数理解析研究所講究録
第 821 巻 1993 年 116-137

WELL-POSEDNESS AND SINGULAR LIMITS IN THE THEORY OF COMPRESSIBLE INVISCID FLUIDS

H. BEIRÃO DA VEIGA

Centro Linceo Interdisciplinare - Accademia Nazionale dei Lincei - Roma

INTRODUCTION. The author's talk at the RIMS Symposium was divided in two parts. In the first one we have considered the problem of the strong well posedness, in Hadamard's sense, for nonlinear hyperbolic equations and systems. In the second part we have considered the problem of the incompressible limit for the compressible Euler equations. In this note we put the accent on the first part, giving a concise explanation of its main ideas. The second part consists on a description of the results proved in reference [BV8].

PART I

1. PRELIMINARIES. Here, we describe a method for studying the continuous dependence (in the strong norm) of solutions to nonlinear equations on the initial data, on the esternal forces, and on the structure of the coefficients. It applies as well to Cauchy and mixed problems, and to higher order equations. For problems (2.1) and (4.1) below, it establishes the continuous dependence of the solution u (in the $\mathcal{C}_T(H^k)$ norm) on the initial data f (in the H^k norm) on the "external forces" F (in the $\mathcal{L}^2_T(H^k)$ norm) and on the coefficient $A(\cdot)$ (in the C^k norm on compact subsets of $\mathbb{R}^m$). The crucial point is, however, related to the continuous dependence of the solutions to linear problems on the coefficients. This method was introduced in references [BV3] and [BV4] and was applied to some other problems in references [BV5],[BV6]. In reference [BV8] it was partially used in order to show strong

convergence of solutions to singular limit problems. The method is technically and conceptually very simple. By this reason, it is easily adaptable to a class of problems that seems to us larger than that covered by the other methods known in the literature. The simplicity of the method makes convenient its application case by case. In fact, the very readable proofs given in the sequel can be pleasantly adapted to particular problems in which the reader could be interested in.

Here, we will illustrate our method by considering first order symmetric hyperbolic systems. It is convenient to begin by considering the Cauchy problem having, however, the mixed problem also in mind. So, let us consider the symmetric hyperbolic system

$$\text{(1.1)} \qquad \partial_t u + A(u)\, \partial_x u = F \qquad \text{in } Q_T \,, \qquad u(0) = f \,,$$

where $$Q_T \equiv [0,T] \times \mathbb{R}^n \,, \quad u \equiv (u_1,\dots,u_m) \,, \quad A(u) \equiv \sum_{i=1}^{n} A^i(u)\, \partial_{x_i} u \,,$$

and the $A^i(u)$ are $m \times m$ symmetric matrices with coefficients $a^i_{q,\ell}(\cdot)$ of class $C^k(\mathbb{R}^m;\mathbb{R})$, $i = 1,\dots,n$, $q,\ell = 1,\dots,m$. In the following, $k > 1 + n/2$ is a fixed integer. Assume that $f \in H^k \equiv H^k(\mathbb{R}^n)$ and that $F \in \mathcal{L}^2_{T_0}(H^k)$. We denote by $\| \;\|_\ell$ the canonical norm in H^ℓ. Other notations are :

$$\mathcal{C}_T(H^\ell) \equiv \bigcap_{j=0}^{\ell} C^j([0,T];H^{\ell-j}) \quad ; \quad \mathcal{L}^2_T(H^\ell) \equiv \bigcap_{j=0}^{\ell} W^{j,2}(0,T;H^{\ell-j}) \;;$$

$$|||u|||^2_\ell \equiv \sum_{j=0}^{\ell} \|\partial_t^j u\|^2_{\ell-j} \;; \quad |||u|||^2_{\ell,T} \equiv \operatorname*{ess\,sup}_{0\le t\le T} |||u(t)|||^2_\ell \;;$$

$$[u]^2_{\ell,T} \equiv \int_0^T |||u(t)|||^2_\ell \, dt.$$

Next, consider the family of problems ($\nu \in \mathbb{N}$)

$$(1.1)_\nu \qquad \partial_t u^\nu + A_\nu(u^\nu)\, \partial_x u^\nu = F^\nu \ , \quad u^\nu(0) = f^\nu \ ,$$

where $A_\nu(\cdot)$, f^ν, and F^ν are as $A(\cdot)$, f , and F above. Assume that

$$(1.2) \qquad \lim_{\nu\to 0} \|f^\nu - f\|_k = 0 \quad , \quad \lim_{\nu\to 0} [F^\nu - F]_{k,T_0} = 0 \ ;$$

$$(1.3) \qquad \lim_{\nu\to 0} A_\nu(\cdot) = A(\cdot) \quad \text{in } C^k \ ,$$

on compact subsets of $\mathbb{R}^m$. Under the above hypothesis there are $T > 0$ and $C > 0$ such that the problem (1.1) has a unique solution $u \in \mathcal{C}_T(H^k)$ (it would be sufficient here to assume that $u \in \mathcal{L}^\infty_T(H^k)$ since continuity follows then easily by applying the estimates proved in the sequel). Upper bounds for T^{-1} and for C depend (non decreasingly) on the norms $\|f\|_k$ and $[F]_{k,T_0}$, and on the C^k norms of the matrices $A^i(\cdot)$ on a fixed compact set that strictly contains the set $\left\{ f(x) : x \in \mathbb{R}^n \right\}$. By applying this result to the system $(1.1)_\nu$, under the hypothesis (1.2),(1.3), it follows that the constants T and C are independent of ν. In particular

$$(1.4) \qquad |||u^\nu|||_{k,T} \le C \quad , \quad \forall \ \nu \in \mathbb{N}.$$

It readily follows that $\lim_{\nu\to 0} \|u^\nu - u\|_{0,T} = 0$ and, by interpolation, that

$$(1.5) \qquad \lim_{\nu\to 0} |||u^\nu - u|||_{k-\vartheta,T} = 0 \ ,$$

for each $\vartheta > 0$. In bounded domains one also has $u^\nu \rightharpoonup u$ in $\mathcal{L}^\infty_T(H^k)$ with respect to the weak-* topology. These convergence results are unsatisfactory. In fact, since the solutions u and u^ν belong to $\mathcal{C}_T(H^k)$, the natural result is

$$(1.6) \qquad \lim_{\nu\to \infty} |||u^\nu - u|||_{k,T} = 0 \ .$$

From (1.6) it follows, in particular, that when the solution $u(t)$ exists on

$[0,T^*]$, for some T^*, then (for sufficiently large values of ν) the solutions u^ν exist on $[0,T^*]$ and satisfy (1.6) on $[0,T^*]$. This results are well known, at least if $A^\nu = A$. However, the particular problem treated here is used only as a vehicle for illustrating the main ideas. In the next sections we will prove (1.6) for the above nonlinear problem and for related linear and mixed problems.The proofs can be done directly to the nonlinear problem. However, since the results for linear problems are interesting by themselves, we study the linear problem and then apply the results to the nonlinear problem.

