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ON THE INSTABILITY OF THE UNIFORM ROTATION OF A BODY WITH LIQUID INSIDE

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1. Introduction

The present work deals with the stability of a uniformly rotating body with a cavity filled with an incompressible viscous fluid. Various statements of the problem of the motion of a rigid body with a cavity filled with fluid have been discussed in the literature [1-5]. In the present work we shall follow [4-5].

Given a rigid body G with a cavity Ω entirely filled with an incompressible viscous liquid. Let O be the center of mass of the entire system "body+liquid", $Ox_1x_2x_3$ be an orthogonal system of coordinates rigidly connected with the body. We assume that the undisturbed motion of the system relative to the point O be a uniform rotation of whole system with constant angular velocity $\omega_0\mathbf{e}_3$ around the axis Ox_3. We shall examine the perturbed motion assuming that its deviations from the unperturbed one are small. Then in the rotating system of coordinates $Ox_1x_2x_3$ the linearization of the Navier-Stokes equations and the equations of moments with respect to the point O can be written as follows [4-5]

$$\frac{\partial \mathbf{u}}{\partial t} + 2\omega_0\mathbf{e}_3 \times \mathbf{u} + \frac{d\mathbf{w}}{dt} \times \mathbf{r} = -\frac{1}{\rho_l}\nabla p + \nu\Delta\mathbf{u} + \mathbf{f} \tag{1}$$

$$\operatorname{div} \mathbf{u} = 0 \tag{2}$$

$$\mathbf{J}\cdot\frac{d\mathbf{w}}{dt} + \omega_0\mathbf{w}\times\mathbf{J}\cdot\mathbf{e}_3 + \omega_0\mathbf{e}_3\times\mathbf{J}\cdot\mathbf{w} + \rho_l\int_\Omega \mathbf{r}\times\frac{\partial\mathbf{u}}{\partial t}\,d\Omega + \omega_0\mathbf{e}_3\times\left(\rho_l\int_\Omega \mathbf{r}\times\mathbf{u}\,d\Omega\right) = \mathbf{M} \tag{3}$$

where

$$\mathbf{J}\cdot\mathbf{w} = \rho_l\int_\Omega \mathbf{r}\times(\mathbf{w}\times\mathbf{r})\,d\Omega + \rho_b\int_G \mathbf{r}\times(\mathbf{w}\times\mathbf{r})\,dG$$

is the inertia tensor of the entire system with respect to the point O. Here t denotes time; $\mathbf{r} = \overrightarrow{(x_1, x_2, x_3)}$; ρ_l, ρ_b are the constant densities of the liquid and the body; ν is the kinematic viscosity of the liquid; $\mathbf{u}$ is a relative velocity of the liquid; $\mathbf{w}$ is a relative angular velocity of the body; p is a relative pressure; $\mathbf{M}(t)$ is a central moment of exterior forces with respect to O; $\mathbf{f}(t,x)$ is a field of external force.

To complete the system (1)-(3) we consider the following boundary and initial conditions

$$\mathbf{u}|_{\partial\Omega} = 0 \tag{4}$$

$$\mathbf{u}(0,x) = \mathbf{u}_0(x)\,, \qquad \mathbf{w}(0) = \mathbf{w}_0 \tag{5}$$

As in [5] the closure in the norm of $\mathbf{L}_2(\Omega)$ of the set of all smooth solenoidal fields $\mathbf{u}$, satisfying the condition $\mathbf{u}\cdot\mathbf{n} = 0$ on $\partial\Omega$, will be referred to as the Hilbert space $\mathbf{J}_0(\Omega)$. Then equations (1),(2),(4) can be rewritten in the operator form [4-5]

$$\frac{d\mathbf{u}}{dt} + \mathbf{P}_0\left(\frac{d\mathbf{w}}{dt}\times\mathbf{r}\right) + \nu\mathbf{A}\mathbf{u} + 2iw_0\mathbf{T}\mathbf{u} = \mathbf{P}_0\mathbf{f} \tag{6}$$

