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ON QUASI-CONVEX FUNCTIONS OF COMPLEX ORDER

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Abstract

The class Q of quasi-convex functions was studied by K.I.Noor. The authors, using the Sälägean differential operator, introduce the class Q(b) of functions quasi-convex of complex order b , $b \neq 0$ and the class $Q_n(b)$ which is the generalization of Q(b), where n is a nonnegative interger. Sharp coefficient bounds are determined for $Q_n(b)$. The authors also obtain some sufficient conditions for functions to belong to $Q_n(b)$ and a distortion theorem.

1. Introduction

Let A denote the class of functions f(z) analytic in the unit disk $E = \{ z : |z| < 1 \}$ having the power series

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m \quad , z \in E \quad . \tag{1.1}$$

Aouf and Nasr [2] introduce the class S*(b) of starlike functions of order b, where b is a non zero complex number, as follows :

$$S^*(b) = \left\{ f : f \in A \text{ and } \mathrm{Re} \left[1 + \frac{1}{b} \left(\frac{zf'(z)}{f(z)} - 1 \right) \right] > 0 \, , \, z \in E \right\} \tag{1.2}$$

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We define the class $K(b)$ of close-to-convex functions of complex order b as follows : $f \in K(b)$ iff $f \in A$ and

$$\mathrm{Re}\left[1+\frac{1}{b}\left(\frac{zf'(z)}{g(z)}-1\right)\right] > 0 \quad , z \in E \tag{1.3}$$

for some starlike function g.

And we define the class $Q(b)$ of quasi-convex functions of complex order b as follows : $f \in Q(b)$ iff $f \in A$ and

$$\mathrm{Re}\left[1+\frac{1}{b}\left(\frac{(zf'(z))'}{g'(z)}-1\right)\right] > 0 \,, z \in E \tag{1.4}$$

for some convex function g.

The class S_n, $n \in N_0 = \{0,1,2,\cdots\}$, was introduced by Sălăgean [7], that is, $f \in S_n$ iff $f \in A$ and

$$\mathrm{Re}\left\{\frac{D^{n+1} f(z)}{D^n f(z)}\right\} > 0 \quad , z \in E \tag{1.5}$$

where the operator $f \longrightarrow D^n f$ is defined by

(1) $D^0 f(z) = f(z)$,
(2) $Df(z) = zf'(z)$,
(3) $D^n f(z) = D(D^{n-1}f(z))$ $\quad (n \in N = \{1,2,\cdots\})$.

It may be noted that S_0 is the class S^* of starlike functions while $S_1 = C$ is formed with all convex functions. More, it is known [7] that $S_{n+1} \subset S_n$, $n \in N_0$.

Let $Q_n(b)$, $n \in N_0$, b is a nonzero complex number, denote the class of functions $f \in A$ satisfying

$$\mathrm{Re}\left\{1+\frac{1}{b}\left[\frac{D^{n+1}f(z)}{D^n g(z)}-1\right]\right\} > 0 \quad , z \in E \tag{1.6}$$

for some $g \in S_n$. Here $Q_0(b) = K(b)$, $Q_1(b) = Q(b)$.

In this paper, we determine coefficient estimates of functions in $Q_n(b)$, $n \in N_0$. Further, we obtain some sufficient conditions for $f \in Q_n(b)$ and a distortion theorem.

2. Coefficient Inequalities

We determine coefficient estimates of functions in $Q_n(b)$, $n \in N_0$. First, we need the following lemmas.

Lemma 2.1 Let $g(z) = z + \sum_{m=2}^{\infty} c_m z^m \in S_n$, where $n \in N_0$.

Then
$$|c_m| \leq \frac{1}{m^{n-1}} \qquad (m \geq 2).$$

proof. Noting that

$$D^n g(z) = z + \sum_{m=2}^{\infty} m^n c_m z^m . \tag{2.1}$$

Since $g \in S_n$, $D^n g(z) \in S^*$. Thus, using the well known coefficient estimates for starlike functions one gets,

$$m^n |c_m| \leq m \ , \quad m \geq 2 .$$

Lemma 2.2 For $n \in N_0$, let

$$D^{n+1} f(z) = \frac{z(1 + (2b - 1)z)}{(1 - z)^3} .$$

Then $f \in Q_n(b)$ and $f(z) = z + \sum_{m=2}^{\infty} \frac{1}{m^n} [(m - 1)b + 1] z^m$ in E.

proof. Let $g \in A$ be defined so that

$$D^n g(z) = \frac{z}{(1-z)^2} \quad .$$

The definitions of S_n implies $g \in S_n$. Therefore,

$$1 + \frac{1}{b}\left[\frac{D^{n+1}f(z)}{D^n\ g(z)} - 1\right] = \frac{1+z}{1-z} \quad , \ z \in E \ .$$

This proves that $f \in Q_n(b)$.

