数理解析研究所講究録
第 825 巻 1993 年 101-117

Canonical metrics and function families

標準距離と関数空間

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"Home-keeping youth have ever homely wits."
-Shakespeare, *The Two Gentlemen of Verona*,
Act 1, Scene 1

1. __Introduction; canonical distances.__ We set $S_E = \mathbb{C}^* \equiv \{|z| \le +\infty\} = \mathbb{C} \cup \{\infty\}$, the one-point compactification of $S_P = \mathbb{C} \equiv \{|z| < +\infty\}$, the complex plane, and, $S_H = D \equiv \{|z| < 1\}$, the unit disk. The letters E, P, and H mean "elliptic", "parabolic", and "hyperbolic", respectively. We shall consider some families of meromorphic functions f: $D \to S_X$ whose "derivative" $\partial_X f$ defined soon safisfies some properties. Let $\delta_X(z,w) = |z - w|/|1 + q(X)\bar{z}w|$ for z, $w \in S_X$, where $q(E) = 1$, $q(P) = 0$, and $q(H) = -1$. From the complex-analytic viewpoint each space S_X has the canonical distance $d_X(z,w)$: the spherical distance $d_E(z,w) = \tan^{-1}\delta_E(z,w)$ $\left(\tan^{-1}(+\infty) = \pi/2\right)$, the Euclidean distance $d_P(z,w) = \delta_P(z,w) = |z - w|$, and the non-Euclidean hyperbolic or the Poincaré distance $d_H(z,w) = \tanh^{-1}\delta_H(z,w)$. The distance $d_X(z,w)$ is, geometrically, the integral of the differential $\left(1 + q(X)|\zeta|^2\right)^{-1}|d\zeta|$ along the geodesic, where one can immediately observe that the density is $\left(1 + q(X)|\zeta|^2\right)^{-1} = \lim_{|w-\zeta|\to 0} d_X(w,\zeta)/|w - \zeta|$. The following "derivative" $\partial_X f$ will play a fundamental role. For a meromorphic f: $D \to S_X$, we set at $z \in D$,

$$(\partial_X f)(z) = \lim_{|w-z|\to 0} d_X(f(w),f(z))/|w - z|,$$

so that $\partial_X f = |f'|/\left(1 + q(X)|f|^2\right)$. We note that $d_X(w,z)/\delta_X(w,z) \to 1$ as $\delta_X(w,z) \to 0$. We begin with the Lipschitz continuity in the next section.

2. α-**Lipschitz condition.** $0 < \alpha \leq 1$. To define $d_E(z,w)$ we set

$$\begin{aligned}
\delta_E(z,w) &= |z - w|/|1 + \bar{z}w| && \text{for } z,\ w \in \mathbb{C} \text{ with } \bar{z}w \neq -1; \\
&= +\infty && \text{for } z,\ w \in \mathbb{C} \text{ with } \bar{z}w = -1; \\
\delta_E(z,\infty) &= \delta_E(\infty,z) = 1/|z| && \text{for } z \in \mathbb{C} \setminus \{0\}; \\
&= +\infty && \text{for } z = 0; \\
\delta_E(z,w) &= 0 && \text{for } z = w \in \mathbb{C}^*.
\end{aligned}$$

Then

$$\begin{aligned}
(\partial_E f)(z) &= |f'(z)|/\left(1 + |f(z)|^2\right) && \text{if } f(z) \neq \infty; \\
&= |(1/f)'(z)| && \text{if } f(z) = \infty,
\end{aligned}$$

is called the spherical derivative. Let Σ be the sphere of diameter 1 in the space $\mathbb{R}^3$ touching $\mathbb{C} = \mathbb{R}^2$ at the origin from above with the stereographic projection from the north pole (0,0,1), in notation, $z^* \to z \in \mathbb{C}^*$. Then $d_E(z,w)$ in case $z \neq w$ is the smaller of the lengths of the arcs with terminal points z^* and w^* on the (or a) great circle of the Riemann sphere Σ passing through z^* and w^*. Hence $0 \leq d_E(z,w) \leq \pi/2$. The spherical chordal distance $\chi(z,w) = \sin d_E(z,w)$ is popular in Complex Analysis. We actually have

$$d_E(z,w)\cos d_E(z,w) \leq \chi(z,w) \leq d_E(z,w) \leq \chi(z,w)\left(1 - \chi(z,w)^2\right)^{-1/2}.$$

Obviously $d_E(z,w)$ satisfies the postulates of metric.

We begin with

Theorem I. *Let* $0 < \alpha \leq 1$ *and let* $f\colon D \to S_X$ *be meromophic. Then* f *satisfies the* α*-Lipschitz condition*:

(2.1) $$d_X(f(z),f(w)) \leq K|z - w|^{\alpha} \quad (K > 0)$$

in D *if and only if* $\left(1 - |z|^2\right)^{1-\alpha}(\partial_X f)(z)$ *is bounded in* D.

The case: $X = P$ is a classical result of G. H. Hardy and J. E. Littlewood (see [D, p. 74, Theorem 5.1]). The case $X = E$ is in [Y34] and $X = H$ is in [Y14]; see also [Y17,36].

