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Extension problems for spinors on S^4

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S^4上のスピノールに対する延長問題

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1. The space of spinors on S^3

Here we shall explain the complex analytic point of view of Dirac operator on S^4 and discuss the eigenvalues of Hamiltonian acting on spinors on the equator$\simeq S^3$. These were obtained in [K].

a. Let us consider two copies of complex planes C_z^2 and $\widehat{C}_w^2$ and a smooth bijection $v: C_z^2 \setminus \{0\} \to \widehat{C}_w^2 \setminus \{0\}$ given by $w = v(z) = -\frac{\overline{z}}{|z|^2}$. We patch C_z^2 and C_w^2 by v to obtain a differentiable manifold $M = C^2 \coprod_v \widehat{C}^2$, which is homeomorphic to S^4.

We endow M with a riemannian metric defined by

$$g = \begin{cases} (1+|z|^2)^{-2} \sum_{i=1}^2 dz_i \otimes d\overline{z}_i & \text{on } C_z^2 \\ (1+|w|^2)^{-2} \sum_{i=1}^2 dw_i \otimes d\overline{w}_i & \text{on } \widehat{C}_w^2 \,. \end{cases}$$

The Levi-Civita connection on M is given by gauge potentials

$$\Gamma(z) = \frac{|z|^2}{1+|z|^2}\,\sigma(z)^{-1}\cdot(d\sigma)_z \quad \text{for } z \in C_z^2$$

$$\widehat{\Gamma}(w) = \frac{|w|^2}{1+|w|^2}\,\sigma(w)^{-1}\cdot(d\sigma)_w \quad \text{for } w \in \widehat{C}_w^2,$$

where $\sigma(z) = |z|^2(v_*)_z$, v_* being the differential of v , and $\sigma(z)^{-1}(d\sigma)_z$ is a one-form valued in $\mathcal{G} = \{X \in gl(4,\mathrm{C}) : {}^tXK + KX = 0\} \simeq o(4,\mathrm{C})$, $K = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}$.

On M there are a unique spin-structure $Spin(M)$ and the associated spinor bundle $S = Spin(M) \times_{Spin(4)} \Delta$. Δ is a basic representation space of $Spin(4)$ which is the direct sum of two irreducible representations of Δ^+ and Δ^- each of dimension 2. Let S^+ and S^- be the corresponding bundles whose cross sections are spinors of positive (respectively negative) chirality . We shall choose a frame of $S^\pm$ and denote the spinors in matrix form

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix} \in \Gamma(S), \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in \Gamma(S^+), \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in \Gamma(S^-),$$

where Γ signifies the sections of a bundle. The inner product of two spinors $\phi, \varphi \in \Gamma(S^\pm)$ is defined by $< \phi(z), \varphi(z) > = \phi_1(z)\overline{\varphi}_1(z) + \phi_2(z)\overline{\varphi}_2(z)$.

b The Dirac operator acting on the spinors is defined as the composition $\mathcal{D} = \mu \cdot \nabla$ where ∇ is the covariant derivative induced by the Levi-Civita connection and μ is Clifford multiplication . The Dirac operator switches S^+ and S^- and is of the form $\mathcal{D} = \begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix}$ where $D : \Gamma(S^+) \to \Gamma(S^-)$.

We gave in [K] the following matrix representation of the Dirac operator.

$$D = \begin{pmatrix} (1+|z|^2)\frac{\partial}{\partial z_1} - \frac{3}{2}\overline{z}_1 & -(1+|z|^2)\frac{\partial}{\partial \overline{z}_2} + \frac{3}{2}z_2 \\ (1+|z|^2)\frac{\partial}{\partial z_2} - \frac{3}{2}\overline{z}_2 & (1+|z|^2)\frac{\partial}{\partial \overline{z}_1} - \frac{3}{2}z_1 \end{pmatrix}$$

$$D^\dagger = \begin{pmatrix} (1+|z|^2)\frac{\partial}{\partial \overline{z}_1} - \frac{3}{2}z_1 & (1+|z|^2)\frac{\partial}{\partial \overline{z}_2} - \frac{3}{2}z_2 \\ -(1+|z|^2)\frac{\partial}{\partial z_2} + \frac{3}{2}\overline{z}_2 & (1+|z|^2)\frac{\partial}{\partial z_1} - \frac{3}{2}\overline{z}_1 \end{pmatrix}$$