where $\mathbf{P}_0$ is the operator of the orthogonal projection of space $\mathbf{L}_2(\Omega)$ onto $\mathbf{J}_0(\Omega)$,

$$\mathbf{T}\mathbf{u} = i\mathbf{P}_0(\mathbf{u}\times\mathbf{e}_3)\,, \quad \mathbf{T} = \mathbf{T}^*\,, \quad \sigma(\mathbf{T}) = [-1,1] \tag{7}$$

and $\mathbf{A}$ is the Friedrichs extension [5] of the following symmetric operator

$$\mathbf{A}_0 = -\mathbf{P}_0\Delta \quad , \quad D(\mathbf{A}_0) = \{\mathbf{u}\in\mathbf{W}_2^2(\Omega)\,|\,\mathrm{div}\;\mathbf{u} = 0\;;\;\mathbf{u}|_{\partial\Omega} = 0\}$$

The operator $\mathbf{A}$ is usually called Stokes operator [5].

Following [5] we denote

$$\mathbf{v}(t) = (\mathbf{u}(t,x),\mathbf{w}(t)) \in \mathbf{H} = \mathbf{J}_0(\Omega)\oplus\mathbb{R}^3 \tag{8}$$

Then in the Hilbert space $\mathbf{H}$ system (1)-(5) can be written in the form of abstract Cauchy problem [5]

$$\frac{d}{dt}\mathbf{N}\mathbf{v} + (\mathbf{C}+\mathbf{B})\mathbf{v} = \mathbf{g}(t) \quad , \quad \mathbf{g}(t) = ((\mathbf{P}_0\mathbf{f})(t),\mathbf{M}(t)) \tag{9}$$

$$\mathbf{v}(0) = \mathbf{v}_0\,, \qquad \mathbf{v}_0 = (\mathbf{u}_0(x),\mathbf{w}_0) \tag{10}$$

where

$$\mathbf{N}\mathbf{v} = \left(\mathbf{u} + \mathbf{P}_0(\mathbf{w}\times\mathbf{r})\,,\,\rho_l\int_\Omega \mathbf{r}\times\mathbf{u}\,d\Omega + \mathbf{J}\cdot\mathbf{w}\right)$$

$$\mathbf{C} = \mathrm{diag}\,(\nu\mathbf{A},\mathbf{I})$$

$$\mathbf{B}\mathbf{v} = \begin{pmatrix} 2iw_0\mathbf{T}\mathbf{u} \\ w_0\mathbf{e}_3\times\left(\rho_l\int_\Omega \mathbf{r}\times\mathbf{u}\,d\Omega\right) + w_0\,(\mathbf{e}_3\times\mathbf{J}\cdot\mathbf{w} + \mathbf{w}\times\mathbf{J}\cdot\mathbf{e}_3) - \mathbf{w} \end{pmatrix}$$

It is shown in [5] that if $\mathbf{g}(t)$ satisfies Hölder condition then for any $\mathbf{v}_0 \in \mathbf{H}$ there exists unique solution $\mathbf{v}(t) \in \mathbf{H}$ of the problem (9),(10).

Consider the spectral problem corresponding to the operator equation (9). Let $\mathbf{v}(t) = \exp(-\lambda t)\mathbf{V}$, $\mathbf{V} \in \mathbf{H}$; $\mathbf{g}(t) \equiv 0$. Then $\mathbf{V}$ should satisfy

$$(\mathbf{C}+\mathbf{B})\mathbf{V} = \lambda \mathbf{N}\mathbf{V} \quad , \quad \mathbf{V} \in \mathbf{H} \tag{11}$$

Using the properties of the operators $\mathbf{C}, \mathbf{B}, \mathbf{N}$ it has been proved in [5] that:

(1) Problem (11) has discrete spectrum $\{\lambda_j\}_{j=1}^{\infty}$. Each eigenvalue λ_j , $j \in \mathbb{N}$ has finite multiplicity.
(2) There exist positive constants $C_1, C_2 > 0$ such that

$$\mathrm{Re}\ \lambda_j \geq -C_1 \quad ; \quad |\mathrm{Im}\ \lambda_j| \leq C_2 \quad , \quad j \in \mathbb{N} \tag{12}$$

(3) The sequence of eigenfunctions and associated functions corresponding to the eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ is complete in $\mathbf{H}$.

