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Global Existence and Asymptotic Behavior of Solutions for the Zakharov Equations

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In the present paper we consider the global existence and asymptotic behavior in time of solutions for the Zakharov equations:

$$i\frac{\partial}{\partial t}E + \Delta E = nE, \quad t > 0, \quad x \in \mathbf{R}^N, \tag{1}$$

$$\frac{\partial^2}{\partial t^2}n - \Delta n = \Delta|E|^2, \quad t > 0, \quad x \in \mathbf{R}^N, \tag{2}$$

$$E(0,x) = E_0(x), \quad n(0,x) = n_0(x), \quad \frac{\partial}{\partial t}n(0,x) = n_1(x), \quad x \in \mathbf{R}^N, \tag{3}$$

where E is a function from $\mathbf{R}^+ \times \mathbf{R}^N$ to $\mathbf{C}^N$, n is a function from $\mathbf{R}^+ \times \mathbf{R}^N$ to $\mathbf{R}$ and $1 \le N \le 3$. Equations (1)-(2) describe the propagation of Langmuir turbulence in an unmagnetized, completely ionized hydrogen plasma (see Zakharov [20]). $E(t,x)$ is the slowly varying complex amplitude of the electric field $\mathcal{E}$ of the Langmuir wave with plasma frequency $\omega_p > 0$:

$$\mathcal{E}(t,x) = \Re(E(t,x)\exp(-it\omega_p)).$$

$n(t,x)$ is the deviation of the ion density from its equilibrium. The right hand side of (1) represents the shift of plasmon frequency caused by the slow density variation $n(t,x)$, and the right hand side of (2) represents the driving force caused by the pressure of plasmon gas.

There are many papers concerning the global existence in time of solutions for (1)-(3). In [17] C. Sulem and P.L. Sulem proved by using the Glerkin method that if $N = 1$ and $(E_0, n_0, n_1) \in H^m \oplus H^{m-1} \oplus (H^{m-2} \cap \dot{H}^{-1}), m \geq 3$, then (1)-(3) have the unique global solutions $(E, n) \in L^\infty(0,\infty; H^m) \oplus L^\infty(0,\infty; H^{m-1})$ and that if $(E_0, n_0, n_1) \in H^1 \oplus L^2 \oplus \dot{H}^{-1}$ and the L^2 norm and the H^1 norm of E_0 are small for $N = 2$ and $N = 3$, respectively, then (1)-(3) have the global weak solutions $(E, n) \in L^\infty(0,\infty; H^1) \oplus L^\infty(0,\infty; L^2)$ for $1 \leq N \leq 3$. But the uniqueness of weak solutions in $H^1 \oplus L^2$ is not yet known. Here H^m denotes the standard Sobolev space $H^m(\mathbf{R}^N)$. $\dot{H}^m$ denotes the homogeneous Sobolev space consisting of all tempered distributions u with $|\xi|^m \hat{u} \in L^2 \equiv L^2(\mathbf{R}^N)$, where $\hat{u}$ is the Fourier transform of u. In [15] Schochet and Weinstein showed a similar result by the different method. When $N = 2$, H. Added and S. Added [1] improved the global existence results due to C. Sulem and P.L. Sulem [17] and showed that if $N = 2$, $(E_0, n_0, n_1) \in H^m \oplus H^{m-1} \oplus (H^{m-2} \cap \dot{H}^{-1})$, $m \geq 3$ and the L^2 norm of E_0 is small, then (1)-(3) have the unique global solutions $(E, n) \in H^m \oplus H^{m-1}$. Recently in [11] the authors have improved the local existence and regularity results due to C. Sulem and P.L. Sulem [17] and Schochet and Weinstein [15] and have brought down the lower bound $m = 3$ of regularity of the solutions to $m = 2$ concerning the unique local existence of solutions for (1)-(3). However, the uniqueness of the weak solutions in $H^1 \oplus L^2$ for $1 \leq N \leq 3$ and the global existence of strong solutions for $N = 3$ are still open. On the other hand, it is conjectured that for $N = 2, 3$, there exist solutions blowing up in finite time (see [20]). In the present paper, for $N = 3$ we consider solving (1)-(2) with the final data given at $t = +\infty$ instead of the initial value problem (1)-(3). This leads to the construction of the wave operator.

