

PROPERTIES OF CERTAIN INTEGRAL OPERATORS

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ABSTRACT

Two integral operators P^α and Q_β^α for analytic functions in the open unit disk are introduced. The object of the present paper is to derive some properties of integral operators P^α and Q_β^α .

I. INTRODUCTION

Let A be the class of functions of the form

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $U = \{z: |z| < 1\}$. Recently, Jung, Kim and Srivastava [4] have introduced the following one-parameter families of integral operators:

$$(1.2) \quad P^\alpha f = P^\alpha f(z) = \frac{2^\alpha}{z\Gamma(\alpha)} \int_0^z \left(\log \frac{z}{t} \right)^{\alpha-1} f(t) dt \quad (\alpha > 0),$$

$$(1.3) \quad Q_\beta^\alpha f = Q_\beta^\alpha f(z) = \left(\frac{\alpha+\beta}{\beta} \right) \frac{\alpha}{z^\beta} \int_0^z \left(1 - \frac{t}{z} \right)^{\alpha-1} t^{\beta-1} f(t) dt$$

($\alpha > 0, \beta > -1$),

and

$$(1.4) \quad J_\alpha f = J_\alpha f(z) = \frac{\alpha+1}{z^\alpha} \int_0^z t^{\alpha-1} f(t) dt \quad (\alpha > -1),$$

where $\Gamma(\alpha)$ is the familiar Gamma function, and (in general)

$$(1.5) \quad \binom{\alpha}{\beta} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\beta+1)\Gamma(\beta+1)} = \binom{\alpha}{\alpha-\beta}.$$

For $\alpha \in \mathbb{N} = \{1, 2, 3, \dots\}$, the operators P^α , Q_1^α , and J_α were considered by Bernardi ([1], [2]). Further, for a real number $\alpha > -1$, the operator J_α was used by Owa and Srivastava [6], and by Srivastava and Owa ([7], [8]).

REMARK. For $f(z) \in \mathcal{A}$ given by (1.1), Jung, Kim and Srivastava [4] have shown that

$$(1.6) \quad P^\alpha f(z) = z + \sum_{n=2}^{\infty} \left(\frac{2}{n+1} \right)^\alpha a_n z^n \quad (\alpha > 0),$$

$$(1.7) \quad Q_\beta^\alpha f(z) = z + \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \sum_{n=2}^{\infty} \frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)} a_n z^n \quad (\alpha > 0, \beta > -1),$$

and

$$(1.8) \quad J_\alpha f(z) = z + \sum_{n=2}^{\infty} \left(\frac{\alpha+1}{\alpha+n} \right) a_n z^n \quad (\alpha > -1).$$

By virtue of (1.6) and (1.8), we see that

$$(1.9) \quad J_\alpha f(z) = Q_\alpha^1 f(z) \quad (\alpha > -1).$$

2. AN APPLICATION OF MILLER-MOCANU LEMMA

To derive some properties of operators, we have to recall here the following lemma due to Miller and Mocanu [5].

LEMMA I. Let $w(u,v)$ be a complex valued function,

$$w: D \longrightarrow \mathbb{C}, \quad D \subset \mathbb{C} \times \mathbb{C} \quad (\mathbb{C} \text{ is the complex plane}),$$

and let $u = u_1 + iu_2$ and $v = v_1 + iv_2$. Suppose that the function $w(u,v)$ satisfies the following conditions:

(i) $w(u,v)$ is continuous in D ;

(ii) $(1,0) \in D$ and $\operatorname{Re}\{w(1,0)\} > 0$;

(iii) $\operatorname{Re}\{w(iu_2, v_1)\} \leq 0$ for all $(iu_2, v_1) \in D$ and such that

$$v_1 \leq -(1 + u_2^2)/2.$$

Let $p(z)$ be regular in \mathbb{U} and $p(z) = 1 + p_1z + p_2z^2 + \dots$ such that $(p(z), zp'(z)) \in \mathbb{D}$ for all $z \in \mathbb{U}$. If $\operatorname{Re}\{w(p(z), zp'(z))\} > 0$ ($z \in \mathbb{U}$), then $\operatorname{Re}\{p(z)\} > 0$ ($z \in \mathbb{U}$).

