases} \displaystyle\int_a^{x-a} [f(x) - f(t)]\mu'_\alpha(x-t)dt & \text{if } a+\epsilon < x < b, \\ 0 & \text{if } a < x < a+\epsilon, \end{cases} \tag{50}$$

It is sufficient to show that the sequence $\psi_\epsilon(x)$ is fundamental in the space $L_p(a,b)$. We suppose that $\epsilon_1 < \epsilon_2$ and put $x > a + \epsilon_2$

$$\begin{aligned} \psi_{\epsilon_1}(x) - \psi_{\epsilon_2}(x) &= \int_a^{x-\epsilon_1} [f(x) - f(t)]\mu'_\alpha(x-t)dt - \int_a^{x-\epsilon_2} [f(x) - f(t)]\mu'_\alpha(x-t)dt \\ &= \int_{\epsilon_1}^{\epsilon_2} [f(x) - f(x-t)]\mu'_\alpha(t)dt. \end{aligned}$$

Since $\omega(f,t) \leqq c\omega(t)t^\alpha|\log(t)|$ and by (17) $|\mu'_\alpha(t)| \leqq ct^{-\alpha-1}|\log(t)|^{-1}$, then

$$|\psi_{\epsilon_1}(x) - \psi_{\epsilon_2}(x)| \leqq c\int_{\epsilon_1}^{\epsilon_2} \frac{\omega(t)}{t}dt \to 0 \quad (\epsilon_2 \to 0).$$

The cases $x < a + \epsilon_1$ and $a + \epsilon_1 < x < a + \epsilon_2$ are considered similarly. Thus, the sequence $\psi_\epsilon(x)$ in (50) is fundamental in the norm of the space $C([a,b])$, and hence also in the norm of $L_p(a,b)$. According to (45) and (49) the function ϕ in the repesentation

$$f = I_{a+}^{\alpha,1}\phi, \ \phi \in L_p(a,b), \ 1 < p < \infty,$$

belongs to H_0^ω. This completes the proof of the theorem.

Corollary 2. *If* $0 < \alpha < 1$, $\lambda > 0$ *and* $\lambda + \alpha < 1$, *then the operator* $I_{a+}^{\alpha,1}$ *maps the space* H_0^λ *isomorphically onto the space* $H_0^{\lambda+\alpha,1}$.

Remark 3. In [6] the statement more general than Corollary 2 was proved giving the conditions for the operator $I_{a+}^{\alpha,\beta}$, $\beta = 1, 2, ...,$ to be an isomorphism between the generalized weighted Hölder spaces $H_0^\lambda([a,b];\rho)$ and $H_0^{\lambda+\alpha,\beta}([a,b];\rho)$, where ρ is the weight (2).

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