Notes for Starlikeness Conditions of Analytic Functions

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1. Introduction

Let A denote the class of functions f which are analytic in the unit disc $D=\{z: |z|<1\}$, with

(1.1) $f(0)=0$ and $f'(0)=1$.

And let S^* be the usual class of starlike functions in D, i.e.

$$S^* = \left\{ f \in A:\ \mathrm{Re}\frac{z f'(z)}{f(z)} > 0,\ z \in D \right\}.$$

On the starlikeness condition for functions of A, we now investigate the following well known problems:

Problem 1. Find the maximum value of r for which $f \in A$ and

(1.2) $|f'(z)-1| < 1,\ z \in D$

imply that f is starlike in

(1.3) $D_r = \{z: |z| < r\}$.

   Problem 2. Find the maximum value of r for which $f \in A$ and

(1.4) $|f'(z) - 1| < r$, $z \in D$

imply that $f \in S^*$.

   We denote by r_1 and r_2 the maximum value of the first problem and of the second problem, respectively.

   For r_1, T. MacGregor [1] showed that

$$r_1 \geqq 2/\sqrt{5} = 0.894\cdots.$$

Recently, M. Nunokawa in [3] proved that $r_1 > 0.926\cdots$, and in [4] Nunokawa et all improved, that is $r_1 > 0.933\cdots$. On the other hand, P. Mocanu [2] showed that the function

(1.5) $f(z) = -2z + z^2 + 6\log \dfrac{2+z}{2}$

belongs to A and satisfies the condition (1.2), but does not belong to S^*. Therefore $r_1 < 1$. For r_2, in the same article Mocanu also proved that

$$r_2 \geqq 2/\sqrt{5} = 0.894\cdots.$$

   In this short paper, modifying the Mocanu's example, we show that

(1.6) $r_1 < 0.9982$

and

(1.7) $r_2 < 0.9962.$

2. Examples

Example 1. Let

(2.1) $f(z) = (2 - a^2)z + \dfrac{a}{2}z^2 + a(a^2 - 1)\log \dfrac{z + a}{a}, \quad z \in D,$

where $\log 1 = 0.$ Since

(2.2) $f'(z) = 2 - a^2 + a\,z + \dfrac{a(a^2 - 1)}{z + a} = 1 + z\dfrac{a\,z + 1}{z + a},$

we easily deduce

$$f(0) = f'(0) - 1 = 0$$

and

$$|f'(z) - 1| < 1, \quad z \in D.$$

If we let

$$f(z) = s + i\,t, \quad z\,f'(z) = u + i\,v$$

and $z = r\,e^{i\theta}$, then we have

$$s = (2 - a^2)\,r\cos\theta + \dfrac{a}{2}r^2\cos 2\theta + \dfrac{a(a^2 - 1)}{2}\log(r^2 + a^2 + 2a\,r\cos\theta)$$

$$- a(a^2 - 1)\log a,$$

$$t = (2 - a^2)\,r\sin\theta + \dfrac{a}{2}r^2\sin 2\theta + a(a^2 - 1)\mathrm{Tan}^{-1}\dfrac{r\sin\theta}{a + r\cos\theta},$$

$$u = (2 - a^2)\,r\cos\theta + a\,r^2\cos 2\theta + a(a^2 - 1) - \dfrac{a^2(a^2 - 1)(r\cos\theta + a)}{r^2 + a^2 + 2a\,r\cos\theta}$$

and

$$v = (2 - a^2) \, r \sin\theta + a \, r^2 \sin 2\theta + \frac{a^2 (a^2 - 1) \, r \sin\theta}{r^2 + a^2 + 2a \, r \cos\theta}.$$

And it is evident that $\mathrm{Re}[z f'/f] > 0$ if and only if

(2.3) $\qquad s \, u + t \, v > 0.$

But by putting

(2.4) $\qquad a = 1.1, \quad r = 0.9982, \quad \theta = 2.743,$

we deduce $s \, u + t \, v = -9.13 \times 10^{-5} < 0$, which shows that f is not

starlike in D_r, i.e. $r_1 < 0.9982$.

Example 2. Let

(2.5) $\qquad g(z) = (1 - \dfrac{b}{a}) z + \dfrac{b}{2(a^2 - 1)} z^2 + b \log (z + a) - b \log a.$

Since

(2.6) $\qquad g'(z) = 1 + \dfrac{b}{a(a^2 - 1)} \cdot \dfrac{z(a z + 1)}{z + a},$

evidently $g \in A$, and

(2.7) $\qquad | \, g'(z) - 1 < k = \left| \dfrac{b}{a(a^2 - 1)} \right|, \quad z \in D.$

Let

$$g(z) = s + i \, t, \quad z f'(z) = u + i \, v$$

and $z = r \, e^{i\theta}$. Then we deduce

$$s = (1 - \frac{b}{a}) \, r \cos\theta + \frac{b}{2(a^2 - 1)} r^2 \cos 2\theta + \frac{b}{2} \log(r^2 + a^2 + 2a \, r \cos\theta) - b \log a,$$

$$t = (1-\frac{b}{a})\, r\sin\theta + \frac{b}{2(a^2-1)}\, r^2\sin 2\theta + b\,\mathrm{Tan}^{-1}\frac{r\sin\theta}{a+r\cos\theta},$$

$$u = (1-\frac{b}{a})\, r\cos\theta + \frac{b}{a^2-1}\, r^2\cos 2\theta + \frac{b\,(r^2+a\,r\cos\theta)}{r^2+a^2+2a\,r\cos\theta},$$

$$v = (1-\frac{b}{a})\, r\sin\theta + \frac{b}{a^2-1}\, r^2\sin 2\theta + \frac{a\,b\,r\sin\theta}{r^2+a^2+2a\,r\cos\theta}.$$

Substituting

(2.8) $a = 1.1,\quad b = 0.2301,\quad r = 1,\quad \theta = 2.743$

we obtain $s\,u + t\,v = -4.36\times10^{-5} < 0$, and

(2.9) $k = 0.9961038\cdots,$

which yield $r_2 < 0.9962.$

References

[1] T. H. MacGregor, A Class of univalent functions, Proc. Amer. Math. Soc., 15(1964), 311-317.

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[3] M. Nunokawa, On the starlike boundary of univalent functions, Sûgaku, 31(1979), 255-256 (Japanese).

[4] M. Nunokawa, S. Fukui, S. Owa, H. Saitoh and T. Sekine, On the starlikeness bound of univalent functions, Math. Japonica, 33, No. 5(1988), 763-767.