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A generalization class of certain subclasses of p-valently analytic functions with negative coefficients*

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Abstract

Recently we [5] have discussed a new generalization class $A(n,\alpha,\beta)$ of certain subclasses of analytic functions with negative coefficients in the unit disk and have proved some properties of functions belonging to the class $A(n,\alpha,\beta)$. In the present paper we introduce a new generalization class $A_p(n,\alpha,\beta)$ of certain subclasses of p-valently analytic functions with negative coefficients in the unit disk and discuss some properties of functions belonging to the class $A_p(n,\alpha,\beta)$.

1. Introduction

Let p be a positive integer, and let $A_p(n)$ denote the class of fuctions of the form

$$(1.1)\quad f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k \qquad (a_k \geqq 0,\ n \in N = \{1,2,3,\cdots\}),$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$.

A function $f(z)$ in the class $A_p(n)$ is said to be a member of the class $R_p(n,\alpha)$ if it satisfies

$$(1.2)\qquad \operatorname{Re}\left\{\frac{pf(z)}{z^p}\right\} > \alpha \qquad (z \in U)$$

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for some $\alpha (0 \leqq \alpha < p)$. Further, a function $f(z)$ in the class $A_p(n)$ is said to be in the class $P_p(n, \alpha)$ if it satisfies

$$\text{(1.3)} \qquad \operatorname{Re}\left\{\frac{f'(z)}{z^{p-1}}\right\} > \alpha \qquad (z \in U)$$

for some $\alpha (0 \leqq \alpha < p)$.

By generalization of some results due to Sarangi and Uralegaddi [2], we see that

LEMMA A. *A function $f(z) \in A_p(n)$ is in the class $R_p(n, \alpha)$ if and only if*

$$\text{(1.4)} \qquad \sum_{k=n+p}^{\infty} \frac{p}{p-\alpha} a_k \leqq 1.$$

LEMMA B. *A function $f(z) \in A_p(n)$ is in the class $P_p(n, \alpha)$ if and only if*

$$\text{(1.5)} \qquad \sum_{k=n+p}^{\infty} \frac{k}{p-\alpha} a_k \leqq 1.$$

Now, we define

DEFINITION. *Suppose that $f(z) \in A_p(n)$, $0 \leqq \alpha < p$ and $\beta \geqq 0$. Then the function $f(z)$ is said to be a member of the class $A_p(n, \alpha, \beta)$ if it satisfies*

$$\text{(1.6)} \qquad \operatorname{Re}\left\{(1-\beta)\frac{pf(z)}{z^p} + \beta\frac{f'(z)}{z^{p-1}}\right\} > \alpha \qquad (z \in U).$$

We note that $A_p(n, \alpha, 0) = R_p(n, \alpha)$ and $A_p(n, \alpha, 1) = P_p(n, \alpha)$. We have

Lemma 1. *Suppose that $f(z) \in A_p(n), 0 \leqq \alpha < p$ and $\beta \geqq 0$. Then the function $f(z)$ is in the class $A_p(n,\alpha,\beta)$ if and only if*

$$\sum_{k=n+p}^{\infty} \left\{ \frac{(1-\beta)p+\beta k}{p-\alpha} \right\} a_k \leqq 1. \tag{1.7}$$

Proof: Let $f(z) \in A_p(n,\alpha,\beta)$. Then we have ,by (1.6),

$$\begin{aligned} &\operatorname{Re}\left\{(1-\beta)\frac{pf(z)}{z^p} + \beta\frac{f'(z)}{z^{p-1}}\right\} \\ &= \operatorname{Re}\left\{p - \sum_{k=n+p}^{\infty} \{(1-\beta)p+\beta k\} a_k z^{k-p}\right\} \\ &> \alpha \qquad (z \in U). \end{aligned} \tag{1.8}$$

Letting $z \to 1$ through real values, we obtain (1.7). Conversely, let $f(z) \in A_p(n)$ satisfy inequality (1.7). Then we have

$$\begin{aligned} &\left|\left\{(1-\beta)\frac{pf(z)}{z^p} + \beta\frac{f'(z)}{z^{p-1}}\right\} - p\right| \\ &= \left|\sum_{k=n+p}^{\infty} \{(1-\beta)p+\beta k\} a_k z^{k-p}\right| \\ &\leqq \sum_{k=n+p}^{\infty} \{(1-\beta)p+\beta k\} a_k |z|^{k-p} \\ &< \ p-\alpha \qquad (z \in U). \end{aligned} \tag{1.9}$$

This proves that inequality (1.6) holds true. ∎

The class $A_1(n,\alpha,\beta)$ is a special case $\left(B_k = \frac{1+(k-1)\beta}{1-\alpha}\right)$ of the class $A(n, B_k)$ introduced by Sekine [3].