We have a decomposition of D and $D^\dagger$ to their longitudinal parts and radial parts;

$$D = \gamma_0\,(\mathbf{n} - \mathcal{P}) \qquad D^\dagger = (\mathbf{n} + \mathcal{P})\gamma_0.$$

Here γ_0 signifies Clifford multiplication of the radial vector $\mathbf{n}$. We shall explain $\mathcal{P}$ soon after. First we introduce an orthonormal frame on M, but here we shall write down it only on the local coordinate $\mathrm{C}^2 \subset M$, the formulas on $\widehat{\mathbb{C}}^2 \subset M$ are easily obtained by the transition relation . This frame is important not only as it gives a neat expression of Dirac operators on M and on the equator $\simeq S^3$ but also as is associated to the Lie group structure of $S^3 \simeq SU(2)$ (see **c**). Let

$$\nu = \frac{1+|z|^2}{|z|}(z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2}) \qquad \epsilon = \frac{1+|z|^2}{|z|}(-\overline{z}_2\frac{\partial}{\partial z_1} + \overline{z}_1\frac{\partial}{\partial z_2})$$

The radial vector field is given by

$$\mathbf{n} = \frac{1}{2}(\nu + \overline{\nu}).$$

Put

$$\theta_0 = \tfrac{1}{2\sqrt{-1}}(\nu - \overline{\nu}) \quad \theta_1 = \tfrac{1}{2}(\epsilon + \overline{\epsilon}) \quad \theta_2 = \tfrac{1}{2\sqrt{-1}}(\epsilon - \overline{\epsilon}).$$

Then $\sqrt{2}\mathbf{n}$, $\sqrt{2}\theta_0$, $\sqrt{2}\theta_1$, $\sqrt{2}\theta_2$ form an orthonormal frame on M and θ_0, θ_1, θ_2 are tangent to the constant altitude $\{|z| = const\}$.

$\mathcal{P} : S^+ \to S^+$ is given by $\mathcal{P} = -(\gamma_0|S^-)\sum_{i=o}^{2}\theta_i\nabla_{\theta_i}$ with γ_0 coming from Clifford multiplication of $\mathbf{n}$.

The matrix representation of $\mathcal{P}$ is written as

$$\mathcal{P} = \begin{pmatrix} -\sqrt{-1}\theta_0 & \overline{\epsilon} \\ -\epsilon & \sqrt{-1}\theta_0 \end{pmatrix} + \frac{3}{2}|z|\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

Let $E = \{|z| = 1\}$ be the equator of M; $E \simeq S^3$. E is endowed with the riemannian metric $g|E$. Since $Spin(3)$ has the spinor representation on $\Delta^\pm$ the restrictions on E of bundle $S^\pm$ is the spinor bundle corresponding to the spin structure $Spin(E)$. γ_0 gives the isomorphism between $S^\pm$. The Dirac operator on E acting on spinors of positive chirality is given by $-\gamma_0\mathcal{P}|E$. The restriction $\mathcal{P}$ on E is called *Hamiltonian* on E .

c Here we shall discuss a little about infinitesimal representations of $SU(2)$ given by the vector fields $\sqrt{-1}\theta_i$, $i = 0, 1, 2$. First we note the commutation relations same as those of $sl(2)$;

$$[\sqrt{-1}\theta_0, \epsilon] = -2\epsilon, \quad [\sqrt{-1}\theta_0, \overline{\epsilon}] = 2\overline{\epsilon}, \quad [\epsilon, \overline{\epsilon}] = 4\sqrt{-1}\theta_0.$$

We now follow the isomorphism $B \simeq S^3 \simeq SU(2)$ and look the point $z \in B$ as $\ddot{z} = \begin{pmatrix} z_1 & -\overline{z}_2 \\ z_2 & \overline{z}_1 \end{pmatrix} \in SU(2)$. We shall then define the right action on E

of $g \in SU(2)$ by $z \cdot g =$ the first column of $\ddot{z} \cdot g$. Put $R_g f(z) = f(z \cdot g)$ for a continupus function f on E. Then the differentials become $dR(e_k) = -\theta_k, \quad k = 0, 1, 2$, where

$$e_0 = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}$$

are the basis of $su(2)$.