In the present paper we shall consider the following problem: under what conditions there exists an eigenvalue λ_{j_0} such that

$$\mathrm{Re}\ \lambda_{j_0} < 0 \tag{13}$$

If there exists λ_{j_0} satisfying (13) then using results of [6] it can be shown that the uniform rotation with the fixed angular velocity $w_0\mathbf{e}_3$ of the body G with the fluid-filled cavity Ω is unstable.

2. Symmetry assumption

We assume that the boundary $\partial\Omega$ is smooth enough and domains G, Ω are invariant with respect to the turn to the angle $\frac{\pi}{2}$ about the axis Ox_3. Exactly speaking, it means that the following symmetry condition holds:

$$\text{(S)} \qquad \begin{aligned} (x_1, x_2, x_3) \in \Omega &\Longleftrightarrow (-x_2, x_1, x_3) \in \Omega \\ (x_1, x_2, x_3) \in G &\Longleftrightarrow (-x_2, x_1, x_3) \in G \end{aligned}$$

Consider the inertia tensor $\mathbf{J}$. From (S) it follows that for any $\mathbf{w} = (w_1, w_2, w_3)$

$$\mathbf{J} \cdot \mathbf{w} = (a_0 w_1\ ,\ a_0 w_2\ ,\ b_0 w_3) = \mathrm{diag}(a_0, a_0, b_0) \cdot \mathbf{w} \tag{14}$$

where

$$a_0 = \rho_l \int\limits_{\Omega} (x_1^2 + x_3^2)\, dx_1\, dx_2\, dx_3 + \rho_b \int\limits_{G} (x_1^2 + x_3^2)\, dx_1\, dx_2\, dx_3 \tag{15}$$

$$b_0 = 2\rho_l \int\limits_{\Omega} x_1^2\, dx_1\, dx_2\, dx_3 + 2\rho_b \int\limits_{G} x_1^2\, dx_1\, dx_2\, dx_3 \tag{16}$$

Denote

$$c_0 = b_0 - a_0 = \rho_l \int\limits_{\Omega} (x_1^2 - x_3^2)\, dx_1\, dx_2\, dx_3 + \rho_b \int\limits_{G} (x_1^2 - x_3^2)\, dx_1\, dx_2\, dx_3 \tag{17}$$

It is easy to check that if $\lambda = 0$ then $(\mathbf{u}, \mathbf{w}) \equiv \alpha(0, \mathbf{e}_3)$ satisfies (11) for any $\alpha \in \mathbb{C}$. Therefore $\lambda = 0$ is eigenvalue of the spectral problem (11). Let $\lambda \neq 0$. Then (11) can be rewritten as follows [7]:

$$\nu' \mathbf{A}\mathbf{u} + 2i\mathbf{T}\mathbf{u} + m\gamma \left[\frac{\gamma + i}{\gamma + ik} \mathbf{P}_1 \mathbf{u} + \frac{\gamma - i}{\gamma - ik} \mathbf{P}_2 \mathbf{u} + d\mathbf{P}_3 \mathbf{u} \right] = \gamma \mathbf{u} \tag{18}$$

where

$$\gamma = \frac{\lambda}{\omega_0}\ , \quad k = -\frac{c_0}{a_0}\ , \quad \nu' = \frac{\nu}{w_0}\ , \quad m = \frac{\rho_l \|\mathbf{c}_1\|^2_{\mathbf{L}_2(\Omega)}}{2a_0}\ , \quad d = \frac{a_0}{b_0} \frac{\|\mathbf{c}_3\|^2_{\mathbf{L}_2(\Omega)}}{\|\mathbf{c}_1\|^2_{\mathbf{L}_2(\Omega)}} \tag{19}$$