Lemma 2.3 Let $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$. If $f \in Q_n(b)$, $n \in N_0$, then

$$|ma_m - c_m|^2 \le 4 \frac{1}{m^{2n}} |b| \left\{ |b| + \sum_{k=2}^{m-1} k^{2n} \left[|ka_k - c_k||c_k| + |b||c_k|^2 \right] \right\}. \tag{2.2}$$

proof. Let $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$ be in $Q_n(b)$. Then (1.6) implies

$$1 + \frac{1}{b}\left[\frac{D^{n+1}f(z)}{D^n\ g(z)} - 1\right] = \frac{1+w(z)}{1-w(z)} \quad , \ z \in E \tag{2.3}$$

for some $g \in S_n$ and where $w \in A$ such that $w(0) = 0$, $w(z) \neq 1$ and $|w(z)| < 1$ for $z \in E$. Let $g(z) = z + \sum_{m=2}^{\infty} c_m z^m$.

Then (2.3) and (2.1) imply

$$w(z)\left(2bz + \sum_{m=2}^{\infty} m^n (2bc_m + ma_m - c_m) z^m\right)$$
$$= \sum_{m=2}^{\infty} m^n (ma_m - c_m) z^m \tag{2.4}$$

Using clunie's method[3], that is to examine the bracketed quantity of the left-hand side in (2.4) and keep only those terms that z^m for $m \le k-1$ for some fixed k, moving the other terms to the right side one obtains

$$w(z)\left\{2bz+\sum_{m=2}^{k-1} m^n[\,ma_m+(2b-1)c_m\,]\,z^m\right\}$$

$$=\sum_{m=2}^{k} m^n\,(ma_m-c_m)z^m+\sum_{m=k+1}^{\infty} A_m z^m \;.$$

Let

$$\phi(z)=w(z)\left\{2bz+\sum_{m=2}^{k-1} m^n[\,ma_m+(2b-1)c_m\,]\,z^m\right\}$$

$$=\sum_{m=2}^{k} m^n\,(ma_m-c_m)z^m+\sum_{m=k+1}^{\infty} A_m z^m \tag{2.5}$$

and $z=re^{i\theta}$, $0<r<1$.

Computing $\displaystyle \frac{1}{2\pi}\int_0^{2\pi}\phi(z)\,\overline{\phi(z)}\;d\theta$

for both expression of $\phi(z)$ in (2.5) and using $|w(z)|<1$, we get

$$\sum_{m=2}^{k} m^{2n}|\,ma_m-c_m\,|^2\,r^{2m}\;.$$

$$\le 4|b|^2r^2+\sum_{m=2}^{k-1} m^{2n}\,|\,ma_m+(2b-1)c_m\,|^2\,r^{2m}$$

We let $r\longrightarrow 1^-$ and find that

$$|ka_k-c_k|^2\le\frac{1}{k^{2n}}\,4|b|\left\{|b|+\sum_{m=2}^{k-1} m^{2n}[\,|ma_m-c_m|\,|c_m|+|b|\,|c_m|^2]\right\}.$$

In particular, when $m=2$ we have

$$|2a_2-c_2|\;\le\;\frac{1}{2^{n-1}}\;|b| \tag{2.6}$$

Theorem 2.4 Let $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$. If $f \in Q_n(b)$ where $n \in N_0$, then

$$|a_m| \le \frac{1}{m^n} [(m-1)|b| + 1] \qquad (m \ge 2) .$$

This result is sharp. An extremal function is given by

$$f(z) = z + \sum_{m=2}^{\infty} \frac{1}{m^n} [(m-1)b + 1] z^m \ . \tag{2.7}$$

proof. Let $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$ be in $Q_n(b)$ and $g(z) = z + \sum_{m=2}^{\infty} c_m z^m$.

We claim that for $m \ge 2$ and $n \in N_0$,

$$|ma_m - c_m| \le \frac{1}{m^n} 2|b| \left[1 + \sum_{k=2}^{m-1} k^n |c_k| \right] . \tag{2.8}$$

We use the second principle of induction on m on (2.9).