2. THE CAUCHY PROBLEM.Let us consider the linear systems

$$\partial_t u + A(t,x)\, \partial_x u = F \quad , \qquad u(0) = f \quad , \tag{2.1}$$

$$\partial_t u^\nu + A_\nu(t,x)\, \partial_x u^\nu = F^\nu \quad , \quad u^\nu(0) = f^\nu \ , \tag{$(2.1)_\nu$}$$

where $\nu \in \mathbb{N}$. Notations are similar to that in section 1. Now, $A(u)$ is replaced by $A(t,x)$, and so on. We assume that $A, A_\nu \in \mathcal{L}^\infty_{T_0}(H^k)$ and that

$$|||A_\nu|||^2_{k,T_0} \le C \quad , \quad \forall \ n \in \mathbb{N} \ . \tag{2.2}$$

Moreover, we assume that $f, f^\nu \in H^k$, that $F, F^\nu \in \mathcal{L}^2_{T_0}(H^k)$, and that $\|f^\nu\|_k \le C$ and $[F^\nu]_{k,T_0} \le C$, where C is independent of ν. More careful manipulations show that we can replace $\mathcal{L}^2_{T_0}$ and $\mathcal{L}^\infty_{T_0}$ by $\mathcal{L}^p_{T_0}$, $p > 1$.

Differentiation of (2.1) with respect to x_j yields

$$\begin{cases} \partial_t(\partial_{x_j} u) + A(t,x)\, \partial_x(\partial_{x_j} u) = \partial_{x_j} F - \partial_{x_j}(A(x,t))\, \partial_x u \quad , \\ (\partial_{x_j} u)(0) = \partial_{x_j} f \ . \end{cases} \tag{2.3}$$

By setting $U = (\partial_{x_1} u, \dots, \partial_{x_n} u)$, $\phi \equiv \partial_x f \equiv (\partial_{x_1} f, \dots, \partial_{x_n} f)$, and $\hat{A}^i \equiv$ diagonal bloc matrix $(A^i, \dots, A^i)$, where the matrix A^i is repeated n times, one shows that U (note that $U \in \mathscr{C}_T(H^{k-1})$) solves, by the construction, the system

$$\partial_t U + \hat{A}(t,x)\, \partial_x U = \Phi \quad , \quad U(0) = \phi \quad , \tag{2.4}$$

where, by definition,

$$\Phi = \partial_x F - (\partial_x A(t,x))\, \partial_x u \ .$$

Note that $\phi \in H^{k-1}$, $\Phi \in \mathscr{L}^2_{T_0}(H^{k-1})$. We do not write the detailed expression of $\partial_x A(t,x)$ since the only fact used in the sequel is that each of its single element has the form $(\partial_{x_j} a^i_{q,\ell}(t,x))\partial_{x_p} u$, for some integers $i,p,j \in [1,n]$, $q,\ell \in [1,m]$.

Similarly, one gets from (2.1) (notations are obvious)

$$\partial_t U^\nu + \hat{A}_\nu(t,x)\, \partial_x U^\nu = \Phi^\nu \quad , \quad U^\nu(0) = \phi^\nu \quad , \tag{2.4$_\nu$}$$

where $\Phi^\nu \equiv \partial_x F^\nu - (\partial_x A_\nu(t,x))\, \partial_x u^\nu$, $\phi^\nu \equiv \partial_x f^\nu = (\partial_{x_1} f^\nu, \dots, \partial_{x_n} f^\nu)$ and $\hat{A}^i_\nu \equiv$ diagonal bloc matrix $(A^i, \dots, A^i)$. The $\hat{A}_\nu$ satisfy (2.2). Next, for each $\varepsilon > 0$ we fix $\phi^\varepsilon \in H^k$ and $\Phi^\varepsilon \in \mathscr{L}^2_{T_0}(H^k)$ such that

$$\|\phi^\varepsilon - \phi\|^2_{k-1} \le \varepsilon \quad , \quad [\Phi^\varepsilon - \Phi]^2_{k-1,T_0} \le \varepsilon \quad , \tag{2.5}$$

and we consider the solution U^ε of the problem

$$\partial_t U^\varepsilon + \hat{A}(t,x)\, \partial_x U^\varepsilon = \Phi^\varepsilon \ , \quad U^\varepsilon(0) = \phi^\varepsilon \ . \tag{2.4$_\varepsilon$}$$

Since $\hat{A} \in \mathscr{L}^\infty_{T_0}(H^k)$ it follows that $U^\varepsilon \in \mathscr{C}_{T_0}(H^k)$. Note that an upper bound for the norm $|||U^\varepsilon|||_{k,T_0}$ depends only on ε and T_0 and on the given functions ϕ, Φ, and $\hat{A}$. Hence, it depends only on ε, T_0, f, F, and A. We write, for convenience,

$$|||U^\varepsilon|||^2_{k,T_0} \le C(\varepsilon, T_0; f,F,A) \equiv \Lambda(\varepsilon). \tag{2.6}$$

Taking the difference, side by side, between equations $(2.4)_\nu$ and $(2.4)_\varepsilon$ we get

$$(2.7)\qquad \begin{cases} \partial_t(U^\nu-U^\varepsilon) + \hat{A}_\nu\partial_x(U^\nu-U^\varepsilon) = \Phi^\nu - \Phi^\varepsilon + (\hat{A}-\hat{A}_\nu)\partial_x U^\varepsilon, \\ (U^\nu - U^\varepsilon)(0) = \phi^\nu-\phi^\varepsilon . \end{cases}$$

The usual H^{k-1}-energy estimate gives

$$|||(U^\nu-U^\varepsilon)(t)|||^2_{k-1} \le C\left\{\|\phi^\nu-\phi^\varepsilon\|^2_{k-1} + [\Phi^\nu-\Phi^\varepsilon]^2_{k-1,t} + \left[(\hat{A}-\hat{A}_\nu)\ \partial_x U^\varepsilon\right]^2_{k-1,t}\right\},$$

where C depends on T_0. Hence

$$(2.8)\qquad |||(U^\nu-U^\varepsilon)(t)|||^2_{k-1} \le C\left\{\varepsilon + \|\phi^\nu-\phi\|^2_{k-1} + [\Phi^\nu-\Phi]^2_{k-1,t} + \right.$$
$$\left. + [\hat{A}-\hat{A}_\nu]^2_{k-1,t}\ |||\partial_x U^\varepsilon|||^2_{k-1,t}\right\}.$$