Applying the above lemma, we derive

THEOREM I. If $f(z) \in \mathcal{A}$ satisfies

$$(2.1) \quad \operatorname{Re}\left\{ \frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)} \right\} > \beta \quad (\alpha > 2; z \in \mathbb{U})$$

for some β ($\beta < 1$), then

$$(2.2) \quad \operatorname{Re}\left\{ \frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)} \right\} > \frac{4\beta - 1 + \sqrt{16\beta^2 - 8\beta + 17}}{8} \quad (z \in \mathbb{U}).$$

PROOF. Noting that

$$(2.3) \quad z(P^{\alpha}f(z))' = 2P^{\alpha-1}f(z) - P^{\alpha}f(z) \quad (\alpha > 1),$$

we have

$$(2.4) \quad \frac{z(P^{\alpha}f(z))'}{P^{\alpha}f(z)} = 2 \frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)} - 1 \quad (\alpha > 1).$$

Define the function $p(z)$ by

$$(2.5) \quad \frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)} = \gamma + (1 - \gamma)p(z)$$

with

$$(2.6) \quad \gamma = \frac{4\beta - 1 + \sqrt{16\beta^2 - 8\beta + 17}}{8}.$$

Then $p(z) = 1 + p_1z + p_2z^2 + \dots$ is analytic in the open unit disk \mathbb{U} .

Since

$$(2.7) \quad \frac{z(P^{\alpha-1}f(z))'}{P^{\alpha-1}f(z)} - \frac{z(P^{\alpha}f(z))'}{P^{\alpha}f(z)} = \frac{(1 - \gamma)zp'(z)}{\gamma + (1 - \gamma)p(z)},$$

or

$$(2.8) \quad \frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)} = \frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)} + \frac{(1-\gamma)zp'(z)}{2\{\gamma + (1-\gamma)p(z)\}},$$

we have

$$(2.9) \quad \operatorname{Re}\left\{\frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)} - \beta\right\} \\ = \operatorname{Re}\left\{\gamma + (1-\gamma)p(z) + \frac{(1-\gamma)zp'(z)}{2\{\gamma + (1-\gamma)p(z)\}} - \beta\right\} \\ > 0.$$

Therefore, if we define the function $w(u,v)$ by

$$(2.10) \quad w(u,v) = \gamma - \beta + (1-\gamma)u + \frac{(1-\gamma)v}{2\{\gamma + (1-\gamma)u\}},$$

then we see that

(i) $w(u,v)$ is continuous in $D = \left[C - \left\{\frac{\gamma}{\gamma-1}\right\}\right] \times C$;

(ii) $(1,0) \in D$ and $\operatorname{Re}\{w(1,0)\} = 1 - \beta > 0$;

(iii) for all $(iu_2, v_1) \in D$ and such that $v_1 \leq -(1 + u_2^2)/2$,

$$\operatorname{Re}\{w(iu_2, v_1)\} = \gamma - \beta + \frac{\gamma(1-\gamma)v_1}{2\{\gamma^2 + (1-\gamma)^2u_2^2\}} \\ \leq \gamma - \beta - \frac{\gamma(1-\gamma)(1 + u_2^2)}{4\{\gamma^2 + (1-\gamma)^2u_2^2\}} \\ = - \frac{(1-\gamma)(\gamma + 4(1-\gamma)\beta)}{4\{\gamma^2 + (1-\gamma)^2u_2^2\}} u_2^2 \\ \leq 0.$$

This implies that the function $w(u,v)$ satisfies the conditions in Lemma 1. Thus, applying Lemma 1, we conclude that

$$(2.11) \quad \operatorname{Re}\left\{\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\right\} > \gamma = \frac{4\beta - 1 + \sqrt{16\beta^2 - 8\beta + 17}}{8} \quad (z \in U).$$

Taking the special values for β in Theorem 1, we have

COROLLARY I. Let $f(z)$ be in the class A . Then

$$(i) \quad \operatorname{Re} \left\{ \frac{p^{\alpha-2} f(z)}{p^{\alpha-1} f(z)} \right\} > -\frac{1}{2} \quad (z \in U)$$

$$\implies \operatorname{Re} \left\{ \frac{p^{\alpha-1} f(z)}{p^{\alpha} f(z)} \right\} > \frac{1}{4} \quad (z \in U),$$

$$(ii) \quad \operatorname{Re} \left\{ \frac{p^{\alpha-2} f(z)}{p^{\alpha-1} f(z)} \right\} > -\frac{1}{4} \quad (z \in U)$$

$$\implies \operatorname{Re} \left\{ \frac{p^{\alpha-1} f(z)}{p^{\alpha} f(z)} \right\} > \frac{\sqrt{5} - 1}{4} \quad (z \in U),$$