$$\mathbf{c}_1 = \mathbf{P}_0((1, i, 0) \times \mathbf{r})\,, \quad \mathbf{c}_2 = \mathbf{P}_0((1, -i, 0) \times \mathbf{r})\,, \quad \mathbf{c}_3 = \mathbf{P}_0((0, 0, 1) \times \mathbf{r}) \tag{20}$$

$$\mathbf{P}_j \mathbf{u} = \frac{(\mathbf{u}, \mathbf{c}_j)_{\mathbf{L}_2(\Omega)}}{\|\mathbf{c}_j\|^2_{\mathbf{L}_2(\Omega)}} \cdot \mathbf{c}_j\,, \qquad j = 1, 2, 3\,, \quad \mathbf{u} \in \mathbf{J}_0(\Omega) \tag{21}$$

Lemma 1 [7]. $(\mathbf{c}_k, \mathbf{c}_n)_{\mathbf{L}_2(\Omega)} = 0\,, \quad k, n \in \{1, 2, 3\}\,, \quad k \neq n\,; \quad \|\mathbf{c}_1\|_{\mathbf{L}_2(\Omega)} = \|\mathbf{c}_2\|_{\mathbf{L}_2(\Omega)}$.

It is easy to check that the following inequalities hold [7]

$$-1 < k < 1\,, \qquad 0 < m < 1\,, \qquad 0 < md < 1 \tag{22}$$

Since $\omega_0 > 0$ then the spectral problem (11) has an eigenvalue $\lambda \in \mathbb{C}^- = \{z \in \mathbb{C} \mid \mathrm{Re}\, z < 0\}$ if and only if (18) has an eigenvalue $\gamma \in \mathbb{C}^-$.

3. Necessary condition

Let $\gamma \in \mathbb{C}^-$ be an eigenvalue of (69), $\mathbf{u}_\gamma$ be a corresponding eigenfunction. Then (18) implies

$$\nu'(\mathbf{A}\mathbf{u}_\gamma, \mathbf{u}_\gamma)_{\mathbf{L}_2} + 2i(\mathbf{T}\mathbf{u}_\gamma, \mathbf{u}_g)_{\mathbf{L}_2} + m(1-k) \left[\frac{\gamma k + i\gamma^2}{\gamma^2 + k^2} \|\mathbf{P}_1 \mathbf{u}_\gamma\|^2_{\mathbf{L}_2} + \frac{\gamma k - i\gamma^2}{\gamma^2 + k^2} \|\mathbf{P}_2 \mathbf{u}_\gamma\|^2_{\mathbf{L}_2} \right]$$
$$= \gamma \left(\|\mathbf{u}_\gamma\|^2_{\mathbf{L}_2(\Omega)} - m\|\mathbf{P}_1 \mathbf{u}_\gamma\|^2_{\mathbf{L}_2(\Omega)} - m\|\mathbf{P}_2 \mathbf{u}_\gamma\|^2_{\mathbf{L}_2(\Omega)} - md\|\mathbf{P}_3 \mathbf{u}_\gamma\|^2_{\mathbf{L}_2(\Omega)} \right)$$

Since $\mathbf{A} \gg 0\,, \mathbf{T} = \mathbf{T}^*\,, \nu' > 0$ then using (22) and Lemma 1 we obtain