The difficulty of constructing the global solutions of (1)-(2) consists in the quadratic nonlinearity of (1)-(2). In [9] Klainerman introduced the notion of the null condition to show the global existence of small amplitude solutions for the wave equation with quadratic nonlinearity in three space dimensions. Recently, Bachelot [3] and Georgiev [6] have improved the null condition technique to show the global existence of small amplitude solutions for the Dirac-Klein-Gordon equations and the Maxwell-Dirac equations, respectively. However, the null condition technique does not seem to be directly applicable to (1)-(3), because the null condition thechnique is based on the Lorentz invariance of the equations. But the Schrödinger equation does not necessarily have the same invariance as the wave equation, and especially the Schrödinger equation is not invariant under the Lorentz transformation.

On the other hand, in [5] Flato, Simon and Taflin study the global existence and asymptotic behavior of solutions for the Maxwell-Dirac equations with the final data given at $t = +\infty$. This corresponds to the construction of the wave operator, more precisely, the modified wave operator.

In our problem, the wave operator W_+ is difined as follows. For the free solutions $(E_+(t), n_+(t))$ of the Schrödinger and wave equations, we find the solutions $(E(t), n(t))$ of (1)-(2) such that (E, n) exist on $[0, +\infty)$ and satisfy

$$(4) \qquad \|E(t) - E_+(t)\|_{L^2} + \|\nabla n(t) - \nabla n_+(t)\|_{L^2}$$
$$+\|\frac{\partial}{\partial t}n(t) - \frac{\partial}{\partial t}n_+(t)\|_{L^2} \longrightarrow 0 \quad (t \to +\infty).$$

Then, we consider the wave operator W_+ as a mapping from the scattered states $(E_+(0), n_+(0), \frac{\partial}{\partial t}n_+(0))$ to the interacting states $(E(0), n(0), \frac{\partial}{\partial t}n(0))$. However, all the solutions of (1)-(2) may not exist globally in time and so we also consider the pseudo wave operator with the reference time $t = 0$ of the interacting states replaced by some $T > 0$. That is, the pseudo wave operator $\widetilde{W}_+$ is defined as a mapping from the scattered states $(E_+(0), n_+(0), \frac{\partial}{\partial t}n_+(0))$ to the interacting states $(E(T), n(T), \frac{\partial}{\partial t}n(T))$ for some $T > 0$, where (E, n) exist on $[T, +\infty)$ and satisfy (4), and T may change for each scattered states. In the present paper, we announce that when $N = 3$, we can prove the existence of the wave operator W_+ of (1)-(2) for small scattered data and the existence of the pseudo wave operator $\widetilde{W}_+$ of (1)-(2) for the scattered data (not necessarily small) with the support of $\widehat{E}_+(0)$ included in the unit ball centered at the origin. Accordingly, it can be also shown that for the initial data belonging to the range of W_+ or the range of $\widetilde{W}_+$, (1)-(3) have the unique global solutions.

Before we state the theorems we define several notations. Le $\omega = \sqrt{-\Delta}$ and let $U(t) = e^{\frac{i}{2}t\Delta}$ be the evolution operator of the free Schrödinger equation. For nonnegative integers m and s, we define $H^{m,s}$ as follows:

$$H^{m,s} = \{v \in \mathcal{S}'(\mathbf{R}^N); \|(1+|x|^2)^{\frac{s}{2}}(1-\Delta)^{\frac{m}{2}}v\|_{L^2} < +\infty\}$$

with the norms

$$\|v\|_{H^{m,s}} = \|(1+|x|^2)^{\frac{s}{2}}(1-\Delta)^{\frac{m}{2}}v\|_{L^2}.$$

For a multi-index $\alpha = (\alpha_1, \cdots, \alpha_N)$ with nonnegative integers α_j, we put

$$|\alpha| = \alpha_1 + \cdots + \alpha_N,$$

$$(\frac{\partial}{\partial x})^\alpha = (\frac{\partial}{\partial x_1})^{\alpha_1} \cdots (\frac{\partial}{\partial x_1})^{\alpha_N}.$$

For $p \geq 1$ and a nonnegative integer k, we let

$$W^{k,p} = \{u \in L^p(\mathbf{R}^N); (\frac{\partial}{\partial x})^\alpha u \in L^p(\mathbf{R}^N), \quad |\alpha| \leq k\}$$

with the norm

$$\|u\|_{W^{k,p}} = \sum_{|\alpha| \leq k} \|(\frac{\partial}{\partial x})^\alpha u\|_{L^p}.$$

We now state the theorems.