$$(iii) \quad \operatorname{Re} \left\{ \frac{p^{\alpha-2} f(z)}{p^{\alpha-1} f(z)} \right\} > 0 \quad (z \in U)$$

$$\implies \operatorname{Re} \left\{ \frac{p^{\alpha-1} f(z)}{p^{\alpha} f(z)} \right\} > \frac{\sqrt{17} - 1}{8} \quad (z \in U),$$

$$(iv) \quad \operatorname{Re} \left\{ \frac{p^{\alpha-2} f(z)}{p^{\alpha-1} f(z)} \right\} > \frac{1}{4} \quad (z \in U)$$

$$\implies \operatorname{Re} \left\{ \frac{p^{\alpha-1} f(z)}{p^{\alpha} f(z)} \right\} > \frac{1}{2} \quad (z \in U),$$

and

$$(v) \quad \operatorname{Re} \left\{ \frac{p^{\alpha-2} f(z)}{p^{\alpha-1} f(z)} \right\} > \frac{1}{2} \quad (z \in U)$$

$$\implies \operatorname{Re} \left\{ \frac{p^{\alpha-1} f(z)}{p^{\alpha} f(z)} \right\} > \frac{\sqrt{17} + 1}{8} \quad (z \in U).$$

Next, we prove

THEOREM 2. If $f(z) \in A$ satisfies

$$(2.12) \quad \operatorname{Re} \left\{ \frac{Q_{\beta}^{\alpha-2} f(z)}{Q_{\beta}^{\alpha-1} f(z)} \right\} > \gamma \quad (\alpha > 2, \beta > -1; z \in \mathbb{U})$$

for some γ ($(\alpha+\beta-3)/2(\alpha+\beta-1) \leq \gamma < 1$), then

$$(2.13) \quad \operatorname{Re} \left\{ \frac{Q_{\beta}^{\alpha-1} f(z)}{Q_{\beta}^{\alpha} f(z)} \right\} > \delta \quad (z \in \mathbb{U}),$$

where

$$(2.14) \quad \delta = \frac{1 + 2\gamma(\alpha+\beta-1) + \sqrt{(1+2\gamma(\alpha+\beta-1))^2 + 8(\alpha+\beta)}}{4(\alpha + \beta)}.$$

PROOF. By the definition of $Q_{\beta}^{\alpha} f(z)$, we know that

$$(2.15) \quad z(Q_{\beta}^{\alpha} f(z))' = (\alpha + \beta)Q_{\beta}^{\alpha-1} f(z) - (\alpha + \beta - 1)Q_{\beta}^{\alpha} f(z) \quad (\alpha > 1, \beta > -1),$$

so that

$$(2.16) \quad \frac{z(Q_{\beta}^{\alpha} f(z))'}{Q_{\beta}^{\alpha} f(z)} = (\alpha + \beta) \frac{Q_{\beta}^{\alpha-1} f(z)}{Q_{\beta}^{\alpha} f(z)} - (\alpha + \beta - 1).$$

We define the function $p(z)$ by

$$(2.17) \quad \frac{Q_{\beta}^{\alpha-1} f(z)}{Q_{\beta}^{\alpha} f(z)} = \delta + (1 - \delta)p(z).$$

Then $p(z) = 1 + p_1 z + p_2 z^2 + \dots$ is analytic in \mathbb{U} . Making use of the logarithmic differentiations in both sides of (2.17), we have

$$(2.18) \quad \frac{z(Q_{\beta}^{\alpha-1} f(z))'}{Q_{\beta}^{\alpha-1} f(z)} = \frac{z(Q_{\beta}^{\alpha} f(z))'}{Q_{\beta}^{\alpha} f(z)} + \frac{(1 - \delta)zp'(z)}{\delta + (1 - \delta)p(z)}.$$

Applying (2.15) to (2.18), we obtain that

$$(2.19) \quad \frac{Q_{\beta}^{\alpha-2} f(z)}{Q_{\beta}^{\alpha-1} f(z)} = \frac{1}{\alpha + \beta - 1} \left\{ (\alpha + \beta) \frac{Q_{\beta}^{\alpha-1} f(z)}{Q_{\beta}^{\alpha} f(z)} - 1 + \frac{(1 - \delta)zp'(z)}{\delta + (1 - \delta)p(z)} \right\}$$

$$= \frac{1}{\alpha + \beta - 1} \left\{ \delta(\alpha + \beta) - 1 + (\alpha + \beta)(1 - \delta)p(z) + \frac{(1 - \delta)zp'(z)}{\delta + (1 - \delta)p(z)} \right\},$$

that is, that

$$(2.20) \quad \operatorname{Re} \left\{ \frac{Q_{\beta}^{\alpha-2} f(z)}{Q_{\beta}^{\alpha-1} f(z)} - \gamma \right\} \\ = \frac{1}{\alpha + \beta - 1} \operatorname{Re} \left\{ \delta(\alpha + \beta) - 1 + (\alpha + \beta)(1 - \delta)p(z) + \frac{(1 - \delta)zp'(z)}{\delta + (1 - \delta)p(z)} \right\} - \gamma > 0.$$