On the other hand (recall that k-1 > n/2)

$$[\Phi^\nu-\Phi]^2_{k-1,t} \le [F^\nu-F]^2_{k,t} + \left[\partial_x(A-A_\nu)\right]^2_{k-1,t} |||\partial_x u|||^2_{k-1,t} +$$
$$+ |||\partial_x A_\nu|||^2_{k-1,t}[\partial_x(u-u^\nu)]^2_{k-1,t}.$$

Hence,

$$(2.9)\qquad |||(U^\nu-U^\varepsilon)(t)|||^2_{k-1} \le C\left\{\varepsilon + \|f^\nu-f\|^2_k + [F^\nu-F]^2_{k,T_0} + \right.$$
$$\left. + [A_\nu-A]^2_{k,t} + [u^\nu-u]^2_{k,t} + \Lambda(\varepsilon)\ [A_\nu-A]^2_{k-1,t}\right\}.$$

The above calculations, if $(2.4)_\nu$ is replaced by (2.4), yield

$$(2.10)\qquad |||(U-U^\varepsilon)(t)|||^2_{k-1} \le C\ \varepsilon .$$

Hence $|||(U^\nu-U)(t)|||^2_{k-1}$ is bounded by the right hand side of (2.9). Since this quantity is equivalent to $|||(u^\nu-u)(t)|||^2_k$, one has, for each $t \in [0,T_0]$

$$(2.11)\qquad |||(u^\nu-u)(t)|||^2_k \le C\left\{\varepsilon + \|f^\nu-f\|^2_k + [F^\nu-F]^2_{k,T_0} + \right.$$

$$+ [A_\nu - A]^2_{k,t} + \Lambda(\varepsilon)\,[A_\nu - A]^2_{k-1,t}\Big\}\,,$$

where the term C $[u^\nu - u]^2_{k,t}$ was previously droped by using Gronwall's lemma. This proves the following result :

THEOREM 2.1. Assume that (2.2) holds. If $f, f^\nu \in H^k$ and if $F, F^\nu \in \mathcal{L}^2_{T_0}(H^k)$, then the solutions u and u^ν of the linear problems (2.1) and $(2.1)_\nu$ satisfy the estimate (2.11). In particular, if (1.2) holds and if

$$\lim_{\nu \to \infty} [A_\nu - A]^2_{k,T_0} = 0 \tag{2.12}$$

then, as $\nu \to \infty$, u^ν converges in the $\mathcal{C}_{T_0}(H^k)$ norm to u, i.e. (1.6) holds.

In fact, given $\sigma > 0$, we fix $\varepsilon_0 = \varepsilon_0(\sigma)$ in equation (2.11) in such a way that C $\varepsilon_0 < \sigma/2$. Since $\Lambda(\varepsilon_0)$ is a fixed quantity, the desired result follows.

Consider now the nonlinear problems (1.1) and $(1.1)_\nu$. One has the following results

THEOREM 2.2. Under the hypotheses (1.2) and (1.3) the solutions of the nonlinear problems $(1.1)_\nu$ converge in the $\mathcal{C}_{T_0}(H^k)$ norm to the solution u of the nonlinear problem (1.1), i.e. (1.6) holds.

Proof. We assume, for convenience, that $A_\nu(\cdot) = A(\cdot)$, $\forall\ \nu \in \mathbb{N}$, leaving to the reader the proof when this assumption is not fulfiled. Define $A(t,x) \equiv A(u(t,x))$, $A_\nu(t,x) \equiv A(u^\nu(t,x))$. The estimate (2.2) holds for $T_0 = T$, due to (1.4), moreover $[A(u^\nu) - A(u)]^2_{\ell,t} \le C\,[u^\nu - u]^2_{\ell,t}$, for $\ell \le k$, since $A(\cdot)$ is of class C^k on compact sets. Hence, one has

$$(2.13) \qquad |||(u^{\nu}-u)(t)|||_k^2 \le C \left\{ \varepsilon + \|f^{\nu}-f\|_k^2 + [F^{\nu}-F]_{k,T}^2 + \Lambda(\varepsilon)\, [u^{\nu}-u]_{k-1,t}^2 \right\},$$

where the term $C\,[u^{\nu}-u]_{k,t}^2$ was previously droped by using Gronwall's lemma. Dependence of $\Lambda(\varepsilon)$ on $A(t,x)$ becames (since $A(t,x) \equiv A(u(t,x))$) dependence on u, hence dependence on the fixed functions f,F, and $A(\cdot)$. The desired result follows now trivially from (2.13), as in theorem 2.1. Note that $[u^{\nu}-u]_{k-1,T}^2 \longrightarrow 0$ as $\varphi \longrightarrow 0$; see (1.5). □

THE MIXED PROBLEM. Hyperbolic mixed problems on open sets Ω with regular, compact boundary can be, very often, reduced to the half-space case $\mathbb{R}_+^n \equiv \left\{x \in \mathbb{R}^n = x_n > 0\right\}$. According to our aim, emphazising the basic points in our method, we assume here that $\Omega = \mathbb{R}_+^n$. By this same reason we will consider a boundary condition $Mu_{|\Sigma_T} = 0$, where M, a p x m matrix ($p \le m$), has constant coefficients and rank p.

Notations are that used in the previous sections, by simply replacing $\mathbb{R}^n$ by $\mathbb{R}_+^n$. We set $\Gamma \equiv \left\{x \in \mathbb{R}^n : x_n = 0\right\}$, $\Sigma_T \equiv [0,T] \times \Gamma$.

The main strategical difference between the proofs for the Cauchy problem and the proofs done below for the mixed problem follows from the fact that for mixed problems, the system (2.3) is not closed. However, differentiation of the boundary conditions with respect to x_j, $j = 1, \ldots, n-1$, gives boundary conditions on $\partial_{x_j} u$ that make the corresponding system complete. This argument fails for the normal direction x_n. However, it works for the t direction. Moreover estimates for $\partial_{x_1} u, \ldots, \partial_{x_{n-1}} u$, $\partial_t u$ yield estimates for $\partial_{x_n} u$ provided that the boundary matrix A^n is non singular on the boundary Σ_T. On the other hand, the main technical difference between the two problems is due to the appearance of compatibility conditions for the mixed problem.

