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Generalized Classes of Multivalent Functions with Negative Coefficients

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1. Introduction and Representation

Let $A_n(p)$ denote the class of functions of the form

$$f(z) = z^p - \sum_{k=p+n}^{\infty} a_k z^k \ (a_k \geqq 0, \ and \ p, n \in N)$$

which are analytic and p-valent in the unit disk $U = \{z : |z| < 1\}$.

Let $A_n(p; \{B_k\})$ denote the subclass of $A_n(p)$ consisting of functions which satisfy the following inequality

$$\sum_{k=p+n}^{\infty} B_k \leqq 1 \ (B_k > 0, n \in N).$$

By means of $A_n(p; \{B_k\})$ with particular values of n or B_k, we have the following representation for subclasses of analytic and p-valent functions with negative coefficients([1],[2],[3] and [5]).

(1.1) $$T_p^*(A,B) = A_1\left(p; \left\{\frac{(1+B)(k-p)+(B-A)p}{(B-A)p}\right\}\right) \ ([1])$$

(1.2) $$C_p(A,B) = A_1\left(p; \left\{\frac{k[(1+B)(k-p)+(B-A)p]}{(B-A)p^2}\right\}\right) \ ([1])$$

(1.3) $T_p(a,b)=A_1\left(p;\left\{\frac{k(1+b)}{2b(1-a)p}\right\}\right)$ ([2]).

(1.4) $C_p(a,b)=A_1\left(p;\left\{\frac{k^2(1+b)}{2b(1-a)p^2}\right\}\right)$ ([2]).

(1.5) $T^*(p,\alpha)=A_1\left(p;\left\{\frac{k-\alpha}{p-\alpha}\right\}\right)$ ([3]).

(1.6) $C(p,\alpha)=A_1\left(p;\left\{\frac{k(k-\alpha)}{p(p-\alpha)}\right\}\right)$ ([3]).

(1.7) $N_{p,m}=A_1\left(p;\left\{(2k-2p+1)\binom{m-1+k}{k-p}\right\}\right)$ ([5]).

2. Distortion theorems

Theorem 1. If $f(z)\in A_n(p;\{B_k\})$ and $B_k\leqq B_{k+1}$, then we have

(2.1) $$Max\left\{0,|z|^p-\frac{1}{B_{p+n}}|z|^{p+n}\right\}\leqq|f(z)|\leqq|z|^p+\frac{1}{B_{p+n}}|z|^{p+n}.$$

The estimate is sharp for the function

(2.2) $$f(z)=z^p-\frac{1}{B_{p+n}}z^{p+n}.$$

Let p=1 in Theorem 1. Then we have

Corollary 1([4], Theorem 1). If $f(z)\in A_n(1;\{B_k\})$ and $B_k\leqq B_{k+1}$ then we have

(2.3) $$Max\left\{0,|z|-\frac{1}{B_{n+1}}|z|^{n+1}\right\}\leqq|f(z)|\leqq|z|+\frac{1}{B_{n+1}}|z|^{n+1}.$$

The estimate is sharp for the function

(2.4) $$f(z)=z-\frac{1}{B_{n+1}}z^{n+1}.$$

Let n=1 in Theorem 1. Then we have

Corollary 2. If $f(z)\in A_1(p;\{B_k\})$ and $B_k \leqq B_{k+1}$, then we have

(2.5) $$Max\left\{0, |z|^p-\frac{1}{B_{p+1}}|z|^{p+1}\right\}\leqq|f(z)|\leqq|z|^p+\frac{1}{B_{p+1}}|z|^{p+1}.$$

The estimate is sharp for the function

(2.6) $$f(z)=z^p-\frac{1}{B_{p+1}}z^{p+1}.$$

Theorem 2. If $f(z)\in A_n(p;k\{B_k\})$ and $B_k \leqq B_{k+1}$, then we have

(2.7) $$Max\left\{0, p|z|^{p-1}-\frac{1}{B_{p+n}}|z|^{p+n-1}\right\}\leqq|f'(z)|$$
$$\leqq p|z|^{p-1}+\frac{1}{B_{p+n}}|z|^{p+n-1}.$$

The estimate is sharp for the function

(2.8) $$f(z)=z^p-\frac{1}{(p+n)B_{p+n}}z^{p+n}.$$

Let p=1 in Theorem 2. Then we have

Corollary 3([4], Theorem 2). If $f(z)\in A_n(1;k\{B_k\})$ and $B_k \leqq B_{k+1}$, then we have

