

A LINEAR OPERATOR AND ITS APPLICATIONS
 TO CERTAIN SUBCLASSES OF MULTIVALENT FUNCTIONS

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I. INTRODUCTION

Let A_p denote the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad (p \in \mathbb{N} = \{1, 2, 3, \dots\}) \quad (1.1)$$

which are analytic in the open unit disk $U = \{z: |z| < 1\}$. A function $f(z)$ belonging to the class A_p is said to be in the class $R_p(\alpha)$ if it satisfies

$$\operatorname{Re}\{f^{(p)}(z)\} > \alpha \quad (1.2)$$

for some α ($\alpha < p!$) and for all $z \in U$. A function $f(z)$ belonging to the class A_p is said to be p -valently starlike of order α if and only if it satisfies

$$\operatorname{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (1.3)$$

for some α ($0 \leq \alpha < p$) and for all $z \in U$. We denote by $S_p^*(\alpha)$ the class of all functions in A_p which are p -valently starlike of order α in U .

A function $f(z)$ belonging to the class A_p is said to be p -valently convex of order α if and only if it satisfies

$$\operatorname{Re}\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha \quad (1.4)$$

for some α ($0 \leq \alpha < p$) and for all $z \in U$. Denoting by $C_p(\alpha)$ the class of all functions in A_p which are p -valently convex of order α in U , it is easily seen that

$$f(z) \in C_p(\alpha) \iff zf'(z) \in S_p^*(\alpha) \quad (0 \leq \alpha < p) \quad (1.5)$$

Further, a function $f(z)$ belonging to A_p is said to be p -valently close-to-convex of order α if and only if there exists a p -valently starlike function $g(z)$ such that

$$\operatorname{Re}\left\{\frac{zf'(z)}{g(z)}\right\} > \alpha \quad (1.6)$$

for some α ($0 \leq \alpha < p$) and for all $z \in U$.

For the functions $f_j(z)$ ($j = 1, 2$) defined by

$$f_j(z) = \sum_{n=0}^{\infty} a_{j,n+p} z^{n+p} \quad (p \in \mathbb{N}), \quad (1.7)$$

we denote the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$f_1 * f_2(z) = \sum_{n=0}^{\infty} a_{1,n+p} a_{2,n+p} z^{n+p}. \quad (1.8)$$

For a function $f(z)$ belonging to the class A_p , we define the generalized Libera integral operator $J_{c,p}$ by

$$J_{c,p}(f(z)) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t) dt, \quad c > -p. \quad (1.9)$$

For $p = 1$ and $c \in \mathbb{N}$, the operator $J_{c,1}$ was introduced by Bernardi [1]. In particular, the operator $J_{1,1}$ was studied earlier by Libera [4] and Livingston [5]. Some interesting results for the operator $J_{c,p}$ was proved by Saitoh [11] and Saitoh et al. [12].

Now, let the function $\phi_p(a, c)$ be defined by

$$\phi_p(a, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+p} \quad (z \in U), \quad (1.10)$$

for $c \neq 0, -1, -2, \dots$, where $(a)_n$ is the Pochhammer symbol given by

$$({}_a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & \text{if } n = 0 \\ a(a+1)\dots(a+n-1), & \text{if } n \in \mathbb{N}. \end{cases} \quad (1.11)$$

Also, we define a linear operator $L_p(a, c)$ on A_p by

$$L_p(a, c; z)f(z) = \left(\sum_{n=0}^{\infty} \frac{({}_a)_n}{(c)_n} z^{n+p} \right) * f(z) \quad (1.12)$$

for $f(z) \in A_p$ and $c \neq 0, -1, -2, \dots$.

The operator $L_1(a, c)$ was introduced by Carlson and Shaffer [2] in their systematic investigation of certain interesting classes of starlike, convex, and prestarlike hypergeometric functions.

$L_p(a, c)$ has the integral representation

$$L_p(a, c; z)f(z) = \int_0^1 u^{-p} f(uz) d\mu(u), \quad (1.13)$$

where μ satisfies

$$d\mu(u) = \frac{u^{a-1}(1-u)^{c-a-1}}{B(a, c-a)} du,$$

and

$$\int_0^1 d\mu(u) = 1.$$

REMARKS. (1) For $f(z) \in A_1 = A$,

$$L_1(n+1, 1; z)f(z) = D^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z)$$

is Ruscheweyh derivative of $f(z)$ ([8]).

