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ON SOME SUFFICIENT CONDITIONS FOR STARLIKENESS AND CONVEXITY

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ABSTRACT

For a function $f(z) = z + a_2z^2 + \ldots$ analytic in the unit disk, we consider the conditions of the form $|f'(z) + zf''(z) - 1| < j$ which imply starlikeness or convexity of it.

I. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{A}$ denote the class of functions $f(z)$ analytic in the unit disk $\mathcal{U} = \{z: |z| < 1\}$ with the conditions $f(0) = f'(0) - 1 = 0$. As usual, we denote by $\mathcal{K}$, $\mathcal{S}^*$, and $\mathcal{C}$ the subclasses of $\mathcal{A}$ whose members are convex, starlike, and close-to-convex, respectively. All these classes are subclasses of univalent functions in the unit disk $\mathcal{U}$ (see, for example [1]).

Let $f(z)$ and $F(z)$ be analytic in the unit disk $\mathcal{U}$. Then we say that $f(z)$ is subordinate to $F(z)$, written by $f(z) \prec F(z)$ or $f \prec F$, if there exists a function $w(z)$ analytic in $\mathcal{U}$ such that $w(0) = 0$, $|w(z)| < 1$ $(z \in \mathcal{U})$, and $f(z) = F(w(z))$. If $F(z)$ is univalent in $\mathcal{U}$, then $f \prec F$ if and only if $f(0) = F(0)$ and $f(\mathcal{U}) \subset F(\mathcal{U})$.

If for a function $f(z) \in \mathcal{A}$ we have

$$|f'(z) + zf''(z) - 1| < 2 \qquad (z \in \mathcal{U}),$$

which is equivalent to

$$(zf'(z))' \prec 1 + 2z ,$$

then by applying Lemma 1, given below, we get

(1) $$f'(z) \prec 1 + z,$$

i.e. $\mathrm{Re}\{f'(z)\} > 0$ $(z \in \mathcal{U})$, and $f(z) \in \mathcal{C}$. But, as Mocanu [4] showed,

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the condition (1) doesn't imply $f(z) \in S^*$.

In that sense, we may ask a question on a constant j, $j < 2$, such that the condition

$$|f'(z) + zf''(z) - 1| < j \qquad (z \in U)$$

implies $f(z) \in S^*$ or $f(z) \in K$. It will be the object of this paper. But previously, we cite the following lemmas that will be used further.

LEMMA 1 ([2]). Let F(z) be a convex function in U (i.e. F(z) is univalent and F(U) is a convex domain). If $\mathrm{Re}\{\gamma\} > 0$ and f(z) is analytic in U, then

$$f(z) \prec F(z) \Rightarrow \frac{1}{z^\gamma}\int_0^z f(t)t^{\gamma-1}dt \prec \frac{1}{z^\gamma}\int_0^z F(t)t^{\gamma-1}dt .$$

LEMMA 2 ([4]). If $f(z) \in A$ and

$$|f'(z) - 1| < \frac{2}{\sqrt{5}} = 0.894... \qquad (z \in U),$$

then $f(z) \in S^*$.

LEMMA 3 ([3]). Let Ω be a subset of the complex plane $\mathbb{C}$ and suppose that the function $\psi: \mathbb{C}^2 \times U \longrightarrow \mathbb{C}$ satisfies the condition $\psi(ix,y;z) \notin \Omega$, for all real x, $y \leqq -(1 + x^2)/2$ and all $z \in U$. If the function p(z) is analytic in U, $p(0) = 1$ and $\psi(p(z),zp'(z);z) \in \Omega$ $(z \in U)$, then $\mathrm{Re}\{p(z)\} > 0$.

2. RESULTS AND CONSEQUENCES

We start with the following statement which easily follows from Lemma 1 and Lemma 2.

THEOREM 1. Let $f(z) \in A$ satisfy the condition

$$(2) \qquad |f'(z) + zf''(z) - 1| < \frac{4}{\sqrt{5}} = 1.788... \qquad (z \in U),$$

then $f(z) \in S^*$.

PROOF. Since the condition (2) may be rewritten in the form

$$(zf'(z))' \prec 1 + \frac{4}{\sqrt{5}} z,$$

an application of Lemma 1 gives that $f'(z) \prec 1 + (2/\sqrt{5})z$, i.e.

$$|f'(z) - 1| < \frac{2}{\sqrt{5}} \qquad (z \in U).$$

Therefore, by using Lemma 2, we have $f(z) \in S^*$.