2. Distortion Theorem

Theorem 1. *If $f(z) \in A_p(n,\alpha,\beta)$ for $0 \leqq \alpha < p$ and $\beta \geqq 0$, then*

$$(2.1) \quad |z|^p - \frac{p-\alpha}{p+n\beta}|z|^{n+p} \leqq |f(z)| \leqq |z|^p + \frac{p-\alpha}{p+n\beta}|z|^{n+p} \qquad (z \in U)$$

for $\beta \geqq 0$, and

$$(2.2) \quad \begin{aligned} |f'(z)| &\leqq p|z|^{p-1} + \frac{(p-\alpha)(n+p)}{p+n\beta}|z|^{n+p-1} \qquad (z \in U) \\ |f'(z)| &\geqq p|z|^{p-1} - \frac{(p-\alpha)(n+p)}{p+n\beta}|z|^{n+p-1} \qquad (z \in U) \end{aligned}$$

for $\beta \geqq 1$. The equalities in (2.1) and (2.2) are attained for the function

$$(2.3) \qquad f(z) = z^p - \frac{p-\alpha}{p+n\beta}z^{n+p}.$$

Proof: Note that

$$(2.4) \qquad \sum_{k=n+p}^{\infty} a_k \leqq \frac{p-\alpha}{p+n\beta} \qquad (\beta \geqq 0)$$

and

$$(2.5) \quad \frac{p+n\beta}{n+p} \sum_{k=n+p}^{\infty} ka_k \leqq \sum_{k=n+p}^{\infty} \{(1-\beta)p+\beta k\}a_k \leqq p-\alpha \quad (\beta \geqq 1)$$

for $f(z) \in A_p(n,\alpha,\beta)$. Therefore, we have (2.1) and (2.2). ∎

Remark. Putting $p = 1$ in Theorem 1, we have the corresponding result due to Yaguchi, Sekine, Saitoh, Owa, Nunokawa and Fukui [5].

3. Inclusion Relations

Theorem 2. *If*

$$(3.1) \quad \begin{aligned} &0 \leqq \alpha_1 < p, \quad 0 \leqq \alpha_2 < p, \\ &0 \leqq \beta_1, \quad 0 \leqq \beta_2, \quad p(\beta_1 - \beta_2) < \alpha_2\beta_1 - \alpha_1\beta_2, \\ &p\{\alpha_1 - \alpha_2 + (\beta_1 - \beta_2)n\} \leqq n(\alpha_2\beta_1 - \alpha_1\beta_2), \end{aligned}$$

then we have

$$A_p(n,\alpha_2,\beta_2) \subsetneqq A_p(n,\alpha_1,\beta_1). \tag{3.2}$$

PROOF: Suppose $f(z) \in A_p(n,\alpha_2,\beta_2)$. Since by Lemma 1

$$\sum_{k=n+p}^{\infty} \frac{(1-\beta_2)p+k\beta_2}{p-\alpha_2} a_k \leqq 1, \tag{3.3}$$

we have only to prove the inequality

$$\frac{(1-\beta_1)p+k\beta_1}{p-\alpha_1} \leqq \frac{(1-\beta_2)p+k\beta_2}{p-\alpha_2} \qquad (k \geqq n+p), \tag{3.4}$$

which is equivalent to the inequality

$$k \geqq \frac{p\{(\beta_2-\beta_1)p+\alpha_1-\alpha_2+\alpha_2\beta_1-\alpha_1\beta_2\}}{(\beta_2-\beta_1)p+\alpha_2\beta_1-\alpha_1\beta_2} \qquad (k \geqq n+p). \tag{3.5}$$

But conditions (3.1) lead to the inequality

$$\frac{p\{(\beta_2-\beta_1)p+\alpha_1-\alpha_2+\alpha_2\beta_1-\alpha_1\beta_2\}}{(\beta_2-\beta_1)p+\alpha_2\beta_1-\alpha_1\beta_2} \leqq n+p, \tag{3.6}$$

which proves (3.5). The function $f_0(z)$ defined by

$$f_0(z) = z^p - \frac{p-\alpha_1}{p+(n+1)\beta_1} z^{p+n+1} \tag{3.7}$$

belongs to the class $A_p(n,\alpha_1,\beta_1) - A_p(n,\alpha_2,\beta_2)$, which proves

$$A_p(n,\alpha_1,\beta_1) \neq A_p(n,\alpha_2,\beta_2). \quad \blacksquare \tag{3.8}$$