Now, we let

$$(2.21) \quad w(u, v) = \frac{1}{\alpha + \beta - 1} \left\{ \delta(\alpha + \beta) - 1 + (\alpha + \beta)(1 - \delta)u + \frac{(1 - \delta)v}{\delta + (1 - \delta)u} \right\} - \gamma$$

with $u = u_1 + iu_2$ and $v = v_1 + iv_2$. Then $w(u, v)$ satisfies that

(i) $w(u, v)$ is continuous in $D = \left[\mathbb{C} - \left\{ \frac{\delta}{\delta - 1} \right\} \right] \setminus \mathbb{C}$;

(ii) $(1, 0) \in D$ and $\operatorname{Re}\{w(1, 0)\} = 1 - \gamma > 0$;

(iii) for all $(iu_2, v_1) \in D$ and such that $v_1 \leq -(1 + u_2^2)/2$,

$$\operatorname{Re}\{w(iu_2, v_1)\} = \frac{1}{\alpha + \beta - 1} \left\{ \delta(\alpha + \beta) - 1 + \frac{\delta(1 - \delta)v_1}{\delta^2 + (1 - \delta)^2 u_2^2} \right\} - \gamma \\ \leq \frac{1}{\alpha + \beta - 1} \left\{ \delta(\alpha + \beta) - 1 - \gamma(\alpha + \beta - 1) - \frac{\delta(1 - \delta)(1 + u_2^2)}{2\{\delta^2 + (1 - \delta)^2 u_2^2\}} \right\} \\ \leq 0.$$

Thus, the function $w(u, v)$ satisfies the conditions in Lemma 1. This shows that $\operatorname{Re}\{p(z)\} > 0$ ($z \in U$), or

$$(2.22) \quad \operatorname{Re} \left\{ \frac{Q_{\beta}^{\alpha-1} f(z)}{Q_{\beta}^{\alpha} f(z)} \right\} > \delta \quad (z \in U).$$

If we take $\gamma = (\alpha + \beta - 3)/2(\alpha + \beta - 1)$ in Theorem 2, then we have

COROLLARY 2. If $f(z) \in A$ satisfies

$$\operatorname{Re} \left\{ \frac{Q_{\beta}^{\alpha-2} f(z)}{Q_{\beta}^{\alpha-1} f(z)} \right\} > \frac{\alpha + \beta - 3}{2(\alpha + \beta - 1)} \quad (\alpha > 2, \beta > -1; z \in U),$$

then

$$\operatorname{Re} \left\{ \frac{Q_{\beta}^{\alpha-1} f(z)}{Q_{\beta}^{\alpha} f(z)} \right\} > \frac{1}{2} \quad (z \in U).$$

Further, letting $\alpha = 2 - \beta$ and $\gamma = 1/2$ in Theorem 2, we have

COROLLARY 3. If $f(z) \in A$ satisfies

$$\operatorname{Re} \left\{ \frac{Q_{\beta}^{-\beta} f(z)}{Q_{\beta}^{1-\beta} f(z)} \right\} > \frac{1}{2} \quad (\beta > -1; z \in U),$$

then

$$\operatorname{Re} \left\{ \frac{Q_{\beta}^{1-\beta} f(z)}{Q_{\beta}^{2-\beta} f(z)} \right\} > \frac{1 + \sqrt{5}}{4} \quad (z \in U).$$

3. AN APPLICATION OF JACK'S LEMMA

We need the following lemma due to Jack [3] (also, due to Miller and Mocanu [5]).

LEMMA 2. Let $w(z)$ be analytic in U with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in U$, then we can write

$$(3.1) \quad z_0 w'(z_0) = k w(z_0),$$

where k is a real number and $k \geq 1$.