(2.9) $$Max\left\{0, 1-\frac{1}{B_{n+1}}|z|^n\right\}\leqq|f'(z)|\leqq1+\frac{1}{B_{n+1}}|z|^n.$$

The estimate is sharp for the function

$$f(z) = z - \frac{1}{(n+1)\,B_{n+1}} z^{n+1}. \tag{2.10}$$

Let n=1 in Theorem 2. Then we have

Corollary 4. If $f(z) \in A_1(p;k\{B_k\})$ and $B_k \leqq B_{k+1}$, then we have

$$Max\left\{0,\ p|z|^{p-1} - \frac{1}{B_{p+1}}|z|^{p}\right\} \leqq |f'(z)| \leqq p|z|^{p-1} + \frac{1}{B_{p+1}}|z|^{p+1}. \tag{2.11}$$

The estimate is sharp for the function

$$f(z) = z^{p} - \frac{1}{(p+1)\,B_{p+n}} z^{p+1}. \tag{2.12}$$

We now show the distortion theorems for $f^{(j)}(z)(2 \leqq j \leqq n+1)$. First we need the following lemma.

Lemma 1. Let

$$\prod_{i=1}^{j}(k-1+i) = \sum_{i=1}^{j} A_i k^{i} \quad (j \geqq 2,\ k \in N).$$

Then

$$\sum_{i=1}^{j} A_i (n+p)^{i-1} = \prod_{i=2}^{j}(n+p-1+i) \quad (n,\ p \in N).$$

The proof is the same as that of Lemma 1([4]).
Putting p=1 in above Lemma 1, we get Lemma 1([4]).

Using Lemma 1, we have

Theorem 3. If $f(z) \in A_n(p:k^{q}\{B_k\})$, $B_k \leqq B_{k+1}$ and $2 \leqq q \leqq p+n$, then we have

$$(2.13)\ (\mathrm{I})\ |f^{(j)}(z)| \leqq \prod_{i=1}^{j}(p+1-i)|z|^{p-j} + \frac{\prod_{i=2}^{j}(n+p-1+i)}{(p+n)^{q-1}B_{p+n}}|z|^{p+n-j},$$

$$|f^{(j)}(z)| \geqq \prod_{i=1}^{j}(p+1-i)|z|^{p-j} - \frac{\prod_{i=2}^{j}(n+p-1+i)}{(p+n)^{q-1}B_{p+n}}|z|^{p+n-j}.$$

$$(2 \leqq j \leqq p),$$

$$(2.14)\ (\mathrm{II})\ |f^{(j)}(z)| \leqq \frac{\prod_{i=2}^{j}(n+p-1+i)}{(p+n)^{q-1}B_{p+n}}|z|^{p+n-j}$$

$$(p+1 \leqq j \leqq p+n).$$

Let p=1 in Theorem 3 (II) . Then we have

Corollary 5([4], Theorem 3). If $f(z)A_n \in (1;k^q\{B_k\})$, $B_k \leqq B_{k+1}$ and $2 \leqq q \leqq n+1$, then we have

$$(2.15)\ q|f^{(j)}(z)| \leqq \frac{\prod_{i=2}^{j}(n+i)}{(n+1)^{q-1}B_{n+1}}|z|^{n-j+1} \quad (2 \leqq j \leqq n+1).$$

Let n=1 in Thorem 3. Then we have

Corollary 6. If $f(z) \in A_1(p;k^q\{B_k\})$, $B_k \leqq B_{k+1}$ and $2 \leqq q \leqq p+1$, then we have

$$(2.16)\ |f^{(j)}(z)| \leqq \prod_{i=1}^{j}(p+1-i)|z|^{p-j} + \frac{\prod_{i=2}^{j}(p+i)}{(p+1)^{q-1}B_{p+1}}|z|^{p-j+1},$$

$$|f^{(j)}(z)| \geqq \prod_{i=1}^{j}(p+1-i)|z|^{p-j} - \frac{\prod_{i=2}^{j}(p+i)}{(p+1)^{q-1}B_{p+1}}|z|^{p-j+1},$$

$$(2 \leqq j \leqq p),$$

$$|f^{(j)}(z)| \leqq \frac{\prod_{i=2}^{j}(p+i)}{(p+1)^{q-1}B_{p+1}} \quad (j=p+1).$$

Remark 1. Substituting some values for B_{p+1} in Corollary 2, we have the following estimates of f(z).