As for the Cauchy problem we start by studying the linear problem. Let us consider linear systems

(3.1) $$\partial_t u + A(t,x)\, \partial_x u = F \ , \quad u(0) = f \ , \quad M\, u_{|\Sigma_T} = 0 \ ,$$

and

$(3.1)_\nu$ $$\partial_t u^\nu + A_\nu(t,x)\, \partial_x u^\nu = F^\nu, \ u^\nu(0) = f^\nu, \ M\, u^\nu_{|\Sigma_T} = 0 \ ,$$

where $A, A^\nu, f, f^\nu, F, F^\nu$ are as in section 2, provided that $\mathbb{R}^n_+$ replaces $\mathbb{R}^n$ in all the assumptions, equations and definitions. We suppose that the matrices A^i, A^i_ν (i=1,...,n) are symmetric; that there is a positive constant μ such that

(3.2) $$|\det A^n| > \mu \quad \text{and} \quad |\det A^n_\nu| > \mu \quad , \quad \forall\, \varphi \in \mathbb{N} \ ,$$

on Σ_{T_0} ; and that the set $\mathcal{N} \equiv \left\{ v \in \mathbb{R}^m : M\, v = 0 \right\}$ is maximal non-positive with respect to $A^n(t,x)$ and $A^n_\nu(t,x)$, for each $(t,x) \in \Sigma_{T_0}$. These assumptions are done just for fixing ideas; in fact the only essential assumption is the existence of regular solutions $u \in \mathcal{L}^\infty_{T_0}(H_\ell)$ satisfying the classical H^ℓ-energy estimates, for $\ell = k-1, k$. Finally, we assume that the couples (f,F) and (f^ν, F^ν) satisfy the compatibility conditions up to order k-1 with respect to the systems (3.1) and $(3.1)_\nu$ respectivelly.

Since $H^k \subset C^{1,\alpha}$, for some $\alpha > 0$, it readily follows from (2.2) and (4.2) that $|\det A^n| > \mu/2$ and $|\det A^n_\nu| > \mu/2$ in a neighbourhood S_{T_0} of Σ_{T_0}, independent of ν. This leads to consider a cut-off function $\vartheta = \vartheta(x_n)$, $x_n \geq 0$, depending only on x_n, which is equal to 1 in a neighbourhood of $x_n = 0$ and vanishes far from the boundary. Since the main point is the regularity up to boundary, there is no inconvenient in assuming that (4.2) holds on the whole domain $Q_{T_0} \equiv [0,T_0] \times \Omega$.

Let us go into the proof of theorem 3.1 below. Together to the equations (2.3) for j = 1, ..., n-1, we also consider the equation

$$(3.3)\qquad \begin{cases} \partial_t(\partial_t u) + A(t,x)\,\partial_x(\partial_t u) = \partial_t F - (\partial_t A(t,x))\,\partial_x u \;, \\ (\partial_t u)(0) = F(0) - A(0,x)\,\partial_x f \equiv \partial_t f \;, \end{cases}$$

obtained from equation (3.1) by differentiation with respect to t. Differentiation of the boundary condition $(3.1)_3$ yields

$$(3.4)\qquad M(\partial_{x_j} u)_{|\Sigma_{T_0}} = 0 \;,\quad j=1,\dots,n-1 \;;\quad M(\partial_t u)_{|\Sigma_{T_0}} = 0 \;.$$

Set, by convenience, $\partial_\tau \equiv (\partial_{x_1},\dots,\partial_{x_{n-1}},\partial_t)$, and define $U \equiv \partial_\tau u$ (i.e., $U \equiv (\partial_{x_1} u, \dots, \partial_{x_{n-1}} u, \partial_t u))$, $\phi \equiv \partial_\tau f$, and

$$(3.5)\qquad \Phi = \partial_\tau F - (\partial_\tau A(x,t))\,\partial_x u \;.$$

Set also $\hat{M} \equiv$ diagonal bloc matrix $(M,\dots,M)$, M repeated n times. Equations (2.3) for $j=1,\dots,n-1$, (3.3) and (3.4) can be written in the abreviate form

$$(3.6)\qquad \partial_t U + \hat{A}(t,x)\,\partial_x U = \Phi \;.\quad U(0) = \phi \;,\quad \hat{M}\,U_{|\Sigma_{T_0}} = 0 \;,$$

that corresponds to (2.4). By replacing in the above arguments the system (3.1) by the system $(3.1)_\nu$ we get

$$(3.6)_\nu\qquad \partial_t U^\nu + \hat{A}_\nu(t,x)\,\partial_x U^\nu = \Phi^\nu \;,\quad U^\nu(0) = \phi^\nu \;,\quad \hat{M}\,U^\nu_{|\Sigma_{T_0}} = 0 \;,$$

where U^ν, Φ^ν and ϕ^ν are defined in the obvious way.By the construction, the couples (ϕ,Φ) and (ϕ^ν,Φ^ν) satisfy the compatibility conditions up to order k-2 for the system (3.6) and $(3.6)_\nu$. Next, we prove (2.9) and (2.10) just as done in section 2. The construction of the couples $(\phi^\varepsilon,\Phi^\varepsilon) \in H^k \times \mathscr{L}^2_{T_0}(H^k)$ must be done here much more carefully. In fact, besides (2.5), each couple must satisfy the compatibility conditions up to order k-1 for the system $(2.4)_\varepsilon$ endowed with the boundary conditions $\hat{M}\,U^\varepsilon_{|\Sigma_T} = 0$. See [BV3], proposition 4.1.

Equation (2.9) and (2.10) show that

$$(3.7)\qquad |||\partial_t(u^\nu-u)(t)|||^2_{k-1} + \sum_{j=1}^{n-1} |||\partial_{x_j}(u^\nu-u)(t)|||^2_{k-1} \le$$

$$\le C\left\{\varepsilon + \|f^\nu-f\|^2_k + [F^\nu-F]^2_{k,T_0} + [A_\nu-A]^2_{k,t} + [u^\nu-u]^2_{k,t} + \Lambda(\varepsilon)\,[A_\nu-A]^2_{k-1,t}\right\}.$$

Finally, we use the equations (3.1) and $(3.1)_\nu$ to express $\partial_{x_n}(u^\nu-u)$ in terms of the other n first order derivatives of $u^\nu-u$. This is done by taking into account the hypothesis (3.2). One has $\partial_{x_n} u = (A^n)^{-1}\left(\sum_{j=1}^{n-1} A^j\partial_{x_j} u - \partial_t u - F\right)$, and similarly for $\partial_{x_n} u^\nu$. It readily follows from the expression of $\partial_{x_n}(u^\nu-u)$ that

$$(3.8)\qquad |||\partial_{x_n}(u^\nu-u)(t)|||^2_{k-1} \le C\Big(|||\partial_t(u^\nu-u)(t)|||^2_{k-1} +$$

$$+\sum_{j=1}^{n-1} |||\partial_{x_j}(u^\nu-u)(t)|||^2_{k-1} + |||(A^\nu-A)(t)|||^2_{k-1} +$$

$$+ |||(F^\nu-F)(t)|||^2_{k-1} + |||(u^\nu-u)(t)|||^2_{k-1}\Big)\,.$$

Hence, by (3.7),

$$(3.9)\qquad |||(u^\nu-u)(t)|||^2_k \le C_{T_0}\Big(\varepsilon + \|f^\nu-f\|^2_k +$$

$$+ [F^\nu-F]^2_{k,T_0} + [A^\nu-A]^2_{k,t} + \Lambda(\varepsilon)[A^\nu-A]^2_{k-1,t}\Big)\,,$$

for each $t \in [0,T_0]$. Note that $|||\cdot|||^2_{k-1,T_0} \le C_{T_0}[\cdot]^2_{k,T_0}$. In equation (3.9), the term $[u^\nu-u]^2_{k,t}$ was previously droped by using Gronwall's lemma. This estimate corresponds to (2.11), in section 2. By using (3.9) and by arguing as in section 2 for proving the theorems 2.1 and 2.2, one proves here the following results.