THEOREM 1. *Let $N = 3$. Assume that $E_{+0} \in H^{6,7}$. We put $E_+(t) = U(t)E_{+0}$. Assume that $n_{+0} \in H^{5,2}$ and $n_{+1} \in H^{4,2}$. We put $n_+(t) = (\cos \omega t)n_{+0} + (\omega^{-1} \sin \omega t)n_{+1}$. Then, there exists $\eta > 0$ such that if*

(5) $$\|E_{+0}\|_{H^{6,7}} + \|n_{+0}\|_{H^{5,2}} + \|n_{+1}\|_{H^{4,2}} \leq \eta,$$

(1)-(2) have the unique solutions $(E(t), n(t))$ satisfying

(6) $$E(t) \in \bigcap_{j=0}^{1} C^j([0,\infty); H^{3-2j}),$$

(7) $$n(t) \in \bigcap_{j=0}^{2} C^j([0,\infty); H^{2-j}), \quad \frac{\partial}{\partial t} n(t) \in C([0,\infty); \dot{H}^{-1}),$$

(8) $$\|E(t) - E_+(t)\|_{H^3} + \|n(t) - n_+(t)\|_{H^2}$$
$$+ \|\frac{\partial}{\partial t} n(t) - \frac{\partial}{\partial t} n_+(t)\|_{H^1 \cap \dot{H}^{-1}} = O(t^{-1/2}) \quad (t \to +\infty),$$

(9) $$(\int_t^{+\infty} \|E(s) - E_+(s)\|_{W^{2,4}}^{8/3} ds)^{3/8}$$

$$+\left(\int_t^{+\infty}\|n(s)-n_+(s)\|_{W^{1,4}}^{8/3}ds\right)^{3/8}=O(t^{-1/2})\quad(t\to+\infty),$$

Remark 1. (i) For $N=3$, $H^{1,1}\subset\dot{H}^{-1}$. Therefor, n_{+0} and n_{+1} belong to $\dot{H}^{-1}$ under the assumptions in Theorem 1.

(ii) Theorem 1 implies that for any scattered data satisfying the assumptions in Theorem 1, there exist the solutions of (1)-(2) approaching asymptotically the free solutions prescribed by those scattered data as $t\to+\infty$. This is quite different from the case of the Maxwell-Schrödinger equations, although (1)-(2) are the system of the Schrödinger equation and the wave equation with quadratic nonlinear coupling like the Maxwell-Schrödinger system. In the case of the Maxwell-Schrödinger equations, the solutions do not approach the free solutions in the sense of (8) (see [18]).

THEOREM 2. *Let $N=3$, $\delta>0$ and $0<\varepsilon<\frac{1}{41}$. Assume that $E_{+0}\in H^{6,7}$ and $\operatorname{supp}\widehat{E}_{+0}\subset\{\xi;|\xi|\leq 1-\delta\}$. We put $E_+(t)=U(t)E_{+0}$. Assume that $n_{+0}\in H^{5,4}$ and $n_{+1}\in H^{4,4}$. We put $n_+(t)=(\cos\omega t)n_{+0}+(\omega^{-1}\sin\omega t)n_{+1}$. Then, there exists a $T>0$ such that (1)-(2) have the unique solutions $(E(t),n(t))$ satisfying*

$$E(t)\in\bigcap_{j=0}^{1}C^j([T,\infty);H^{3-2j}),\tag{10}$$

$$n(t)\in\bigcap_{j=0}^{2}C^j([T,\infty);H^{2-j}),\quad\frac{\partial}{\partial t}n(t)\in C([T,\infty);\dot{H}^{-1}),\tag{11}$$

$$\|E(t)-E_+(t)\|_{L^2}=O(t^{-2+2\varepsilon})\quad(t\to+\infty),\tag{12}$$

$$\|E(t)-E_+(t)\|_{H^1}+\|n(t)-n_+(t)\|_{L^2}+\|\frac{\partial}{\partial t}n(t)-\frac{\partial}{\partial t}n_+(t)\|_{\dot{H}^{-1}}=O(t^{-3/2+\varepsilon})\quad(t\to+\infty),\tag{13}$$