A polynomials that satisfies

$$P(az_1, bz_2, b\overline{z}_1 . a\overline{z}_2) = a^k b^l P(z_1, z_2, \overline{z}_1, \overline{z}_2)$$

is called of class (k, l). The set of polynomials of class (k, l) is denoted by $S_{k,l}$. Let $\mathcal{H}$ be the set of harmonic polynomials on C^2 and put $\mathcal{H}_{k.l} = \mathcal{H} \cap S_{k,l}$. We have $S_{k,l} = \mathcal{H}_{k,l} \oplus |z|^2 S_{k-1,l-1}$, hence $\dim \mathcal{H}_{k,l} = k + l + 1$. It foollws also that , on E, every polynomial is a sum of harmonic polynomials in $\mathcal{H}_{k,l}$'s. This ensures the fact that our family of eigenspinors of Hamiltonian on E obtained later is a complete system.

Put, for $r \geq 0$ and $0 \leq k, q \leq r$,

$$h^q_{k,r-k}(z) = \epsilon^q (z_1^k z_2^{r-k}).$$

For each pair r and $k \leq r$ the set $\{h^q_{k,r-k};\ q = 0, \cdots, r\}$ forms a basis of $\mathcal{H}_{k,r-k}$.

Proposition.

(1) $\sqrt{-1}\theta_0\, h^q_{k,r-k} = (r - 2q) h^q_{k,r-k}$

(2) $\epsilon\, h^q_{k,r-k} = h^{q+1}_{k,r-k}$

(3) $\overline{\epsilon}\, h^q_{k,r-k} = -4q(r - q + 1) h^{q-1}_{k,r-k}$

Hence the space of harmonic polynomials $\mathcal{H}$ (restricted on B) is decomposed by the right action R of $SU(2)$ into $\mathcal{H} = \sum_r \sum_{k=0}^r \mathcal{H}_{k,r-k}$. Each induced representation $R_{k,r-k} = (R,\ \mathcal{H}_{k,r-k})$ is an irrreducible representation with the highest weight $\frac{r}{2}$.

d Put, for $r \leq 0$, $0 \leq k \leq r$, and $0 \leq q \leq r + 1$,

$$\phi^q_{k,r-k} = \begin{pmatrix} q 2^{-q+1} h^{q-1}_{k,r-k} \\ -2^{-q} h^q_{k,r-k} \end{pmatrix}.$$

Then we have from the matrix representation of the Hamiltonian and the Proposition in **c**;

$$\mathcal{P}\phi^q_{k,r-k} = (r + \frac{3}{2})\phi^q_{k,r-k}.$$

Thus the positive eigenvalues and eigenfunctions of $\mathcal{P}$ are obtained . In particular the multiplicity of the eigenvalue r is $(r+1)(r+2)$.

The investigation of negative eigenspinors is related to the left action of $SU(2)$ on the harmonic polynomials. The left action of a $g \in SU(2)$ on E is defined by $g \cdot z =$ the first column of $g \cdot \ddot{z}$. Let $L_g f(z) = f(g^{-1} \cdot z)$ for a continuous function on E.

We introduce the following vector fields on $M - \{0, \hat{0}\}$, that have the following local expressions on $\mathbb{C}^2 - \{0\}$:

$$\mu = \frac{1+|z|^2}{|z|}(z_2\frac{\partial}{\partial z_2} + \overline{z}_1\frac{\partial}{\partial \overline{z}_1}), \qquad \delta = \frac{1+|z|^2}{|z|}(\overline{z}_2\frac{\partial}{\partial \overline{z}_1} - z_1\frac{\partial}{\partial z_2}).$$

$$\tau_0 = \frac{1}{2\sqrt{-1}}(\mu - \overline{\mu}), \quad \tau_1 = \frac{1}{2}(\delta + \overline{\delta}), \quad \tau_2 = \frac{1}{2\sqrt{-1}}(\delta - \overline{\delta}).$$