Applying Lemma 2 for the operator P^{α} , we have

THEOREM 3. If $f(z) \in A$ satisfies

$$(3.2) \quad \operatorname{Re} \left\{ \frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)} \right\} > \beta \quad (\alpha > 2, \beta \leq 1/4; z \in \mathbb{U}),$$

then

$$(3.3) \quad \operatorname{Re} \left\{ \frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)} \right\} > \gamma \quad (z \in \mathbb{U}),$$

where

$$(3.4) \quad \gamma = \frac{3 + 4\beta + \sqrt{16\beta^2 - 40\beta + 9}}{8}.$$

PROOF. Defining the function $w(z)$ by

$$(3.5) \quad \frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)} = \frac{1 - (1 - 2\gamma)w(z)}{1 + w(z)},$$

we see that $w(z)$ is analytic in \mathbb{U} and $w(0) = 0$. It follows from (3.5) that

$$(3.6) \quad \frac{z(P^{\alpha-1}f(z))'}{P^{\alpha-1}f(z)} = \frac{z(P^{\alpha}f(z))'}{P^{\alpha}f(z)} - \frac{(1 - 2\gamma)zw'(z)}{1 - (1 - 2\gamma)w(z)} - \frac{zw'(z)}{1 + w(z)}.$$

Using (2.3), we obtain that

$$(3.7) \quad \frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)} = \frac{1 - (1 - 2\gamma)w(z)}{1 + w(z)} - \frac{1}{2} \frac{zw'(z)}{w(z)} \left(\frac{(1 - 2\gamma)w(z)}{1 - (1 - 2\gamma)w(z)} + \frac{w(z)}{1 + w(z)} \right).$$

If we suppose that there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1 \quad (w(z_0) \neq -1),$$

then Lemma 2 gives us that

$$z_0 w'(z_0) = k w(z_0) \quad (k \geq 1).$$

Therefore, letting $w(z_0) = e^{i\theta}$ ($\theta \neq \pi$), we have

$$\begin{aligned}
 (3.8) \quad & \operatorname{Re} \left\{ \frac{P^{\alpha-2}f(z_0)}{P^{\alpha-1}f(z_0)} \right\} \\
 &= \operatorname{Re} \left\{ \frac{1 - (1 - 2\gamma)w(z_0)}{1 + w(z_0)} \right. \\
 &\quad \left. - \frac{k}{2} \left[\frac{(1 - 2\gamma)w(z_0)}{1 - (1 - 2\gamma)w(z_0)} + \frac{w(z_0)}{1 + w(z_0)} \right] \right\} \\
 &= \operatorname{Re} \left\{ \frac{1 - (1 - 2\gamma)e^{i\theta}}{1 + e^{i\theta}} - \frac{k}{2} \left[\frac{(1 - 2\gamma)e^{i\theta}}{1 - (1 - 2\gamma)e^{i\theta}} + \frac{e^{i\theta}}{1 + e^{i\theta}} \right] \right\} \\
 &= \gamma - \frac{k}{2} \left[\frac{(1 - 2\gamma)(\cos\theta - (1 - 2\gamma))}{1 + (1 - 2\gamma)^2 - 2(1 - 2\gamma)\cos\theta} + \frac{1}{2} \right] \\
 &\leq \gamma - \frac{k}{2} \left[\frac{1}{2} - \frac{1 - 2\gamma}{2(1 - \gamma)} \right] \\
 &\leq \frac{\gamma(3 - 4\gamma)}{4(1 - \gamma)} = \beta.
 \end{aligned}$$

This contradicts our condition (3.2). Therefore, we have

$$(3.9) \quad |w(z)| = \left| \frac{\frac{P^{\alpha-1}f(z)}{P^\alpha f(z)} - 1}{\frac{P^{\alpha-1}f(z)}{P^\alpha f(z)} + (1 - 2\gamma)} \right| < 1 \quad (z \in \mathbb{U}),$$

which implies (3.3). This completes the proof of Theorem 3.

Taking $\beta = 0$ and $\beta = 1/4$ in Theorem 3, we have

COROLLARY 3. Let $f(z)$ be in the class A . Then

$$\operatorname{Re} \left\{ \frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)} \right\} > 0 \quad (\alpha > 2; z \in \mathbb{U})$$

$$\implies \operatorname{Re} \left\{ \frac{P^{\alpha-1}f(z)}{P^\alpha f(z)} \right\} > \frac{3}{4} \quad (z \in \mathbb{U}),$$

and

$$\operatorname{Re}\left\{\frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)}\right\} > \frac{1}{4} \quad (\alpha > 2; z \in \mathbb{U})$$

$$\implies \operatorname{Re}\left\{\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\right\} > \frac{1}{2} \quad (z \in \mathbb{U}).$$