Corollary 1. *If*

(3.9) $0 \leqq \alpha_1 \leqq \alpha_2 < p, \quad 0 \leqq \beta_1 \leqq \beta_2, \quad (\beta_2 - \beta_1) + (\alpha_2 - \alpha_1) > 0,$

then we have

$$A_p(n, \alpha_2, \beta_2) \subsetneqq A_p(n, \alpha_1, \beta_1) \tag{3.10}$$

Proof: By Theorem 2, we have

$$\begin{aligned} &A_p(n, \alpha_2, \beta_1) \subsetneqq A_p(n, \alpha_1, \beta_1) \qquad (0 \leqq \alpha_1 < \alpha_2 < p), \\ &A_p(n, \alpha_2, \beta_2) \subsetneqq A_p(n, \alpha_2, \beta_1) \qquad (0 \leqq \beta_1 < \beta_2), \end{aligned} \tag{3.11}$$

which prove Corollary 1. ∎

Corollary 2. *If* $0 < \beta_1 < 1 < \beta_2$, *then*

$$A_p(n, \alpha, \beta_2) \subsetneqq P_p(n, \alpha) \subsetneqq A_p(n, \alpha, \beta_1) \subsetneqq R_p(n, \alpha). \tag{3.12}$$

4. Starlikeness

A function $f(z)$ in the class $A_p(n)$ is said to be p-valently starlike of order α if it satisfies

$$\operatorname{Re} \frac{z f'(z)}{f(z)} > \alpha \qquad (z \in U) \tag{4.1}$$

for some $\alpha (0 \leqq \alpha < p)$. We need the following lemma which is a generalization of a result due to Chatterjea [1] (also Srivastava, Owa and Chatterjea [4]).

Lemma C. *A function $f(z) \in A_p(n)$ is p-valently starlike of order γ if and only if*

$$\sum_{k=n+p}^{\infty} \frac{k - \gamma}{p - \gamma} a_k \leqq 1 \tag{4.2}$$

for some $\gamma (0 \leqq \gamma < p)$.

Lemma C is proved by using the similar method as in Chatterjea [1]. Using Lemma C, we have

THEOREM 3. *If $f(z) \in A_p(n,\alpha,\beta)$ for $0 \leqq \alpha < p$ and $\beta \geqq 1$, then $f(z)$ is starlike of order $(1-\frac{1}{\beta})p$.*

PROOF: It follows from $f(z) \in A_p(n,\alpha,\beta)$ that

$$\sum_{k=n+p}^{\infty} \{k-(1-\frac{1}{\beta})p\}a_k \leqq \frac{p-\alpha}{\beta} \leqq p-(1-\frac{1}{\beta})p. \tag{4.3}$$

Therefore, by Lemma C, we have the assertion of Theorem 3. ▮

5. Quadi-Hadamard product

For functions $f_1(z)$ and $f_2(z)$ defined by

$$f_j(z) = z^p - \sum_{k=n+p}^{\infty} a_{j,k}z^k \qquad (a_{j,k} \geqq 0,\ n \in N,\ j=1,2) \tag{5.1}$$

in the class $A_p(n)$, we denote by $f_1 * f_2(z)$ the quasi-Hadamard product of functions $f_1(z)$ and $f_2(z)$, that is,

$$f_1 * f_2(z) = z^p - \sum_{k=n+p}^{\infty} a_{1,k}a_{2,k}z^k. \tag{5.2}$$

THEOREM 4. *If $f_j(z) \in A_p(n,\alpha_j,\beta)$ for $0 \leqq \alpha_j < p, \beta \geqq 0$ and $j=1,2$, then $f_1 * f_2(z) \in A_p(n,\alpha,\beta)$, where*

$$\alpha = p - \frac{(p-\alpha_1)(p-\alpha_2)}{p+\beta n}. \tag{5.3}$$

The result is sharp for functions $f_1(z)$ and $f_2(z)$ defined by

$$f_j(z) = z^p - \frac{p-\alpha_j}{p+\beta n}z^{n+p} \qquad (j=1,2). \tag{5.4}$$

PROOF: We have to find the largest α such that

$$\sum_{k=n+p}^{\infty} \frac{(1-\beta)p+\beta k}{p-\alpha}a_{1,k}a_{2,k} \leqq 1. \tag{5.5}$$

For functions $f_j(z) \in A_p(n, \alpha_j, \beta)$, we have

$$\sum_{k=n+p}^{\infty} \left\{ \frac{(1-\beta)p+\beta k}{p-\alpha} \right\} a_{j,k} \leqq 1 \qquad (j=1,2). \tag{5.6}$$