(2) For $f(z) \in A_p$,

$$L_p(n+p, 1; z)f(z) = D^{n+p-1} f(z) = \frac{z^p}{(1-z)^{n+p}} * f(z)$$

is Ruscheweyh derivative introduced by Goel and Sohi [3].

(3) For $f(z) \in A_p$,

$$L_p(c+p, c+p+1; z)f(z) = J_{c,p}(f(z))$$

is the generalized Libera integral operator ([11], [12]).

(4) $\phi_1(a, c; z)$ is an incomplete beta function, related to the Gauss hypergeometric functions by

$$\phi_1(a, c; z) = z {}_2F_1(1, a; c; z).$$

2. SOME RESULTS

We shall now prove the following results.

THEOREM I. Let $f(z) \in A_p$ and $c > a > 0$. If $f(z) \in R_p(\alpha)$ ($z \in U$, $\alpha < p!$), then we have

$$L_p(a, c; z)f(z) \in R_p(\alpha).$$

PROOF. It is sufficient to show that

$$\operatorname{Re} \left\{ \frac{d^p}{dz^p} L_p(a, c; z)f(z) \right\} > \alpha \quad (z \in U).$$

Using the integral representation, we have

$$\begin{aligned} \operatorname{Re} \left\{ \frac{d^p}{dz^p} L_p(a, c; z)f(z) \right\} &= \frac{1}{B(a, c-a)} \int_0^1 u^{a-1} (1-u)^{c-a-1} \operatorname{Re}\{f^{(p)}(uz)\} du \\ &> \frac{\alpha}{B(a, c-a)} \int_0^1 u^{a-1} (1-u)^{c-a-1} du \\ &= \frac{\alpha}{B(a, c-a)} B(a, c-a) \\ &= \alpha, \end{aligned}$$

which evidently completes the proof of Theorem 1.

COROLLARY I. Let $f(z) \in A_p$ and $c > -p$. If $f(z) \in R_p(\alpha)$ ($z \in U$, $\alpha < p!$), then we have

$$J_{c,p}(f(z)) \in R_p(\alpha).$$

In order to prove our main results depicting properties of the function $L_p(a, c; z)f(z)$, we shall need the following lemma.

LEMMA. Let $\psi(z)$ and $g(z)$ be analytic in \mathbb{U} and satisfy

$$\begin{aligned} \psi(0) = \psi'(0) = \dots = \psi^{(p-1)}(0) = 0, \quad \psi^{(p)}(0) \neq 0, \\ g(0) = g'(0) = \dots = g^{(p-1)}(0) = 0, \quad g^{(p)}(0) \neq 0. \end{aligned}$$

Suppose that for each σ ($|\sigma| = 1$) and ρ ($|\rho| = 1$) we have

$$\psi(z) * \frac{1 + \rho\sigma z}{1 - \sigma z} g(z) \neq 0 \quad (z \in \mathbb{U} - \{0\}). \quad (2.1)$$

Then for each function $F(z)$ analytic in \mathbb{U} and satisfying

$$\operatorname{Re}\{F(z)\} > 0 \quad (z \in \mathbb{U}), \quad (2.2)$$

we have

$$\operatorname{Re}\left\{ \frac{(\psi * Fg)(z)}{(\psi * g)(z)} \right\} > 0 \quad (z \in \mathbb{U}). \quad (2.3)$$

REMARK. In the case $p = 1$, this lemma was given by Ruscheweyh and Small [9]. The proof of this lemma is similar to their proof.

Applying the above lemma, we now have

THEOREM 2. Let $f(z) \in S_p^*(\alpha)$ ($0 \leq \alpha < p$) and let

$$L_p(a, c; z) \left[\frac{1 + \rho\sigma z}{1 - \sigma z} \right] f(z) \neq 0 \quad (z \in \mathbb{U} - \{0\}) \quad (2.4)$$

for each σ ($|\sigma| = 1$) and ρ ($|\rho| = 1$), and for $c \neq 0, -1, -2, \dots$.