But we can get the precise result for a stronger condition than (2). Namely, we have

THEOREM 2. If $f(z) \in A$ satisfies

(3) $$|f'(z) + zf''(z) - 1| < \frac{3}{2} \qquad (z \in U),$$

then $f(z) \in S^*$ and

(4) $$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 \qquad (z \in U).$$

PROOF. At the begining, we note that we use the method given by Mocanu [4]. From (3) we have $(zf'(z))' \prec 1 + (3/2)z$, and by using Lemma 1, $f'(z) \prec 1 + (3/4)z$, and so (by using the same lemma once again)

(5) $$\frac{f(z)}{z} \prec 1 + \frac{3}{8} z .$$

If we put

(6) $$\frac{zf'(z)}{f(z)} = \frac{2p(z)}{p(z) + 1} \quad \text{and} \quad g(z) = \frac{f(z)}{z},$$

then the inequality (3) is equivalent to

(7) $$\left| g(z) \frac{2zp'(z) + 4p(z)^2}{(p(z) + 1)^2} - 1 \right| < \frac{3}{2} \qquad (z \in U).$$

To prove (4) from (7), it is enouth to prove that $\mathrm{Re}\{p(z)\} > 0$ $(z \in U)$. By Lemma 3, it is sufficient to prove that

$$(8) \qquad \left| g(z) \frac{2y + 4(ix)^2}{(ix+1)^2} - 1 \right| \geqq \frac{3}{2}$$

for all real x, $y \leqq -(1 + x^2)/2$ and all $z \in \mathbb{U}$. Later, if we let $g(z) = u + iv$, then (8) is equivalent to

$$(9) \qquad 16(u^2 + v^2)y^2 - 16\{(4(u^2 + v^2) - u)x^2 + 2vx + u\}y$$
$$+ \{64(u^2 + v^2) - 32u - 5\}x^4 + 64vx^3 + (32u - 10)x^2 - 5 \geqq 0.$$

From the relation (5), we have

$$(10) \qquad u^2 + v^2 - 2u + \frac{55}{64} < 0.$$

Also, from (10), we easily obtain the following inequalities which we will be used:

$$(11) \qquad 4(u^2 + v^2) - u > 0, \; 10(u^2 + v^2) + 3u > 0,$$
$$20(u^2 + v^2) - 8u - 1 > 0, \quad \frac{5}{8} < u < \frac{11}{8}.$$

By using (11), we deduce that

$$\{4(u^2 + v^2) - u\}x^2 + 2vx + u > 0$$

for all real x. Therefore, if we denote by L the left-side of (9) and if we use $y \leqq -(1 + x^2)/2$, then we obtain

$$(12) \qquad L \geqq Ax^4 + 2Bx^3 + cx^2 + 2Dx + E \equiv M(x),$$

where $A = 5\{20(u^2 + v^2) - 8u - 1\}$, $B = 40v$, $C = 40(u^2 + v^2) + 32u - 10$, $D = 8v$, and $E = 4(u^2 + v^2) + 8u - 5$.
If we write $M(x) = x^2M_1(x) + M_2(x)$, where $M_1(x) = Ax^2 + 2Bx + C_1$, $M_2(x) = C_2x^2 + 2Dx + E$, $C_1 = 40(u^2 + v^2) + 12u = 4\{10(u^2 + v^2) + 3u\}$, and $C_2 = 10(2u - 1)$, then we shall prove that $M_1(x) \geqq 0$ and $M_2(x) \geqq 0$ for all real x. First, from (11) we get $A > 0$, $C_2 > 0$, and after that if we put $u^2 + v^2 = 2u - a$ with $55/64 < a \leqq 1$, then we have

$$B^2 - AC_1 = -20\{816u^2 - (183 + 780a)u + 90 + 200a^2\} < 0$$

and

$$D^2 - C_2E = -2\{192u^2 - 2(97 + 20a)u + 52a + 25\} < 0.$$

Hence we have $M_1(x) > 0$ and $M_2(x) > 0$, and we conclude that $M(x) > 0$.

REMARK I. In the paper [4], Mocanu has proved that if $\alpha \geqq 1/2$, $f(z) \in A$ and

$$|f'(z) + \alpha z f''(z) - 1| < 1 \qquad (z \in U),$$

then $f(z) \in S^*$ and $|zf'(z)/f(z) - 1| < 1$ $(z \in U)$. This means that we improved Mocanu's result for $\alpha = 1$.

The following theorem gives the results on convexity problem.

THEOREM 3. Let $f(z)$ be in the class A.

(i) If $f(z)$ satisfies

$$(13) \qquad |f'(z) + zf''(z) - 1| < \frac{2}{\sqrt{5}} \qquad (z \in U),$$

then $f(z) \in K$.