Finally, we prove

THEOREM 4. If $f(z) \in \mathcal{A}$ satisfies

$$(3.10) \quad \operatorname{Re}\left\{\frac{Q_{\beta}^{\alpha-2}f(z)}{Q_{\beta}^{\alpha-1}f(z)}\right\} > \gamma \quad (\alpha > 2, \beta > -1; z \in \mathbb{U})$$

for some γ ($\gamma < 1$), then

$$(3.11) \quad \operatorname{Re}\left\{\frac{Q_{\beta}^{\alpha-1}f(z)}{Q_{\beta}^{\alpha}f(z)}\right\} > \delta \quad (z \in \mathbb{U}),$$

where δ ($0 \leq \delta < 1$) is the smallest positive root of the equation

$$(3.12) \quad 2(\alpha+\beta)\delta^2 - \{2(\alpha+\beta)(\gamma+1) - (2\gamma-1)\}\delta + 2\{(\alpha+\beta-1)\gamma + 1\} = 0.$$

PROOF. Define the function $w(z)$ by

$$(3.13) \quad \frac{Q_{\beta}^{\alpha-1}f(z)}{Q_{\beta}^{\alpha}f(z)} = \frac{1 - (1 - 2\delta)w(z)}{1 + w(z)},$$

we obtain that

$$(3.14) \quad \frac{Q_{\beta}^{\alpha-2}f(z)}{Q_{\beta}^{\alpha-1}f(z)} = \frac{1}{\alpha+\beta-1} \left\{ (\alpha+\beta) \frac{1 - (1-2\delta)w(z)}{1 + w(z)} - 1 \right. \\ \left. - \frac{zw'(z)}{w(z)} \left[\frac{(1-2\delta)w(z)}{1 - (1-2\delta)w(z)} + \frac{w(z)}{1 + w(z)} \right] \right\}.$$

Therefore, supposing that there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1 \quad (w(z_0) \neq -1),$$

we see that

$$(3.15) \quad \operatorname{Re} \left\{ \frac{Q_{\beta}^{\alpha-2} f(z_0)}{Q_{\beta}^{\alpha-1} f(z_0)} \right\} \leq \frac{1}{\alpha+\beta-1} \left\{ (\alpha+\beta)\delta - 1 - \frac{\delta}{2(1-\delta)} \right\} \\ = \gamma.$$

This completes the assertion of Theorem 4.

Making $\gamma = 0$ in Theorem 4, we have

COROLLARY 4. If $f(z) \in A$ satisfies

$$(3.16) \quad \operatorname{Re} \left\{ \frac{Q_{\beta}^{\alpha-2} f(z)}{Q_{\beta}^{\alpha-1} f(z)} \right\} > 0 \quad (\alpha > 2, \beta > -1; z \in \mathbb{U}),$$

then

$$(3.17) \quad \operatorname{Re} \left\{ \frac{Q_{\beta}^{\alpha-1} f(z)}{Q_{\beta}^{\alpha} f(z)} \right\} > \frac{-1}{\alpha + \beta - 1} \quad (z \in \mathbb{U}).$$

Letting $\gamma = 1/2$ in Theorem 4, we have

COROLLARY 5. If $f(z) \in A$ satisfies

$$(3.18) \quad \operatorname{Re} \left\{ \frac{Q_{\beta}^{\alpha-2} f(z)}{Q_{\beta}^{\alpha-1} f(z)} \right\} > \frac{1}{2} \quad (\alpha > 2, \beta > -1; z \in \mathbb{U}),$$

then

$$(3.19) \quad \operatorname{Re} \left\{ \frac{Q_{\beta}^{\alpha-1} f(z)}{Q_{\beta}^{\alpha} f(z)} \right\} > \frac{\alpha + \beta - 3}{\alpha + \beta - 1} \quad (z \in \mathbb{U}).$$

Further, if $\alpha = 3 - \beta$, we have

$$\operatorname{Re} \left\{ \frac{Q_{\beta}^{1-\beta} f(z)}{Q_{\beta}^{2-\beta} f(z)} \right\} > \frac{1}{2} \quad (\beta > -1; z \in \mathbb{U})$$

$$\implies \operatorname{Re} \left\{ \frac{Q_{\beta}^{2-\beta} f(z)}{Q_{\beta}^{3-\beta} f(z)} \right\} > 0 \quad (z \in \mathbb{U}).$$

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