THEOREM 3.1 Assume that (2.2) and (3.2) hold and assume that the couples (f,F) and (f^ν,F^ν) belong to $H^k \times \mathcal{L}^2_{T_0}(H^k)$ and satisfy the compatibility conditions up to order k-1 for the systems (3.1) and $(3.1)_\nu$ respectivelly. Let u and u_ν be the solutions of these linear systems. Then, (3.9) holds. In particular, if the assumptions (1.2) and (2.12) are satisfied, then (1.6) holds.

THEOREM 3.2 Assume that $A(\cdot)$ and $A_\nu(\cdot)$ are as in section 1 and that M is as above. Assume that the matrices $A^n(v)$ and $A^n_\nu(v)$ are non singular for each $v \in \mathcal{N}$ and that the set $\mathcal{N}$ is maximal non-positive with respect to $A^n(v)$ and $A^n_\nu(v)$, for each $v \in \mathcal{N}$. Assume that the couples (f,F), (F^ν,F^ν) belong to $H^k \times \mathcal{L}^2_{T_0}(H^k)$ and satisfy the compatibility conditions up to order k-1 for the systems (1.1) and $(1.1)_\nu$ respectivelly, endowed with the boundary conditions $M\, u_{|\Sigma_T} = 0$ and $M\, u^\nu_{|\Sigma_T} = 0$. Finally, assume that (1.2) and (1.3) are satisfied. Let u and u^ν be the solutions of the systems (1.1) and $(1.1)_\nu$ endowed with the above boundary conditions. Then (1.6) holds.

Remark. We want to explain the reason that leads us to work on the systems (2.4) and $(2.4)_\nu$ instead of working directly on the systems (1.1) and $(1.1)_\nu$. At this regard it is more significant to consider the direct approach to the nonlinear problem (i.e., without passing through the linear approach). In this case, instead of (2.4), $(2.4)_\nu$ and $(2.4)_\varepsilon$ one has

(3.10) $$\partial_t U + \hat{A}(U)\, \partial_x U = \Phi \quad , \quad U(0) = \phi \ ,$$

$(3.10)_\nu$ $$\partial_t U^\nu + \hat{A}(U^\nu)\, \partial_x U^\nu = \Phi^\nu \quad , \quad U^\nu(0) = \phi^\nu \ ,$$

and

$(3.10)_\varepsilon$ $$\partial_t U^\varepsilon + \hat{A}(U)\, \partial_x U^\varepsilon = \Phi^\varepsilon \quad , \quad U^\varepsilon(0) = \phi^\varepsilon \ ,$$

where $\Phi = \partial_x F - (\partial_x A(u))\, \partial_x u$, $\phi = \partial_x f$ and similarly for Φ^ν and ϕ^ν. Here Φ^ε and ϕ^ε are defined as done before. Next we estimate $|||(U^\nu - U)(t)|||^2_{k-1}$ by following the proof done in section 2 for the linear Cauchy problem (with obvious modifications borrowed from the extension to the nonlinear problem).

Let us try to follow similar arguments directly for u and u^ν. So, consider the systems

$$(3.11) \qquad \partial_t u + A(u)\, \partial_x u = F \quad , \quad u(0) = f \quad ,$$

$$(3.11)_\nu \qquad \partial_t u^\nu + A(u^\nu)\, \partial_x u^\nu = F^\nu \quad , \quad u^\nu(0) = f^\nu,$$

and the auxiliar system (that corresponds here to $(3.10)_\varepsilon$)

$$(3.12)_\varepsilon \qquad \partial_t u^\varepsilon + A(u)\, \partial_x u^\varepsilon = F^\varepsilon \quad , \quad u^\varepsilon(0) = f^\varepsilon \quad ,$$

where $f^\varepsilon \in H^{k+1}$, $F^\varepsilon \in \mathcal{L}^2_T(H^{k+1})$ and $\|f^\varepsilon - f\|^2_k \le \varepsilon$, $[F^\varepsilon - F]^2_{k,T} \le \varepsilon$. The system $(3.12)_\varepsilon$ is not useful since the solution $u^\varepsilon(t)$ does not belong to H^{k+1}. We can try to overcome this obstacle by considering the alternative auxiliar system

$$(3.13) \qquad \partial_t u^\varepsilon + A(u^\varepsilon)\, \partial_x u^\varepsilon = F^\varepsilon \quad , \quad u^\varepsilon(0) = f^\varepsilon.$$

Note that this device requires an extra-regularity for the coefficients $A(\cdot)$.

Now, $|||(u^\nu - u)(t)|||^2_k \le |||(u^\nu - u^\varepsilon)(t)|||^2_k + |||(u^\varepsilon - u)(t)|||^2_k$. On estimating the first term in the right hand side of this last inequality (by taking the difference between equations $(3.11)_\nu$ and (3.13) and then by applying to the solution $u^\nu - u^\varepsilon$ the H^k-energy estimate) one gets

$$|||(u^\nu - u^\varepsilon)|||^2_k \le C \left\{ \dots + [u^\varepsilon - u]^2_{k-1,t}\ |||u^\varepsilon|||^2_{k-1,t} \right\}$$

since

$[A(u^\varepsilon) - A(u)]_{k-1,t} \le C\, [u^\varepsilon - u]_{k-1,t}$. At this point we need to show that

$$(3.14) \qquad \lim_{\varepsilon \to 0} [u^\varepsilon - u]^2_{k-1,t}\ |||u_\varepsilon|||^2_{k+1,t} = 0.$$

This can be proved for the Cauchy problem and for some mixed problems that do not require compatibility conditions (as the Euler incompressible equations) by making a particular choice for the couples $(f^{\varepsilon},F^{\varepsilon})$. This can still be done for some particular mixed problems that require a very small number of compatibility conditions (for instance, compressible Euler equations in the half space for $n = k = 3$). However, for general mixed problems the compatibility conditions prevent from this way.