(ii) If $f(z)$ satisfies

$$(14) \qquad |f'(z) + zf''(z) - 1| < j \text{ and } |\arg(f'(z))| \leqq \operatorname{arctg}\left(\frac{\sqrt{1 - j^2}}{j}\right)$$

for some $2/\sqrt{5} < j \leqq 1$ and for all $z \in U$, then also $f(z) \in K$.

PROOF. (i) The condition (13) is equivalent to

$$|(zf'(z))' - 1| < \frac{2}{\sqrt{5}} \qquad (z \in U),$$

and by Lemma 2 we obtain $zf'(z) \in S^*$, that is, $f(z) \in K$.

(ii) Let $f(z) \in A$ satisfies the conditions in (14) and let $2/\sqrt{5} < j < 1$ Then from the first condition in (14) we ha e

$$(15) \qquad \left|f'(z)\left(1 + \frac{zf''(z)}{f'(z)}\right) - 1\right| < j \qquad (z \in U).$$

If we put $g(z) = f'(z)$ and $p(z) = 1 + zf''(z)/f'(z)$, then we write $|g(z)p(z) - 1| < j$ $(z \in U)$ instead of (15). To prove that $\operatorname{Re}\{p(z)\} > 0$ $(z \in U)$, by Lemma 3, it is enough to prove that

$$|g(z)(ix) - 1|^2 - j^2 \geqq 0$$

for given j ($2/\sqrt{5} < j < 1$), for all real x, $y \leqq -(1 + x^2)/2$, and for all $z \in U$. In that sense, let's put $g(z) = u + iv$. Then from the second inequality of (14), we have $u/v \leqq \sqrt{1 - j^2}\,/j$ $(z \in U)$, and so

$$\begin{aligned}|g(z)(ix) - 1|^2 - j^2 &= |(u + iv)(ix) - 1|^2 - j^2\\ &= (u^2 + v^2)x^2 + 2vx + 1 - j^2\\ &\geqq 0.\end{aligned}$$

Then, by Lemma 3, we obtain $\mathrm{Re}\{p(z)\} > 0$ $(z \in U)$, i.e. $f(z) \in K$. For $j = 1$ from (14), we get $\arg(f'(z)) = 0$ $(z \in U)$, which gives $f'(z) = $ const., i.e. $f(z) \equiv z$.

COROLLARY 1. Let $f(z) \in A$ satisfy

$$|f'(z) + zf''(z) - 1| < j \quad \text{and} \quad |f'(z) - 1| < \sqrt{1 - j^2}$$

for some j ($2/\sqrt{5} < j \leqq 1$) and for all $z \in U$. Then $f(z) \in K$.

REMARK 2. If we consider the functions

$$f(z) = z + \frac{j}{n^2} z^n \qquad (n > 2,\ 2/\sqrt{5} < j \leqq n/\sqrt{n^2 + 1}\ ),$$

then we have that

$$|f'(z) + zf''(z) - 1| = j|z|^{n-1} < j \qquad (z \in U),$$

and by Part (i) we don't conclude that $f(z) \in K$. But since

$$|f'(z) - 1| = \frac{j}{n}|z|^{n-1} < \frac{j}{n} \leqq \sqrt{1 - j^2} \qquad (z \in U),$$

by Corollary 1, we have $f(z) \in K$. In that sense, the part (ii) of Theorem 3 is justified.

Finally, if we make a summary of all the previous results and considerations we can easily derive

COROLLARY 2. Let f(z) be in the class A with $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$

Then the following implications are true:

(i) $\sum_{n=2}^{\infty} n^2|a_n| \leqq 2 \implies \mathrm{Re}\{f'(z)\} > 0 \ (z \in \mathcal{U})$, i.e. $f(z) \in \mathcal{C}$;

(ii) $\sum_{n=2}^{\infty} n^2|a_n| \leqq \frac{4}{\sqrt{5}} \implies f(z) \in \mathcal{S}^*$;

(iii) $\sum_{n=2}^{\infty} n^2|a_n| \leqq \frac{3}{2} \implies f(z) \in \mathcal{S}^*$ and $\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 \ (z \in \mathcal{U})$;

(iv) $\sum_{n=2}^{\infty} n^2|a_n| \leqq j$ and $\sum_{n=2}^{\infty} n|a_n| \leqq \sqrt{1 - j^2} \quad (\frac{2}{\sqrt{5}} < j \leqq 1)$

$\implies f(z) \in \mathcal{K}$;

(v) $\sum_{n=2}^{\infty} n^2|a_n| \leqq \frac{2}{\sqrt{5}} \implies f(z) \in \mathcal{K}$.

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