$$\|E(t)-E_+(t)\|_{H^2}+\|n(t)-n_+(t)\|_{H^1}+\|\frac{\partial}{\partial t}n(t)-\frac{\partial}{\partial t}n_+(t)\|_{L^2}=O(t^{-2+3\varepsilon})\quad(t\to+\infty),\tag{14}$$

$$\|E(t)-E_+(t)\|_{H^3}+\|n(t)-n_+(t)\|_{H^2}+\|\frac{\partial}{\partial t}n(t)-\frac{\partial}{\partial t}n_+(t)\|_{H^1}=O(t^{-2+4\varepsilon})\quad(t\to+\infty),\tag{15}$$

where T depends only on δ, ε, $\|E_{+0}\|_{H^{6,7}}$, $\|n_{+0}\|_{H^{5,4}}$ and $\|n_{+1}\|_{H^{4,4}}$.

Remark 2. (i) The support condition on the Fourier transform of E_{+0} is indispensable to the proof of Theorem 2. However, we can replace this support condition by the following:

$$\mathrm{supp}\hat{E}_{+0} \subset \{\xi; |\xi| \leq 1-\delta\} \cup \{\xi; |\xi| \geq 1+\delta\}$$

(ii) The support condition on the Fourier transform of E_{+0} seems to have some physical meaning. This condition implies that most part of the Schrödinger wave propagates at speed less than one and so the ion sound speed is faster than that of the Schrödinger wave. This situation seems important from a physical point of view (see Zakharov [20], H. Added and S. Added [2], Shochet and Weinstein [15] and Ozawa and Tsutsumi [13]).

The following corollary is an immediate consequence of Theorems 1 and 2.

COROLLARY 3. *Assume $N = 3$.*

(i) By D_+ we denote the set of all scattered states (E_{+0}, n_{+0}, n_{+1}) satistying (5). Then, for (1)-(2) the wave operator $W_+ : (E_{+0}, n_{+0}, n_{+1}) \mapsto (E(0), n(0), \frac{\partial}{\partial t}n(0))$ is well defined on D_+.

(ii) Let $\delta > 0$. By $\widetilde{D}_+$ we denote the set of all scattered states $(E_{+0}, n_{+0}, n_{+1}) \in H^{6,7} \oplus H^{5,4} \oplus H^{4,4}$ such that $\mathrm{supp}\widehat{E}_{+0} \subset \{\xi; |\xi| \leq 1-\delta\}$. Then, for (1)-(2) the pseudo wave operator $\widetilde{W}_+ : (E_{+0}, n_{+0}, n_{+1}) \mapsto (E(T), n(T), \frac{\partial}{\partial t}n(T))$ is well defined on $\widetilde{D}_+$.

(iii) For any initial data (E_0, n_0, n_1) belonging to the range of W_+ or the range of $\widetilde{W}_+$, there exist the unique global solutions $(E(t), n(t))$ of (1)-(3) such that

$$E(t) \in \bigcap_{j=0}^{1} C^j([0,\infty); H^{3-2j}),$$

$$n(t) \in \bigcap_{j=0}^{2} C^j([0,\infty); H^{2-j}), \quad \frac{\partial}{\partial t}n(t) \in C([0,\infty); \dot{H}^{-1}).$$

Here W_+ and $\widetilde{W}_+$ are defined in (i) and (ii), respectively.

Remark 3. The range of $\widetilde{W}_+$ includes some large data, while the range of W_+ includes only small data. Therefore, Corollary 3 (iii) implies that for some large initial data, there exist the unique global solutions of (1)-(3). However, it is not clear what initial data belong to the range of $\widetilde{W}_+$.

Theorems 1 and 2 follow from the combination of the various energy and decay estimates. The energy estimates are the standard ones and the decay estimates are crucial. The decay estimates needed for the proof of Theorem 1 are based on the $L^p - L^q$ estimate and the Stricahrtz type estimate (see, e.g., [7] and [19] for the Schrödinger equation and [14] for the wave equation). The decay estimates needed for the proof of Theorem 1 are based on the special properties of the nonlinear coupling for the Zakharov equations and the propagation properties of the Schrödinger wave and the acoustic wave. The support condition on the Fourier transform of E_{+0} ensures that the nonlinear coupling term of (1) decays fast enough as $t \to +\infty$ (see Ozawa and Tsutsumi [13] and Tsutsumi [18]). The details of the proofs will appear elsewhere.

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