PART II

4. THE INCOMPRESSIBLE LIMIT. Here, we describe the problem studied and the results obtained in reference [BV8], to which the reader is refered for the proofs. For convenience, we study our problem in the space-periodic case. Hence Ω is the n-dimensional torus. We set $Q_T = \Omega \times [0,T]$. In the sequel k denotes a fixed integer, $k > 1 + n/2$. Moreover, u is the r-vector $(u_1,\dots,u_r)$ and $\lambda > \lambda_0 > 0$ is a parameter. If $u = u(t,x)$, we denote by $u(0)$ the function $u(0,\cdot)$. Let $B^i(u,\lambda)$, $i = 1, \dots, n$ be rxr matrices, of class C^{k+1}, defined for each $\lambda > \lambda_0$ and each $u \in \mathcal{O}$. $\mathcal{O}$ is an open, regular, connected subset of $\mathbb{R}^r$. As in [K Ma 2], we assume that there are n+1 symmetric matrices $A^0(u,\lambda)$ and $A^i(u,\lambda)$, $i = 1,\dots,n$, such that $A^0B^i = A^i$ and that

(4.1) $$(A^0(u,\lambda)\xi,\xi) \geq m\ |\xi|^2 \quad , \quad m > 0 \ ,$$

for all $u \in \mathcal{O}$ (shrink $\mathcal{O}$, if necessary) and all $\xi \in R^r$. Moreover,

(4.2) $$|B(u,\lambda)| \ \leq \ c\lambda \quad , \quad |A^0(u,\lambda)| \ \leq c,$$

(4.3) $$\lambda\ |D_u\ A^0(u,\lambda)| \ \leq c \ ,$$

(4.4) $$\sum_{j=1}^{k+1} |D_u^j\ B(u,\lambda)| \ \leq \ c \ ,$$

for all $u \in \mathcal{O}$, $\lambda > \lambda_0$. Here $B = (B^1,\dots,B^n)$ and $Bu_x \equiv \sum_{i=1}^{n} B^i u_{x_i}$. Next, we consider the system of equations

$$(4.5) \qquad u^\lambda_t + B(u^\lambda,\lambda)\, u^\lambda_x = 0 \quad \text{in } Q_T \quad , \quad u^\lambda(0) = u^\lambda_0,$$

where $u^\lambda_0 \in H^k$ and $\left\{u^\lambda_0(x) : x \in \Omega\right\} \subset \mathcal{O}_0$ for each $\lambda > \lambda_0$. $\mathcal{O}_0$ is a compact subset of $\mathcal{O}$. In the sequel the symbols $C, C_0, C_1, \ldots,$ denote positive constants that are independent of λ. The same symbol may be used to denote distinct constants, even in the same formula.

The following result is due to Klainerman and Majda [K Ma2]:

<u>THEOREM 4.1. Assume that</u> $\|u^\lambda_0\|_k \le C_0$, <u>for all</u> $\lambda > \lambda_0$. <u>Then, there is a positive real</u> T, <u>independent of</u> λ, <u>and a unique solution</u> $u^\lambda \in C_T\,(H^k) \cap C^1_T(H^{k-1})$ <u>of</u> (4.5). <u>Moreover,</u>

$$(4.6) \qquad \|u^\lambda\|_{k,T} + \lambda^{-1}\|\partial_t u^\lambda\|_{k-1,T} \le C.$$

<u>Furthermore, if</u>

$$(4.7) \qquad \|B(u^\lambda_0,\lambda)\, u^\lambda_{0,x}\|_{k-1} \le C_0$$

<u>then</u>

$$(4.8) \qquad \|\partial_t u^\lambda\|_{k-1,T} \le C\,.$$

Note that $B(u^\lambda_0,\lambda)u^\lambda_{0,x} = -\,\partial_t u^\lambda(0)$. Hence (4.7) is also a necessary condition for having (4.8).Next we describe the application of the above result to the Euler compressible and incompressible equations.

Consider a fluid filling Ω and obeying a law of state $p = p(\rho)$. Denote by $\bar\rho > 0$ the mean density of this fluid. By replacing $p(\rho)$ by $p(\rho) - p(\bar\rho)$ one has $p(\rho) = 0$ if and only if $\rho = \bar\rho$. Here and in the sequel we assume that $p \in C^{k+2}(\mathbb{R}^+;\mathbb{R})$ and that $p'(\rho) > 0$ for each $\rho > 0$. Let $\rho(p)$ denote the inverse function of $p(\rho)$, defined on the open interval $I \equiv p(\mathbb{R}^+)$. Set

$$g(p) \equiv \rho'(p) \,/\, \rho(p)\ .$$

Clearly, $g(p) > 0$ for each $p \in I$. The equations of motion are

$$(4.9)\qquad \begin{cases} g(p)\ (\partial_t p + v\cdot\nabla p) + \nabla\cdot v = 0 \ , \\ \rho(p)\ (\partial_t v + (v\cdot\nabla)v) + \nabla p = 0 \ , \\ v(0) = v_0(x) \quad , \quad p(0) = p_0(x) \ . \end{cases}$$

We are interested in considering a family of laws of state $p^\lambda(\rho) = \lambda^2 p(\rho)$ and in studying the behaviour of the solutions as the parameter λ goes to ∞. The parameter λ plays here the part of inverse of the Mach number; see [Ma].

Denoting by ρ^λ the inverse of the function p^λ, one has $\rho^\lambda(p^\lambda) = \rho(p^\lambda/\lambda^2)$, hence $g^\lambda(p^\lambda) = \lambda^{-2} g(p^\lambda/\lambda^2)$. Consequently, the equations of motion under the above λ-law of state and for initial data $v_0^\lambda(x)$, $p_0^\lambda(x)$ are

$$(4.10)\qquad \begin{cases} \lambda^{-1} g(\bar{p}^\lambda/\lambda)\ (\partial_t \bar{p}^\lambda + v^\lambda\cdot\nabla\bar{p}^\lambda) + \nabla\cdot v^\lambda = 0 \ , \\ \rho(\bar{p}^\lambda/\lambda)\ (\partial_t v^\lambda + (v^\lambda\cdot\nabla)v^\lambda) + \lambda\ \nabla\ \bar{p}^\lambda = 0 \ , \\ v^\lambda(0) = v_0^\lambda(x) \quad , \quad \bar{p}^\lambda(0) = \bar{p}_0^\lambda(x) \equiv \lambda^{-1} p_0^\lambda(x) \ , \end{cases}$$

where the "true" pressure p^λ is replaced by $\bar{p}^\lambda \equiv \lambda^{-1} p^\lambda$. Note that $\bar{p}^\lambda(x) = 0$ if and only if $\rho^\lambda(x) = \bar{\rho}$.