Then we have

$$L_p(a, c; z) f(z) \in S_p^*(\alpha).$$

PROOF. It is sufficient to show that

$$\operatorname{Re}\left\{ \frac{z[L_p(a, c; z) f(z)]'}{L_p(a, c; z) f(z)} \right\} > \alpha \quad (2.5)$$

for $z \in \mathbb{U}$. Since

$$\operatorname{Re}\left\{ \frac{z[L_p(a, c; z) f(z)]'}{L_p(a, c; z) f(z)} \right\} = \operatorname{Re}\left\{ \frac{L_p(a, c; z) z f'(z)}{L_p(a, c; z) f(z)} \right\}$$

$$= \operatorname{Re} \left\{ \frac{\phi_p(a, c; z) * (zf'(z))}{\phi_p(a, c; z) * f(z)} \right\}, \quad (2.6)$$

putting

$$\psi(z) = \phi_p(a, c; z), \quad F(z) = \frac{zf'(z)}{f(z)} - \alpha, \quad \text{and } g(z) = f(z)$$

in Lemma, we can see that

$$\begin{aligned} \operatorname{Re} \left\{ \frac{(\psi * Fg)(z)}{(\psi * g)(z)} \right\} &= \operatorname{Re} \left\{ \frac{\phi_p(a, c; z) * \{zf'(z) - \alpha f(z)\}}{\phi_p(a, c; z) * f(z)} \right\} \\ &= \operatorname{Re} \left\{ \frac{z[L_p(a, c; z)f(z)]'}{L_p(a, c; z)f(z)} \right\} - \alpha > 0, \end{aligned} \quad (2.7)$$

which completes the proof of Theorem 2.

COROLLARY 2. Let $f(z) \in S_p^*(\alpha)$ ($0 \leq \alpha < p$) and let

$$L_p(c+p, c+p+1; z) \left(\frac{1 + \rho\sigma z}{1 - \sigma z} \right) f(z) \neq 0 \quad (z \in \mathbb{U} - \{0\}) \quad (2.8)$$

for each σ ($|\sigma| = 1$) and ρ ($|\rho| = 1$), and for $c \neq -p-1, -p-2, -p-3, \dots$.

Then we have

$$J_{c,p}(f(z)) \in S_p^*(\alpha).$$

Setting $p = 1$ in Theorem 2, we have

COROLLARY 3 (Owa et al. [7]). Let $f(z) \in S_1^*(\alpha)$ ($0 \leq \alpha < 1$) and let

$$L_1(a, c; z) \left(\frac{1 + \rho\sigma z}{1 - \sigma z} \right) f(z) \neq 0 \quad (z \in \mathbb{U} - \{0\})$$

for each σ ($|\sigma| = 1$) and ρ ($|\rho| = 1$), and for $c \neq 0, -1, -2, \dots$.

Then we have

$$L_1(a, c; z)f(z) \in S_1^*(\alpha).$$

Next, we prove

THEOREM 3. Let $f(z) \in C_p(\alpha)$ ($0 \leq \alpha < p$) and let

$$L_p(a, c; z) \left\{ \frac{1 + \rho\sigma z}{1 - \sigma z} \right\} z f'(z) \neq 0 \quad (z \in \mathbb{U} - \{0\}) \quad (2.9)$$

for each σ ($|\sigma| = 1$) and ρ ($|\rho| = 1$), and for $c \neq 0, -1, -2, \dots$.

Then we have

$$L_p(a, c; z) f(z) \in \mathcal{C}_p(\alpha).$$

PROOF. Note that $f(z) \in \mathcal{C}_p(\alpha)$ if and only if $z f'(z) \in \mathcal{S}_p^*(\alpha)$.

By using Theorem 2, we know that

$$\begin{aligned} f(z) \in \mathcal{C}_p(\alpha) &\iff z f'(z) \in \mathcal{S}_p^*(\alpha) \\ &\implies L_p(a, c; z) z f'(z) \in \mathcal{S}_p^*(\alpha) \\ &\implies z [L_p(a, c; z) f(z)]' \in \mathcal{S}_p^*(\alpha) \\ &\iff L_p(a, c; z) f(z) \in \mathcal{C}_p(\alpha), \end{aligned} \quad (2.10)$$

which completes the proof of Theorem 3.