For m=2, $|2a_2 - c_2| \le \frac{1}{2^{n-1}} |b|$ is true as shown in (2.6). Now assume that (2.8) is true for all $m \le p$. Taking $m = p + 1$ in (2.2), we get

$$|(p+1)a_{p+1} - c_{p+1}|^2 \le 4 \frac{1}{(p+1)^{2n}} |b| \left\{ |b| + \sum_{k=2}^{p} k^{2n} [|ka_k - c_k||c_k| + |b||c_k|^2] \right\}$$

$$= 4 \frac{1}{(p+1)^{2n}} |b| \left\{ |b| + \sum_{k=2}^{p} k^{2n} [|ka_k - c_k||c_k| + |b| \sum_{k=2}^{p} k^{2n} |c_k|^2 \right\} .$$

Now using (2.8), we have

$$|(p+1)a_{p+1} - c_{p+1}|^2 \le 4 \frac{1}{(p+1)^{2n}} |b|^2 \left\{ 1 + 2\sum_{k=2}^{p} k^n |c^k| \left[1 + \sum_{j=2}^{k-1} j^n |c_j| \right] + \sum_{k=2}^{p} k^{2n} |c_k|^2 \right\}$$

$$= 4\frac{1}{(p+1)^{2n}}|b|^2\left\{1 + 2\sum_{k=2}^{p} k^n|c^k| + 2\sum_{k=2}^{p} k^n\left[|c_k|\sum_{j=2}^{k-1} j^n|c_j|\right] + \sum_{k=2}^{p} k^{2n}|c_k|^2\right\}$$

$$= 4\frac{1}{(p+1)^{2n}}|b|^2\left[1 + \sum_{k=2}^{p} k^n|c_k|\right]^2 .$$

This show that (2.8) is valid for $m = p + 1$. Hence, the claim is correct. From Lemma 2.1 and (2.8) it follows that

$$|ma_m - c_m| \leq \frac{1}{m^n} 2|b| \left[1 + \sum_{k=2}^{m-1} k^n|c_k|\right]$$

$$\leq \frac{1}{m^n} m(m-1)\ |b|\ ,\quad m \geq 2 \tag{2.9}$$

Finally from Lemma 2.1 and (2.9),

$$|a_m| \leq \frac{1}{m^n}[\,(m-1)|b| + 1\,]\ ,\quad m \geq 2 \quad .$$

Putting $n = 1$ in Theorem 2.4, we have the following corollary.

Corollary 2.5 If $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$ is quasi-convex function of complex order b, then

$$|a_m| \leq \frac{1}{m}[\,(m-1)|b| + 1\,]$$

This result is sharp.

Remark 2.6 For b=1, Corollary 2.5 is reduced to coefficient bounds for the quasi-convex functions due to Noor [5].

Taking n = 0 in Theorem 2.4 ,

Corollary 2.7 If $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$ is a close-to-convex function of complex order b, then

$$|a_n| \leq (m-1)|b| + 1 .$$

This result is sharp.

This corollary may be found in [1] .

Remark 2.8 For b = 1, Corollary 2.7 is reduced to the coefficient bounds for the close-to-convex functions due to Reade [6].

Lemma 2.9 ([4]) Let $w(z)$ be regular in the unit disk E and such that $w(0)=0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r$ at a point z_0 , then we have $z_0 w'(z_0) = k\, w(z_0)$ where k is real and $k \geq 1$.

Theorem 2.10 If a function f(z) belonging to A satisfies

$$\left|\frac{D^{n+1} f(z)}{D^n g(z)} - 1\right|^{\alpha} \left|\frac{D^{n+2} f(z)}{D^n g(z)} - \frac{D^{n+1} f(z)\, D^{n+1} g(z)}{[D^n g(z)]^2}\right|^{\beta} < |b|^{\alpha+\beta} \quad (z \in E) \quad (2.10)$$

for some $\alpha \geq 0$, $\beta \geq 0$ and $g(z) \in S_n$, then $f(z) \in Q_n(b)$.

proof. Defining the function w(z) by

$$w(z) = \frac{1}{b}\left[\frac{D^{n+1} f(z)}{D^n g(z)} - 1\right] \quad (2.11)$$

for $g(z) \in S_n$. We see that w(z) is regular in E and $w(0) = 0$.
Noting that

$$bzw'(z) = \frac{D^{n+2}f(z)}{D^n g(z)} - \frac{D^{n+1}f(z)\, D^{n+1}g(z)}{(D^n g(z))^2} \quad . \tag{2.12}$$

We know that (2.10) can be written as

$$|bw(z)|^{\alpha}\, |bzw'(z)|^{\beta} \;<\; |b|^{\alpha+\beta} \quad . \tag{2.13}$$

Suppose that there exists a point $z_0 \in E$ such that

$$\underset{|z|\le|z_0|}{\mathrm{Max}}\ |w(z)| = |w(z_0)| = 1 \quad . \tag{2.14}$$

Then, Lemma 2.9 leads us to

$$|bw(z_0)|^{\alpha}\, |bz_0w'(z_0)|^{\beta} = |b|^{\alpha+\beta} k^{\beta} \ge |b|^{\alpha+\beta} \qquad (k \ge 1)$$

which contradicts our condition (2.10). Therefore, we conclude that $|w(z)| < 1$ for all $z \in E$, that is, that

$$\left| \frac{1}{b}\left[\frac{D^{n+1}f(z)}{D^n\ g(z)} - 1\right] \right| < 1 \qquad (z \in E) \ .$$

This implies that

$$\mathrm{Re}\left\{1 + \frac{1}{b}\left(\frac{D^{n+1}f(z)}{D^n\ g(z)} - 1\right)\right\} > 0 \qquad (z \in E)$$

which proves $f(z) \in Q_n(b)$.