(2.17) *If* $f(z) \in T_p^*(A,B)\left(B_{p+1}=\dfrac{1+B+(B-A)p}{(B-A)p}\right)$, *then*

$$|z|^p-\frac{(B-A)p}{1+B+(B-A)p}|z|^{p+1} \leqq |f(z)|$$

$$\leqq |z|^p+\frac{(B-A)p}{1+B+(B-A)p}|z|^{p+1}$$

([1], Theorem 3).

(2.18) *If* $f(z) \in C_p(A,B)\left(B_{p+1}=\dfrac{(p+1)\{1+B+(B-A)p\}}{(B-A)p^2}\right)$,

then

$$|z|^p-\frac{(B-A)p^2}{(p+1)\{1+B+(B-A)p\}}|z|^{p+1} \leqq |f(z)|$$

$$\leqq |z|^p+\frac{(B-A)p^2}{(p+1)\{1+B+(B-A)p\}}|z|^{p+1}$$

(cf. [1], Theorem 4).

(2.19) *If* $f(z) \in T_p(a,b)\left(B_{p+1}=\dfrac{(p+1)(1+b)}{2b(1-a)p}\right)$, *then*

$$|z|^p-\frac{2b(1-b)p}{(p+1)(1+b)}|z|^{p+1} \leqq |f(z)|$$

$$\leqq |z|^p+\frac{2b(1-b)p}{(p+1)(1+b)}|z|^{p+1}$$

([2], Theorem 6).

(2.20) If $f(z) \in C_p(a,b)$ $\left(B_{p+1}=\dfrac{(p+1)^2(1+b)}{2b(1-a)p^2}\right)$, then

$$|z|^p-\frac{2b(1-b)p^2}{(p+1)^2(1+b)}|z|^{p+1}\leqq|f(z)|$$
$$\leqq|z|^p+\frac{2b(1-b)p^2}{(p+1)^2(1+b)}|z|^{p+1}$$

([2], Theorem 7).

(2.21) If $f(z) \in T^*(p,\alpha)$ $\left(B_{p+1}=\dfrac{p+1-\alpha}{p-\alpha}\right)$, then

$$|z|^p-\frac{p-\alpha}{p+1-\alpha}|z|^{p+1}\leqq|f(z)|\leqq|z|^p+\frac{p-\alpha}{p+1-\alpha}|z|^{p+1}$$

([3], Theorem 3.1).

(2.22) If $f(z) \in C(p,\alpha)$ $\left(B_{p+1}=\dfrac{(p+1)(p+1-\alpha)}{p(p-\alpha)}\right)$, then

$$|z|^p-\frac{p(p-\alpha)}{(p+1)(p+1-\alpha)}|z|^{p+1}\leqq|f(z)|$$
$$\leqq|z|^p+\frac{p(p-\alpha)}{(p+1)(p+1-\alpha)}|z|^{p+1}$$

([3], Theorem 3.2).

(2.23) If $f(z) \in N_{p,m}$ $(B_{p+1}=3(m+p))$, then

$$|z|^p-\frac{1}{3(m+p)}|z|^{p+1}\leqq|f(z)|\leqq|z|^p+\frac{1}{3(m+p)}|z|^{p+1} \quad ([5]).$$

Remark 2. Substituting some values for B_{p+1} in Corollary 4, we have the following estimates of $f'(z)$

(2.24) If $f(z)\in T_p^*(A,B)\left(B_{p+1}=\dfrac{1+B+(B-A)p}{(p+1)(B-A)p}\right)$

$$p|z|^{p-1}-\frac{(p+1)(B-A)p}{1+B+(B-A)p}|z|^p\leqq|f'(z)|$$

$$\leqq p|z|^{p-1}+\frac{(p+1)(B-A)p}{1+B+(B-A)p}|z|^p$$

([1], Theorem 3).

(2.25) q If $f(z)\in C_p(A,B)\left(B_{p+1}=\dfrac{1+B+(B-A)p}{(B-A)p^2}\right)$, then

$$p|z|^{p-1}-\frac{(B-A)p^2}{1+B+(B-A)p}|z|^p\leqq|f'(z)|$$

$$\leqq p|z|^{p-1}+\frac{(B-A)p^2}{1+B+(B-A)p}|z|^p$$

(cf. [1], Theorem 4).