We have $dL(e_i) = -\tau_i|E; \quad i = 0, 1, 2.$

Let

$$\hat{h}^{r-k,k}_q(z) = \delta^q(\overline{z}_1^k z_2^{r-k}).$$

$\{\hat{h}^{l,k}_q;\ q = 0, \cdots, r\}$ give a basis of $\widehat{\mathcal{H}}^{l,k}$: the space of harmonic polynomials that satisfy the condition $P(az_1, az_2, b\overline{z}_1.b\overline{z}_2) = a^l b^k P(z_1, z_2, \overline{z}_1, \overline{z}_2)$. Put, for $r \geq 0$, $0 \leq k \leq r$, and $0 \leq q \leq r+1$,

$$\pi^{r-k,k}_q = \begin{pmatrix} 2^{-q}\widehat{h}^{r-k+1,k}_q \\ 2^{-q}\widehat{h}^{r-k,k+1}_q \end{pmatrix}.$$

By an easy calculus we have

$$\mathcal{P}\pi^{r-k,k}_q = -(r + \frac{3}{2})\pi^{r-k,k}_q.$$

Thus we have

Theorem 1 [K]. *The eigenvalues of* $\mathcal{P}$ *are* $\pm\left(\frac{3}{2}+r\right)$; $r=0,1,2,\cdots$ *with multiplicity* $(r+1)(r+2)$, *in particular, there is no zero mode spinor of* $\mathcal{P}$ *and the spectrum are symmetric relative to* 0.

e Here we note corresponding subjects on the other coordinate neighborhood $\widehat{\mathrm{C}}^2_w$. The transition function to describe the bundle $Spin(M)$ is $-{}^t(\gamma_0) = -\overline{\gamma}_0$ and a spinor on M is a pair of $\varphi(z) \in \Gamma(\mathrm{C}^2_z \times \Delta)$ and $\widehat{\varphi}(w) \in \Gamma(\widehat{\mathrm{C}}^2 \times \Delta)$ that are patched by $\widehat{\varphi}(v(z)) = -\overline{(\gamma_0\varphi)(z)})$. The matrix representations of the Dirac operator on $\widehat{\mathrm{C}}^2 \subset M$ has the same form as those in (1-5) but the first and the second are changed since a section on $\widehat{\mathrm{C}}^2$ of the bundle S^+ (resp. S^-) is valued in Δ^- (resp. Δ^+). This is *"CPT"-theorem* . The counterpart of $\mathcal{P}$ is defined as $\mathcal{P} = (\gamma_0|S^+)\sum\theta_1\nabla_{\theta_i}$ acting on $\hat{\varphi} \in \Gamma(\widehat{\mathrm{C}}^2_w \times \Delta^-)$. For a $\varphi \in \Gamma(\mathrm{C}^2_z \times \Delta^+)$, we have $D\widehat{\varphi} = \widehat{D\varphi}$ and $\widehat{\mathcal{P}\varphi} = \mathcal{P}\widehat{\varphi}$.

2 Extension of spinors from the equator

a Let H be the space of square integrable spinors of positive chirality on E. Let $H_\pm$ be the closed subspace of H spanned by the eigenvectors ϕ_λ corresponding to the positive (resp. negative) eigenvalues λ of $\mathcal{P}$.

Put $c(r,q,k) = \left(\frac{q!k!(r-k)!}{(r+1-q)!}\right)^{-\frac{1}{2}}$. Then a complete orthonormal system of eigenspinors of $\mathcal{P}$ is given by

$$\left\{ c(r,q,k)\phi^q_{k,r-k},\ c(r,q,k)\pi^{r-k,k}_q;\ r \geq 0,\ 0 \leq k \leq r,\ 0 \leq q \leq r+1 \right\}.$$