3. Distortion Theorem

Theorem 3.1 Let $f \in Q_n(b)$, $n \in N_0$. Then for $|z| = r < 1$, and $|2b - 1| \le 1$,

$$\frac{r - |2b-1|\, r^2}{(1+r)^3} \le |D^{n+1}\ f(z)| \le \frac{r + |2b-1|\, r^2}{(1-r)^3} \quad . \tag{3.1}$$

This result is sharp for the function f(z) given by

$$D^{n+1} f(z) = \frac{z(1 + (2b - 1)z)}{(1 - z)^3} .$$

proof. Let $f \in Q_n(b)$. Then (1.6) implies for some $g \in S_n$

$$\frac{D^{n+1} f(z)}{D^n g(z)} = \frac{1 + (2b-1)w(z)}{1 - w(z)} , \quad z \in E ,$$

where $w \in A$ and $|w(z)| \leq |z|$ in E. This gives for $|z| < r = 1$

$$\frac{1 - |2b - 1|r}{1 + r} \leq \left|\frac{D^{n+1} f(z)}{D^n g(z)}\right| \leq \frac{1 + |2b - 1|r}{1 - r} . \qquad (3.2)$$

The definition of S_n implies $D^n g(z)$ is starlike . Hence by the well known bounds on functions which are starlike in E , we get for $|z| = r < 1$

$$\frac{r}{(1 + r)^2} \leq | D^n g(z) | \leq \frac{r}{(1 - r)^2} . \qquad (3.3)$$

Using (3.2) and (3.3), one can get (3.1).

Taking (i) $n = 0$, (ii) $n = 0, b = 1$, (iii) $n = 1$ and (iv) $n = 1, b = 1$ in Theorem 3.1, we have the following corollaries, respectively.

Corollary 3.2 If f is a close-to-convex function of complex order b, where $|2b - 1| \leq 1$, then for $|z| = r < 1$

$$\frac{1 - |2b - 1|r}{(1 + r)^3} \leq |f'(z)| \leq \frac{1 + |2b - 1|r}{(1 - r)^3} . \qquad (3.4)$$

Corollary 3.3 If f is a close-to-convex function, then for $|z| = r < 1$

$$\frac{1 - r}{(1 + r)^3} \leq |f'(z)| \leq \frac{1 + r}{(1 - r)^3} . \qquad (3.5)$$

Corollary 3.4 If f is a quasi-convex function of complex order b, where $|2b - 1| \leq 1$, then for $|z| = r < 1$

$$\frac{(2 + r) - |2b - 1|r}{2(1 + r)^2} \leq |f'(z)| \leq \frac{(2 - r) + |2b - 1|r}{2(1 - r)^2} . \tag{3.6}$$

proof. By n = 1, in Theorem 3.1, we have

$$\frac{1 - |2b - 1|r}{(1 + r)^3} \leq |(zf'(z))'| \leq \frac{1 + |2b - 1|r}{(1 - r)^3} . \tag{3.7}$$

Intergrating the right hand side of (3.7) from 0 to z, we obtain

$$|zf'(z)| \leq \int_0^z |(zf'(z))'|dz$$

$$\leq \int_0^r \frac{1 + |2b - 1|r}{(1 - r)^3} dr = \frac{r\{(2-r) + |2b-1|r\}}{2(1 - r)^2} . \tag{3.8}$$

In order to obtain a lower bound for $|f'(z)|$,we proceed as follows. Let d_1 be the radius of the open disk contained in the map of E by $zf'(z)$. Let z_0 be the point of $|z| = r$ for which $|zf'(z)|$assumes its minimum value. This minimum increases with (r the image of $|z| = r$ by $w = zf'(z)$ expands) and is less than d_1. Hence the line segment connecting the origin with the point $z_0f'(z_0)$ will be covered entirely by the values of $zf'(z)$ in E. Let l be the arc in E which is mapped by $w = zf'(z)$ onto this line segment. Then

$$|zf'(z)| = \int_l |(zf'(z))'||dz|$$

$$\geq \int_0^r \frac{1-|2b - 1|r}{(1 - r)^3} dr = \frac{r\{(2+r) + |2b-1|r\}}{2(1 + r)^2} . \tag{3.9}$$

Using (3.8) and (3.9), one can get (3.6).

Corollary 3.5 If f is a quasi-convex function, then for $|z| = r < 1$

$$\frac{1}{(1 + r)^2} \leq |f'(z)| \leq \frac{1}{(1 - r)^2} .$$

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