The system (4.10) can be written in the above form (4.5), as follows. Denote by b^i_{kj} the "row k column j" element of the matrix $B^i \equiv E^i + \text{diag}\left\{v_i/g, v_i/\rho, v_i/\rho, v_i/\rho\right\}$, $i = 1,2,3$. The matrix E^i is defined by setting $e^i_{1,i+1} = \lambda/g$; $e^i_{i+1,1} = \lambda/\rho$; and $e^i_{kj} = 0$ otherwise. Moreover, if $A^0 \equiv \text{diag}\left\{g,\rho,\rho,\rho\right\}$ one has $A^i = A^0 B^i$ where $A^i = F^i + \text{diag}\left\{v_i, v_i, v_i, v_i\right\}$, $i = 1,2,3$. Here $f^i_{i+1,1} = f^i_{1,i+1} = \lambda$; and $f^i_{kj} = 0$ otherwise. Above, $g = g(\bar{p}^\lambda/\lambda)$ and $\rho = \rho(\bar{p}^\lambda/\lambda)$.

By using the above set up and by defining $u^\lambda \equiv (\bar{p}^\lambda, v^\lambda)$, $u_0^\lambda \equiv (\bar{p}_0^\lambda, v_0^\lambda)$ it readily follows that the system (4.10) has the form (4.5).

Now we assume (see [K Ma2], Eq. (1.7)) that

(4.11) $$v_0^\lambda(x) = v_0(x) + \lambda^{-1} w_0^\lambda(x) \quad , \quad \bar{p}_0^\lambda(x) = \lambda^{-1} p_0^\lambda(x)$$

where

(4.12) $$\nabla\cdot v_0 \equiv 0 \quad \text{and} \quad \|w_0^\lambda\|_k + \|p_0^\lambda\|_k \le C .$$

Obviously, $\lim \|u_0^\lambda - u_0\|_k = 0$ as $\lambda \longrightarrow \infty$, where $u_0 = (0,v_0)$. Moreover (4.7) holds since

(4.13) $$B(u_0^\lambda,\lambda)\, u_{0,x}^\lambda = (v_0^\lambda\cdot\nabla\bar{p}_0^\lambda + (\lambda/g_0^\lambda)\nabla\cdot v_0^\lambda \; , \; (v_0^\lambda\cdot\nabla)v_0^\lambda + (\lambda/\rho_0^\lambda)\nabla\bar{p}_0^\lambda \;) ,$$

where $\rho_0^\lambda = \rho(\bar{p}_0^\lambda/\lambda)$, $g_0^\lambda \equiv g(\bar{p}_0^\lambda/\lambda)$. Hence, by theorem 4.1, one has the estimate

$$\|\bar{p}^\lambda\|_{k,T} + \|v^\lambda\|_{k,T} + \|\bar{p}_t^\lambda\|_{k-1,T} + \|v_t^\lambda\|_{k-1,T} \le C.$$

In particular, subsequences converge in $L_T^\infty(H^k)$ or in $L_T^\infty(H^{k-1})$, with respect to the weak-* topologies, as $\lambda \longrightarrow \infty$. It is not difficult to verify that the limit functions satisfy the incompressible Euler equations

(4.14) $$\begin{cases} \nabla\cdot v = 0 \quad , \\ \bar{\rho}\;(\partial_t v + (v\cdot\nabla)v) + \nabla\pi = 0 \quad , \\ v(0) = v_0(x) \quad , \end{cases}$$

for some $\pi(t,x)$. By the uniqueness of the regular solution of (1.14) it follows that the whole sequence $\{v^\lambda\}$, $\lambda \longrightarrow \infty$, converges to v in the $L_T^\infty(H^k)$ weak-* topology. Similarly, v_t^λ converges to v_t and $\nabla p^\lambda = \nabla(\lambda\bar{p}_\lambda)$ converges to $\nabla\pi$ in $L_T^\infty(H^{k-1})$ weak-* but not (in general) in $C_T(H^{k-1})$. However, we prove that the trajectories (ρ^λ,v^λ) converge to that of the Euler incompressible equation, i.e. to $(\bar{\rho},v)$, in the strong norm H^k, uniformly in time. More precisely, we prove the following result:

<u>THEOREM 4.2</u> <u>Let</u> $(\bar{p}_0^\lambda,v_0^\lambda)$, $\lambda > \lambda_0$, <u>be a family of initial data satisfying the assumptions</u> (4.11), (4.12), <u>and let</u> $(\bar{p}^\lambda,v^\lambda)$ <u>be the corresponding solution to the compressible Euler equations</u> (4.10). <u>Let</u> $\rho^\lambda = \rho(\bar{p}^\lambda/\lambda)$ <u>denote the density of the fluid</u>. <u>Then</u>

(4.15) $$\lim_{\lambda\to\infty}\left(\|v^{\lambda}-v\|_{k,T}+\|\rho^{\lambda}-\bar{\rho}\|_{k,T}\right)=0 \quad ,$$

where v and $\bar{\rho}$ are those appearing in the Euler incompressible equations (4.14).

The theorem 1.2 is a corollary of the general theorem 5.1 below, which guarantees an uniform approximation result (in the strong norm) for the solution u^{λ} of (1.5) by regular solutions $u^{\lambda,\delta}$.

Remarks

(i) The above results of Klainerman and Majda on the incompressible limit have been extended and developed by Schochet [Sc1] for non barotropic fluids in bounded domains. It is worth noting that the presence of the boundary gives rise to serious obstacles (see also (Sc2,3]). It would be interesting to extend the method developed below to Schochet's approach. Or, alternativelly, to get the same extension by using our approach to the compressible equations in bounded domains ([BV5], [BV7], and references).

Other interesting results on the incompressible limit were obtained by Agemi [Ag], Asano [As], Ebin [Eb1,2], and Ukai [U].

(ii) For the viscous, time dependent, problem the reader is refered to [K Ma1], [Ma], and references in there.

(iii) Convergence of compressible viscous solutions to the incompressible one, for the steady equations, was studied by us in references [BV 1,2].

5. THE GENERAL THEOREM. Here Ω can be the n-dimensional thorus or the whole space $\mathbb{R}^n$. In the sequel we consider systems (4.5) enjoying the hypothesis (4.1) to (4.4). Moreover, we assume that

(5.1) $$\lim_{\lambda\to\infty}\|u_0^{\lambda}-u_0\|_k=0 \quad ,$$

for some $u_0 \in H^k$. In the sequel we will consider an auxiliar family

$\left\{ u_0^{\lambda,\delta} \in H^{k+1} : \lambda > \lambda_0 , \delta \in]\, 0,\delta_0] \right\}$, for some fixed $\delta_0 > 0$, such that, for each $\delta \in]\, 0,\delta_0]$.

$$\|u_0^{\lambda,\delta}\|_k \le C_0 , \forall \lambda > \lambda_0 , \quad \|u_0^{\lambda,\delta}\|_{k+1} \le C_0(\delta) , \forall \lambda > \lambda_0,$$

(5.2)

$$\|u_0^{\lambda,\delta} - u_0^{\lambda}\|_k \le \delta , \forall \lambda > \lambda(\delta) , \quad \|u_0^{\lambda,\delta} - u_0^{\lambda}\|_{k-1} \le \delta , \forall \lambda > \lambda_0 .$$