$$\nu'(\mathbf{A}\mathbf{u}_\gamma, \mathbf{u}_\gamma)_{\mathbf{L}_2} + m(1-k)k\mathrm{Re}\gamma \left[\|\mathbf{P}_1 \mathbf{u}_\gamma\|^2_{\mathbf{L}_2} \frac{1}{|\gamma + ik|^2} + \|\mathbf{P}_2 \mathbf{u}_\gamma\|^2_{\mathbf{L}_2} \frac{1}{|\gamma - ik|^2} \right]$$
$$= \mathrm{Re}\gamma \left(\|\mathbf{u}_\gamma\|^2_{\mathbf{L}_2(\Omega)} - m\|\mathbf{P}_1 \mathbf{u}_\gamma\|^2_{\mathbf{L}_2(\Omega)} - m\|\mathbf{P}_2 \mathbf{u}_\gamma\|^2_{\mathbf{L}_2(\Omega)} - md\|\mathbf{P}_3 \mathbf{u}_\gamma\|^2_{\mathbf{L}_2(\Omega)} \right)$$
$$\leq \mathrm{Re}\gamma\, (1 - \max(m, md))\, \|\mathbf{u}_\gamma\|^2_{\mathbf{L}_2(\Omega)} < 0$$

As far as $k \in (-1, 1)\,, m > 0$ then condition $\mathrm{Re}\gamma < 0$ implies

$$k > 0 \tag{23}$$

which is a necessary condition for the existence an eigenvalue $\gamma \in \mathbb{C}^-$. Henceforth we shall assume $k \in (0, 1)$.

4. Instability criterion

We denote

$$\mathbf{D}(\gamma, k) = \left[\frac{\gamma + i}{\gamma + ik}\mathbf{P}_1 + \frac{\gamma - i}{\gamma - ik}\mathbf{P}_2 + d\mathbf{P}_3\right]$$

three-dimensional operator in $\mathbf{J}_0(\Omega)$. Then (18) can be written as follows:

$$\nu'\mathbf{A}\mathbf{u} + 2i\mathbf{T}\mathbf{u} + m\gamma\mathbf{D}(\gamma, k)\mathbf{u} = \gamma\mathbf{u} \tag{24}$$

Using properties of $\mathbf{A}$ and applying usual arguments [5] it is easy to show that the operator $\nu'\mathbf{A} + 2i\mathbf{T}$ has discrete spectrum $\{\lambda_j(\nu')\}_{j=1}^{\infty}$. Since $\mathbf{A}^* = \mathbf{A} \gg 0$, $\mathbf{T} = \mathbf{T}^*$ and $\mathbf{A}^{-1}$ is compact then $\operatorname{Re} \lambda_j(\nu') \geq \nu'\lambda_1(\mathbf{A})$. Therefore for any $\nu' > 0$, $\gamma \in \mathbb{C}^-$ there exists

$$\mathbf{R}(\gamma, \nu') = (\nu'\mathbf{A} + 2i\mathbf{T} - \gamma\mathbf{I})^{-1}$$

Thus for $\gamma \in \mathbb{C}^-$ equation (24) can be written in the form

$$\mathbf{u} + m\gamma\mathbf{R}(\gamma, \nu') \circ \mathbf{D}(\gamma, k)\mathbf{u} = 0 \tag{25}$$

Applying the orthogonal projector $\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3$ we can rewrite (25) as follows:

$$\mathbf{P}\mathbf{u} + m\gamma\mathbf{P} \circ \mathbf{R}(\gamma, \nu') \circ \mathbf{D}(\gamma, k) \circ \mathbf{P}\mathbf{u} = 0 \tag{26}$$

$$(\mathbf{I} - \mathbf{P})\mathbf{u} + m\gamma(\mathbf{I} - \mathbf{P}) \circ \mathbf{R}(\gamma, \nu') \circ \mathbf{D}(\gamma, k) \circ \mathbf{P}\mathbf{u} = 0 \tag{27}$$

It is obvious that the system (26),(27) is solvable if and only if the equation (27) is solvable. Since $\mathbf{P} + m\gamma\mathbf{P} \circ \mathbf{R}(\gamma, \nu') \circ \mathbf{D}(\gamma, k) \circ \mathbf{P}$ is a linear three-dimensional operator mapping $\mathbf{P}\mathbf{J}_0(\Omega)$ into $\mathbf{P}\mathbf{J}_0(\Omega)$ then (27) is solvable if and only if

$$\det \left\| \ (\mathbf{c}_j, \mathbf{c}_n)_{\mathbf{L}_2(\Omega)} + m\gamma(\mathbf{P} \circ \mathbf{R}(\gamma, \nu') \circ \mathbf{D}(\gamma, k)\mathbf{c}_j \, , \, \mathbf{c}_n)_{\mathbf{L}_2(\Omega)} \ \right\|_{j,n=1,2,3} = 0 \tag{28}$$