Take an eigenspinor φ_λ and extend it by $\Phi_\lambda(z) = r_\lambda(|z|)\varphi_\lambda(\frac{z}{|z|})$ to C^2,where $r_\lambda(t) = t^{\lambda-\frac{3}{2}}(\frac{1+t^2}{2})^{\frac{3}{2}}$. Then $\Phi_\lambda(z)$ is a zero-mode spinor of D on C^2. This is proved by the following calculus:

$$\begin{aligned} D\Phi_\lambda(z) &= \gamma_0(\mathbf{n}-\mathcal{P})(\Phi_\lambda(z)) \\ &= \gamma_0\left((1+|z|^2)r'_\lambda(|z|) - (\lambda-\frac{3}{2})\frac{1+|z|^2}{|z|}r_\lambda(|z|) - 3|z|r_\lambda(|z|)\right)\varphi_\lambda(\frac{z}{|z|}). \end{aligned}$$

But $r_\lambda(t)$ satisfies the equation

$$(1+t^2)r'_\lambda(t) - (\lambda-\frac{3}{2})\frac{1+t^2}{t}r_\lambda(t) - 3tr_\lambda(t) = 0.$$

Therefore $D\Phi_\lambda = 0$.

Let $\mathcal{N}(U)$ (resp. $\mathcal{N}^\dagger(U)$) be the space of zero-mode spinors of Dirac operator D (resp. $D^\dagger$) on U that have L^2−boundary values .

Theorem 2 [K]. *Let $R = \{z \in \mathrm{C}^2;\ |z| < 1\}$ and $\hat{R} = \{w \in \widehat{\mathrm{C}}^2;, |w| < 1\}$.*

(1) *H_+ is isomorphic to $\mathcal{N}(R)$,*
(2) *H_- is isomorphic to $\mathcal{N}(\hat{R})$,*
(3) *Every spinor in H is equal to the difference of the restrictions of zero mode spinors on R and on $\hat{R}$.*

Proof: Let $\varphi \in H_+$ and expand it in $\varphi = \sum_{\lambda>0} a_\lambda \phi_\lambda$. The spinor on R ; $\Phi(z) = \sum_{\lambda>0} a_\lambda \Phi_\lambda(z)$ is well defined. In fact , consider the finite sum; $\Phi^n_m = \sum_{\lambda=m+\frac{3}{2}}^{n+\frac{3}{2}} a_\lambda \Phi_\lambda$. Then $< \Phi^n_m, \Phi^n_m > (z)$ is subharmonic on R and is dominated by some constant multiple of its L^2−norm on E, hence converges there to 0 compact uniformly as m, n tend to infinity. If we note the fact that each component of Φ is harmonic we see that it has L^2−boundary value . Conversely let $\Phi \in \mathcal{N}(R)$ and let φ be its restriction to E. We can show that the eigenfunction expansion of φ by $\{\phi_\lambda\}$ can not contain the term with $\lambda < 0$ and $\varphi \in H_+$. As for (2) consider the function $r_{-\mu}(t) = t^{\mu-\frac{3}{2}}(\frac{1+t^2}{2})^{\frac{3}{2}}, \quad t \geq 0$, where $-\mu = -r - \frac{3}{2}$, $r = 0, 1, \cdots$ and do the same argument as in (1) . Relations in **e** transform the result to that on $\hat{R}$.

b Let H^* be the space of square integrable spinors of negative chirality on E. γ_0 switches H and H^*; $(\gamma_0|S^+)H = H^*$, $(\gamma_0|S^-)H^* = H$. We shall define $H^*_+ = (\gamma_0|S^+)H_+$ and $H^*_- = (\gamma_0|S^+)H_-$.

Let $\psi^* \in H^*_-$ and suppose that $\psi = (\gamma_0|S^-)\psi^*$ is an eigenspinor belonging to a negative eigenvalue $\lambda = -(r + \frac{3}{2})$. Let $\Psi(z) = s_\lambda(|z|)\psi(\frac{z}{|z|})$, where $s_\lambda(t) = t^{-(\lambda-\frac{3}{2})}(\frac{2}{1+t^2})^{\frac{3}{2}}$. Then as before we can verify that $\Psi(z)$ extend ψ to C^2 , $\Psi(0) = 0$ and $D^\dagger\psi^* = (\mathbf{n} + \mathcal{P})\gamma_0\psi^* = (\mathbf{n} + \mathcal{P})\psi = 0$.