Under the assumption (5.1) such a family $\{u_0^{\lambda,\delta}\}$ exists. In the sequel we also consider the following two additional hypotheses on the family $\{u_0^{\lambda,\delta}\}$:

For each <u>fixed</u> $\delta \in]\, 0,\delta_0]$ there is a function $u_0^{\delta} \in H^{k+1}$ such that

(5.3) $$\lim_{\lambda \to \infty} \|u_0^{\lambda,\delta} - u_0^{\delta}\|_{k+1} = 0 .$$

And

(5.4) $$\|B(u_0^{\lambda,\delta},\lambda) u_{0,x}^{\lambda,\delta}\|_k \le C(\delta) , \quad \forall \lambda > \lambda(\delta).$$

We remark that in the fluidynamics case the assumptions (4.11),(4.12) are sufficient to guarantee the existence of a family $\{u^{\lambda,\delta}\}$ satisfying (besides (5.2)) (5.3) and (5.4).

Consider a family of initial data $u_0^{\lambda,\delta}$ satisfying (5.2) and the corresponding solutions $u^{\lambda,\delta}$ to the problems

(5.5) $$u_t^{\lambda,\delta} + B(u^{\lambda,\delta},\lambda)\, u_x^{\lambda,\delta} = 0 , \quad u^{\lambda,\delta}(0) = u_0^{\lambda,\delta} .$$

These solutions satisfy the estimates

(5.6) $$\|u^{\lambda,\delta}\|_{k,T} + \lambda^{-1}\|u_t^{\lambda,\delta}\|_{k-1,T} \le C , \quad \|u^{\lambda,\delta}\|_{k-1,T} \le C(\delta) ,$$

for each $\lambda > \lambda_0$ and each $\delta \in]\, 0,\delta_0]$. This follows from theorem 4.1.

One has the following results.

<u>THEOREM 5.1. Let Ω be the n-dimensional thorus or the whole $\mathbb{R}^n$. Assume that the hypotheses</u> (4.1)-(4.4) <u>hold and let u_0, u_0^{λ} satisfy</u> (5.1). <u>Let $u_0^{\lambda,\delta}$ be a family of functions such that</u> (5.2) <u>holds (note that such functions exist).</u>

Denote by u^{λ} and by $u^{\lambda,\delta}$ the solutions of problems (4.5) and (5.5) respectivelly. Let $\varepsilon \in]0,1]$ be given. Then there are positive reals $C(\varepsilon)$, $\lambda(\varepsilon)$, and $\lambda_1(\delta)$ such that for each $\delta>0$ one has

$$(5.7) \qquad \|u^{\lambda,\delta}-u^{\lambda}\|_{k,T} \leq C_0(\varepsilon+C(\varepsilon)\delta) \quad , \quad \forall\lambda > \lambda(\varepsilon,\delta) \equiv \max\{\lambda(\varepsilon),\lambda_1(\delta)\}.$$

THEOREM 5.2 Assume that the hypotheses of theorem 5.1 are satisfied and that, for each fixed δ, the limit: $\lim_{\lambda\to\infty} u^{\lambda,\delta}$ exists in $C_T(H^0)$. Then, the whole sequence $\{u^{\lambda}\}$ is convergent in $C_T(H^k)$, as $\lambda \to \infty$.

The theorem 1.2 on the incompressible limit follows easily from the above theorem 5.1 (or from theorem 5.2). See [BV8].

REFERENCES

[Ag] R. Agemi, "The incompressible limit of compressible fluid motion in a bounded domain", Proc. Japan Acad., Ser.A, 57 (1981) 291-293.

[As] K. Asano, "On the incompressible limit of the compressible Euler equation", Japan J. Applied Math. 4 (1987) 455-488.

[BV1] H.Beirão da Veiga, "Stationary motions and the incompressible limit for compressible viscous fluids", Houston J. Math. 13 (1987), 527-544.

[BV2] H.Beirão da Veiga, "An L^p-theory for the n-dimensional, stationary, compressible Navier-Stokes equations, and the incompressible limit for compressible fluids. The equilibrium solutions", Comm. Math. Physics 109 (1987), 229-248.

[BV3] H.Beirão da Veiga, "Perturbation theory and well-posedness in Hadamard's sense of hyperbolic initial-boundary value problems", preprint #2.71 (582), Department of Mathematics - Univ. Pisa (April 1991).

[BV4] H.Beirão da Veiga, "Data dependence in the mathematical theory of compressible inviscid fluids", Arch. Rat. Mech. Anal., 119 (1992) 109-127.

[BV5] H.Beirão da Veiga, "Perturbation theorems for linear hyperbolic mixed problems and applications to the compressible Euler equations", Comm.Pure Appl. Math., 46 (1993), 221-259.

[BV6] H.Beirão da Veiga, "Structural-stability and data dependence for fully nonlinear hyperbolic problems", Arch.Rat.Mech.Analysis, 120 (1992), 51-60.

[BV7] H.Beirão da Veiga, "The initial boundary value problem for the non-barotropic compressible Euler equations: structural-stability and data dependence", to appear.

[BV8] H.Beirão da Veiga, "On the singular limit for slightly compressible fluids", Preprint # 2.98 (637), Dept. Math. Univ. Pisa (May 1992).

[Eb1] D.G.Ebin, "The motion of slightly compressible fluids viewed as motion with a strong constraining force", Annals of Math. 105 (1977), 141-200.

[Eb2] D.G.Ebin, "Motion of slightly compressible fluids in a bounded domain", Comm. Pure Appl. Math., 35 (1982), 451-485.

[K Ma1] S.Klainerman & A.Majda, "Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids", Comm. Pure Appl. Math. 34 (1981), 481-524.

[K Ma2] S.Klainerman & A.Majda, "Compressible and incompressible fluids", Comm. Pure Appl. Math. 35 (1982), 629-653.

[Ma] A.Majda, "Compressible fluid flow and systems of conservation laws in several space dimensions", Springer-Verlag, Appl. Math. Sci. # 53, New York (1984).

[Sc1] S.Schochet, "The compressible Euler equations in a bounded domain: Existence of solutions and the incompressible limit", Comm. Math. Phys. 104 (1986), 49-75.

[Sc2] S.Schochet, "Singular limits in bounded domains for quasi-linear symmetric hyperbolic systems having a vorticity equation", J. Diff. Eq. 68 (1987), 400-428.

[Sc3] S.Schochet, "Symmetric hyperbolic systems with a large parameter", Comm. Part. Diff. Eq. 12 (1987) 1627-1651.

[U] S. Ukai, "The incompressible limit and the initial layer of the compressible Euler equation", J. Math. Kyoto Univ., 26 (1986) 323-331.