(2.26) If $f(z)\in T_p(a,b)\left(B_{p+1}=\dfrac{1+b}{2b(1-a)p}\right)$, then

$$p|z|^{p-1}-\frac{2b(1-a)p}{1+b}|z|^p\leqq|f'(z)|$$

$$\leqq p|z|^{p-1}+\frac{2b(1-a)p}{1+b}|z|^p$$

([2], Theorem 6).

(2.27) If $f(z)\in C_p(a,b)\left(B_{p+1}=\dfrac{(p+1)(1+b)}{2b(1-a)p^2}\right)$, then

$$p|z|^{p-1}-\frac{2b(1-a)p^2}{(p+1)(1+b)}|z|^p\leqq|f'(z)|$$

$$\leqq p|z|^{p-1}+\frac{2b(1-a)p^2}{(p+1)(1+b)}|z|^p$$

([2], Theorem 7).

(2.28) If $f(z) \in T^*(p,\alpha)$ $\left(B_{p+1}=\dfrac{p+1-\alpha}{(p+1)(p-\alpha)}\right)$, then

$$p|z|^{p-1}-\frac{(p+1)(p-\alpha)}{p+1-\alpha}|z|^p \leqq |f'(z)|$$

$$\leqq p|z|^{p-1}+\frac{(p+1)(p-\alpha)}{p+1-\alpha}|z|^p.$$

([3], Theorem 3.1).

(2.29) If $f(z) \in C(p,\alpha)$ $\left(B_{p+1}=\dfrac{p+1-\alpha}{p(p-\alpha)}\right)$, then

$$p|z|^{p-1}-\frac{p(p-\alpha)}{p+1-\alpha}|z|^p \leqq |f'(z)|$$

$$\leqq p|z|^{p-1}+\frac{p(p-\alpha)}{p+1-\alpha}|z|^p$$

([3], Theorem 3.2).

(2.30) If $f(z) \in N_{p,m}$ $\left(B_{p+1}=\dfrac{3(m+p)}{p+1}\right)$, then

$$p|z|^{p-1}-\frac{p+1}{3(m+p)}|z|^p \leqq |f'(z)| \leqq p|z|^{p-1}+\frac{p+1}{3(m+p)}|z|^p$$

([6]).

Remark 3. Substituting some values for B_{p+1} in Corollary 6, we have the following estimates of $f''(z)$

(2.31) If $f(z) \in C_p(A,B)$ $\left(B_{p+1}=\dfrac{1+B+(B-A)p}{(p+1)(B-A)p^2}\right)$, then

$$p(p-1)|z|^{p-2}-\frac{(p+2)(B-A)p^2}{1+B+(B-A)p}|z|^{p-1}\leqq|f''(z)|$$

$$\leqq p(p-1)|z|^{p-2}+\frac{(p+2)(B-A)p^2}{1+B+(B-A)p}|z|^{p-1}\quad(p\geqq 2),$$

$$|f''(z)|\leqq\frac{3(B-A)}{1+2B-A}\qquad(p=1).$$

(2.32) If $f(z)\in C_p(a,b)\left(B_{p+1}=\dfrac{1+b}{2b(1-a)p^2}\right)$, then

$$p(p-1)|z|^{p-2}-\frac{2(p+2)b(1-a)p^2}{(p+1)(1+b)}|z|^{p-1}\leqq|f''(z)|$$

$$\leqq p(p-1)|z|^{p-2}+\frac{2(p+2)b(1-a)p^2}{(p+1)(1+b)}|z|^{p-1}\quad(p\geqq 2)$$

(cf. [2], Theorem 7),

$$|f''(z)|\leqq\frac{3b(1-a)}{1+b}\quad(p=1).$$

(2.33) If $f(z)\in C(p,\alpha)\left(B_{p+1}=\dfrac{p+1-\alpha}{(p+1)p(p-\alpha)}\right)$, then

$$p(p-1)|z|^{p-2}-\frac{p(p+2)(p-\alpha)}{p+1-\alpha}|z|^{p-1}\leqq|f''(z)|$$

$$\leqq p(p-1)|z|^{p-2}+\frac{p(p+2)(p-\alpha)}{p+1-\alpha}|z|^{p-1}\quad(p\geqq 2)$$

(cf. [3], Theorem 3.3),

$$|f''(z)|\leqq\frac{3(1-\alpha)}{2-\alpha}\qquad(p=1)\ ([4],(3.4)).$$

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