We denote for $\nu > 0, \gamma \in \mathbb{C}^-$

$$b_{jn}(\gamma, \nu) = (\mathbf{R}(\gamma, \nu)\mathbf{c}_j \, , \, \mathbf{c}_n)_{\mathbf{L}_2(\Omega)} \quad , \quad j, n = 1, 2, 3 \tag{29}$$

Lemma 2 [7]. *For any $\gamma \in \mathbb{C}^-$, $\nu > 0$, $j, n \in \{1, 2, 3\}$, $j \neq n$*

$$b_{jn}(\gamma, \nu) = 0 \tag{30}$$

From (29),(30) it follows that equation (28) can be rewritten as follows:

$$0 = \left[\|\mathbf{c}_1\|^2_{\mathbf{L}_2(\Omega)} + m\gamma\frac{\gamma + i}{\gamma + ik}b_{11}(\gamma, \nu')\right] \cdot \left[\|\mathbf{c}_2\|^2_{\mathbf{L}_2(\Omega)} + m\gamma\frac{\gamma - i}{\gamma - ik}b_{22}(\gamma, \nu')\right] \cdot \left[\|\mathbf{c}_3\|^2_{\mathbf{L}_2(\Omega)} + md\gamma b_{33}(\gamma, \nu')\right] \tag{31}$$

We denote for any $\nu > 0$, $m, k \in (0,1)$, $\gamma \in \mathbb{C}^-$

$$(32) \qquad \begin{cases} f_1(\gamma,\nu,m,k) = \|\mathbf{c}_1\|^2_{\mathbf{L}_2(\Omega)} + m\gamma\dfrac{\gamma+i}{\gamma+ik}b_{11}(\gamma,\nu) \\ f_2(\gamma,\nu,m,k) = \|\mathbf{c}_2\|^2_{\mathbf{L}_2(\Omega)} + m\gamma\dfrac{\gamma-i}{\gamma-ik}b_{22}(\gamma,\nu) \\ f_3(\gamma,\nu,m,k) = \|\mathbf{c}_3\|^2_{\mathbf{L}_2(\Omega)} + m\gamma b_{33}(\gamma,\nu) \end{cases}$$

Functions $f_j(\gamma,\nu,m,k)$, $j = 1,2,3$ are analytic in $\gamma \in \mathbb{C}^-$ for any $\nu > 0$, $m, k \in (0,1)$. Since

$$\lim_{\substack{|\gamma|\to\infty \\ \gamma\in\mathbb{C}^-}} |\gamma| \cdot \|\mathbf{R}(\gamma,\nu)\| = 1$$

then for any $\nu > 0$, $m \in (0,1)$ there exists $r(\nu,m) > 0$ such that

$$(33) \qquad f_j(\gamma,\nu,m,k) \neq 0 \quad , \quad \gamma \in \mathbb{C}^-, \quad |\gamma| \geq r(\nu,m)$$

for any $j = 1,2,3$, $k \in (0,1)$.

Thus we obtain the following instability criterion:

spectral problem (18) has an eigenvalue $\gamma \in \mathbb{C}^-$ if and only if

$$f_1(\gamma,\nu',m,k) \cdot f_2(\gamma,\nu',m,k) \cdot f_3(\gamma,\nu',md,k) = 0$$

All the eigenvalues $\gamma \in \mathbb{C}^-$ are situated inside the half-disk

$$\{z \in \mathbb{C}^- \mid |z| \leq \max(r(\nu',m), r(\nu',md))\}$$

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