By the Cauchy-Schwarz inequality, inequality (5.6) lead tothe inequality

$$\sum_{k=n+p}^{\infty} \frac{(1-\beta)p+\beta k}{\sqrt{(p-\alpha_1)(p-\alpha_2)}} \sqrt{a_{1,k}a_{2,k}} \leqq 1. \tag{5.7}$$

Therefore, it is sufficient to prove that

$$\begin{aligned} &\frac{(1-\beta)p+\beta k}{p-\alpha} a_{1,k}a_{2,k} \\ &\leqq \frac{(1-\beta)p+\beta k}{\sqrt{(p-\alpha_1)(p-\alpha_2)}} \sqrt{a_{1,k}a_{2,k}} \qquad (k \geqq n+p), \end{aligned} \tag{5.8}$$

that is, that

$$\sqrt{a_{1,k}a_{2,k}} \leqq \frac{p-\alpha}{\sqrt{(p-\alpha_1)(p-\alpha_2)}} \qquad (k \geqq n+p). \tag{5.9}$$

From (5.7), we need to show that

$$\frac{\sqrt{(p-\alpha_1)(p-\alpha_2)}}{(1-\beta)p+\beta k} \leqq \frac{p-\alpha}{\sqrt{(p-\alpha_1)(p-\alpha_2)}} \qquad (k \geqq n+p) \tag{5.10}$$

or

$$\alpha \leqq p - \frac{(p-\alpha_1)(p-\alpha_2)}{(1-\beta)p+\beta k} \qquad (k \geqq n+p). \tag{5.11}$$

Noting that the function

$$\phi(k) = p - \frac{(p-\alpha_1)(p-\alpha_2)}{(1-\beta)p+\beta k} \qquad (k \geqq n+p) \tag{5.12}$$

is increasing on k, we have

$$\alpha \leqq \phi(n+p) = p - \frac{(p-\alpha_1)(p-\alpha_2)}{p+\beta n}. \quad \blacksquare \tag{5.13}$$

Finally, we derive

THEOREM 5. *Let $f_j(z)(j=1,2)$ define by (5.1). If $f_j(z)\in A_p(n,\alpha_j,\beta)(j=1,2)$, then the function*

$$f(z)=z^p-\sum_{k=n+p}^{\infty}\left\{(a_{1,k})^2+(a_{2,k})^2\right\}z^k \tag{5.14}$$

is in the class $A_p(n,\alpha,\beta)$, where

$$\alpha=p-\frac{2(p-\alpha_0)^2}{p+\beta n}\qquad(\alpha_0=\min\{\alpha_1,\alpha_2\}). \tag{5.15}$$

The result is sharp for the function $f(z)$ defined by

$$f_j(z)=z^p-\frac{p-\alpha_0}{p+\beta n}z^{n+p}\qquad(j=1,2), \tag{5.16}$$

when $\alpha_0=\alpha_1=\alpha_2$.

PROOF: Since
(5.17)

$$\sum_{k=n+p}^{\infty}\left\{\frac{(1-\beta)p+\beta k}{p-\alpha_j}a_{j,k}\right\}^2\leqq\left\{\sum_{k=n+p}^{\infty}\frac{(1-\beta)p+\beta k}{p-\alpha_j}a_{j,k}\right\}^2$$
$$\leqq 1\qquad(j=1,2),$$

we obtain that
(5.18)

$$\sum_{k=n+p}^{\infty}\left\{\frac{(1-\beta)p+\beta k}{p-\alpha_0}\right\}^2\left\{(a_{1,k})^2+(a_{2,k})^2\right\}$$
$$\leqq\sum_{k=n+p}^{\infty}\left\{\frac{(1-\beta)p+\beta k}{p-\alpha_1}a_{1,k}\right\}^2+\sum_{k=n+p}^{\infty}\left\{\frac{(1-\beta)p+\beta k}{p-\alpha_2}a_{2,k}\right\}^2$$
$$\leqq 2,$$

where α_0 is defined by (5.15). This implies that weonly find the largest α such that

$$\frac{(1-\beta)p+\beta k}{p-\alpha} \leqq \frac{1}{2}\left\{\frac{(1-\beta)p+\beta k}{p-\alpha_0}\right\}^2 \qquad (k \geqq n+p) \tag{5.19}$$

or

$$\alpha \leqq p - \frac{2(p-\alpha_0)^2}{(1-\beta)p+\beta k} \qquad (k \geqq n+p). \tag{5.20}$$

Since the function

$$\phi(k) = p - \frac{2(p-\alpha_0)^2}{(1-\beta)p+\beta k} \qquad (k \geqq n+p). \tag{5.21}$$

is increasing on k, we have

$$\alpha \leqq \phi(n+p) = p - \frac{2(p-\alpha_0)^2}{p+\beta n}. \quad \blacksquare \tag{5.22}$$

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