Thus in the same manner as in Theorem 2 we have the following;

Theorem 3.

(1) *H^*_- is isomorphic to $\{\phi \in \mathcal{N}^\dagger(R);\ \phi(0) = 0\}$,*
(2) *H^*_+ is isomorphic to $\{\psi \in \mathcal{N}^\dagger(\hat{R});\ \psi(\hat{0}) = 0\}$,*
(3) *Every spinor in H^* is equal to the difference of the restrictions of zero mode spinors on R and on $\hat{R}$.*

c From the definition $< \phi, \psi >= 0$ for all $\phi \in H$ and $\psi \in H^*$.
Let ϕ and ψ be spinors on $R = \{|z| \leq 1\}$, Stokes' theorem statts;

$$\int_R \frac{1}{(1+|z|^2)^4} \left(< D\phi, \psi > + < \phi, D^\dagger\psi >\right) dV = \frac{1}{8}\int_E < \phi, \gamma_0\psi > d\sigma.$$

Theorems 2,3 and Stokes' theorem yield immediately that

$$\int_E < \phi, \gamma_0 \psi > d\sigma = 0 \quad \text{for } \phi \in H_+,\ \psi \in H_-^*.$$

Similarly

$$\int_E < \phi, \gamma_0 \psi > d\sigma = 0 \quad \text{for } \phi \in H_-,\ \psi \in H_+^*.$$

The coupling betweeen $H_\pm^*$ and $H_\pm$ does not vanish and is important to construct the geometric model of conformal field theory on S^4 which will be treated in the next paper.

d Actually eigenspinors $\phi_\lambda\,;\lambda > 0$ are extended to $\mathcal{N}(\mathrm{C}^2)$ and those for $\lambda < 0$ are extended to $\mathcal{N}(\widehat{\mathrm{C}}^2)$. We list here a table of expansion formula for ϕ_λ, $\phi_\lambda^* = \gamma_0\phi_\lambda$ for $\lambda > 0$ and π_λ $\pi_\lambda^* = \gamma_0\phi_\lambda$ for $\lambda < 0$.

(1) $\Phi_\lambda(z) = |z|^{\lambda-\frac{3}{2}}(\frac{1+|z|^2}{2})^{\frac{3}{2}}\phi_\lambda(\frac{z}{|z|}) \in \mathcal{N}(\mathrm{C}^2)$, $\lambda > 0$ and $\Phi_\lambda(z) = \phi_\lambda(z)$ for $|z| = 1$.

(2) $\widehat{\Phi_\lambda^*}(w) = |w|^{\lambda+\frac{3}{2}}(\frac{2}{1+|w|^2})^{\frac{3}{2}}\widehat{\phi_\lambda^*}(\frac{w}{|w|}) \in \mathcal{N}^\dagger(\widehat{\mathrm{C}}^2)_0$, $\lambda > 0$ and $\widehat{\Phi_\lambda^*}(-\bar{z}) = -\overline{\gamma_0\phi_\lambda^*}(z)$ for $|z| = 1$.

(3) $\widehat{\Pi}_\lambda(w) = |w|^{-\lambda-\frac{3}{2}}(\frac{1+|w|^2}{2})^{\frac{3}{2}}\widehat{\pi}_\lambda(\frac{w}{|w|}) \in \mathcal{N}(\widehat{\mathrm{C}}^2)$, $\lambda < 0$ and $\widehat{\Pi}_\lambda(-\bar{z}) = -\overline{\gamma_0\pi_\lambda}(z)$ for $|z| = 1$.

(4) $\Pi_\lambda^*(z) = |z|^{-\lambda+\frac{3}{2}}(\frac{2}{1+|z|^2})^{\frac{3}{2}}\pi_\lambda^*(\frac{z}{|z|}) \in \mathcal{N}^\dagger(\mathrm{C}^2)_0$, $\lambda < 0$ and $\Pi_\lambda^*(z) = \pi_\lambda^*(z)$ for $|z| = 1$.

References

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