COROLLARY 4. Let $f(z) \in \mathcal{C}_p(\alpha)$ ($0 \leq \alpha < p$) and let

$$L_p(c+p, c+p+1; z) \left\{ \frac{1 + \rho\sigma z}{1 - \sigma z} \right\} z f'(z) \neq 0 \quad (z \in \mathbb{U} - \{0\}) \quad (2.11)$$

for each σ ($|\sigma| = 1$) and ρ ($|\rho| = 1$), and for $c \neq -p-1, -p-2, -p-3, \dots$.

Then we have

$$J_{c,p}(f(z)) \in \mathcal{C}_p(\alpha).$$

Finally, we prove

THEOREM 4. Let $f(z) \in \mathcal{K}_p(\alpha)$, i.e., there exists $g(z) \in \mathcal{S}_p^*(0)$ such that

$$\operatorname{Re} \left\{ \frac{z f'(z)}{g(z)} \right\} > \alpha \quad (0 \leq \alpha < p; z \in \mathbb{U}).$$

Further, let

$$L_p(a, c; z) \left\{ \frac{1 + \rho\sigma z}{1 - \sigma z} \right\} g(z) \neq 0 \quad (z \in \mathbb{U} - \{0\}) \quad (2.12)$$

for each σ ($|\sigma| = 1$) and ρ ($|\rho| = 1$), and for $c \neq 0, -1, -2, \dots$.

Then we have

$$L_p(a, c; z)f(z) \in K_p(\alpha). \quad (2.13)$$

PROOF. By Theorem 2, if $g(z) \in S_p^*(0)$, then $L_p(a, c; z)g(z) \in S_p^*(0)$.

It is sufficient to show that

$$\operatorname{Re} \left\{ \frac{z[L_p(a, c; z)f(z)]'}{L_p(a, c; z)g(z)} \right\} > \alpha \quad (2.14)$$

for $z \in \mathbb{U}$. Since

$$\begin{aligned} \operatorname{Re} \left\{ \frac{z[L_p(a, c; z)f(z)]'}{L_p(a, c; z)g(z)} \right\} &= \operatorname{Re} \left\{ \frac{L_p(a, c; z)zf'(z)}{L_p(a, c; z)g(z)} \right\} \\ &= \operatorname{Re} \left\{ \frac{\phi_p(a, c; z)^*(zf'(z))}{\phi_p(a, c; z)g(z)} \right\}, \end{aligned} \quad (2.15)$$

setting

$$\psi(z) = \phi_p(a, c; z), \quad F(z) = \frac{zf'(z)}{g(z)} - \alpha, \quad \text{and } g(z) = g(z)$$

in Lemma, we observe that

$$\begin{aligned} \operatorname{Re} \left\{ \frac{(\psi^*Fg)(z)}{(\psi^*g)(z)} \right\} &= \operatorname{Re} \left\{ \frac{\phi_p(a, c; z)^*\{zf'(z) - \alpha g(z)\}}{\phi_p(a, c; z)^*g(z)} \right\} \\ &= \operatorname{Re} \left\{ \frac{z[L_p(a, c; z)f(z)]'}{L_p(a, c; z)g(z)} \right\} - \alpha > 0, \end{aligned} \quad (2.16)$$

which completes the proof of Theorem 4.

COROLLARY 5. Let $f(z) \in K_p(\alpha)$, i.e., there exists $g(z) \in S_p^*(0)$ such that

$$\operatorname{Re} \left\{ \frac{zf'(z)}{g(z)} \right\} > \alpha \quad (0 \leq \alpha < p; z \in \mathbb{U}).$$

Further, let

$$L_p(c+p, c+p+1; z) \left(\frac{1 + \rho\sigma z}{1 - \sigma z} \right) g(z) \neq 0 \quad (z \in \mathbb{U} - \{0\}) \quad (2.17)$$

for each σ ($|\sigma| = 1$) and ρ ($|\rho| = 1$), and for $c \neq -p-1, -p-2, -p-3, \dots$.
Then we have

$$J_{c,p}(f(z)) \in K_p(\alpha).$$

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