What can we say in the limiting case $\alpha \to 0$, namely, in case

(2.2) $$\left(1 - |z|^2\right)(\partial_X f)(z) \text{ is bounded in } D\ ?$$

Answers are in the next section.

3. ∂_X-**boundedness and the family** B_X. Let $f\colon D \to S_X$. Then f is said to be ∂_X-bounded, or $f \in B_X$ in notation, if

$$\left(1 - |z|^2\right)(\partial_X f)(z) = \lim_{|w-z|\to 0} d_X(f(w),f(z))/d_H(w,z)$$

is bounded in D.

Case $X = H$: Each holomorphic $f\colon D \to D$ is ∂_H-bounded in view of the Pick differential form of the Schwarz lemma (*SPL*):

$$\left(1 - |z|^2\right)(\partial_H f)(z) = \left(1 - |z|^2\right)|f'(z)|/\left(1 - |f(z)|^2\right) \leq 1,\ z \in D.$$

Case $X = E$: The family B_E is just the family N of normal meromorphic functions in the sense of O. Lehto and K. I. Virtanen [LV] in D; see also [Y1, 2, 5, 6, 13, 18, 28, 29, 30, 32, 39].

Case $X = P$: Each member $f \in B_P$ is called a Bloch function; we denote $B_P = B$. The Bloch functions appear in the proof of the Bloch theorem: There exists a universal constant $c_B > 0$, called the Bloch constant, such that if f is holomorphic in D and $f'(0) = 1$, then the Riemann image surface of D by f over $\mathbb{C}$ contains an open one-sheeted disk of (Euclidean) radius c_B. See [La]. For B see also [Y4, 7, 10, 18, 28, 29, 32, 33, 35, 39].

It is not difficult to prove that $f\colon D \to S_X$ is in B_X if and only if f is Lipschitz-continuous in the sense that

$$d_X(f(z),f(w)) \leq K d_H(z,w) \quad (K > 0) \tag{3.1}$$

in D. This is trivial for $X = H$ where we always have $K = 1$.

We consider here area criteria for $f\colon D \to S_X$ to be in B_X in case $X = E, P$. Let Q be a nonempty subset of D for which the double integral

$$A_X(f,Q) = \iint_Q (\partial_X f)^2(z)\,dxdy \leq +\infty \quad (z = x + iy)$$

is significant; if Q is a domain, then $A_X(f,Q)$ is the area of the Riemann image surface of Q by f covering the set-theoretic image $f(Q) \subset S_X = (S_X, d_X)$. We denote by $A_X^*(f,Q)$ the area of $f(Q)$; thus $A_X^*(f,Q) = \iint_{f(Q)} \left(1 + q(X)|z|^2\right)^{-2} dxdy \leq A_X(f,Q)$. Suppose that $Q \equiv \Delta(z,r) = \{w \in \mathbb{C};\ \delta_H(w,z) < r\}$, $z \in D$, $0 < r \leq 1$. This is the disk $\{w \in S_H;\ d_H(w,z)$

$< \tanh^{-1} r\}$ in the space $S_H \equiv (S_H, d_H)$ on the one hand, and the disk $\{w \in S_P;\ d_P(w, z_o) < r_o\}$ (the Appolonius disk) of center $z_o = z(1 - r^2)/(1 - |z|^2 r^2)$ and radius $r_o = r(1 - |z|^2)/(1 - |z|^2 r^2)$ in the space $S_P \equiv (S_P, d_P)$, on the other, $z \in D$, $0 < r \leq 1$.

Theorem II. *Let $X = E$ or P and $f\colon D \to S_X$. Then $f \in B_X$ if and only if one of the following holds*:

(3.2) *For each fixed $r \in (0,1)$,*

$$\sup_{z \in D} A_X(f, \Delta(z, r)) < \pi \text{ in case } X = E;$$

$$< +\infty \text{ in case } X = P.$$

(3.3) *There exists $r \in (0,1)$ such that*

$$\sup_{z \in D} A_X^*(f, \Delta(z, r)) < \pi \text{ in case } X = E;$$

$$< +\infty \text{ in case } X = P.$$

The proof depends on the estimate of $(1 - |z|^2)(\partial_X f)(z)$ by $A_X^*(f, \Delta(z, r))$; see [Y18] and also [Y35, Theorem 6, Y4, Y32].

What can we say in case $X = H$ in terms of $A_H^*(f, \Delta(z, r))$?

(3.4) For each $z \in D$ and each $r \in (0,1)$, we have

$$\left(1 - |z|^2\right)(\partial_H f)(z) \leq \Phi_r\left(A_H^*(f, \Delta(z, r))\right) \leq 1,$$

where $\Phi_r(x) = r^{-1} x^{1/2} (x + \pi)^{-1/2}$, $x \geq 0$; see [Y22, Theorem 1]; see also [Y15]. One might say that this is an improvement of the Pick version of the Schwarz lemma.

All the equalities hold in case f is a conformal homeomorphism from D onto D.

4. X-**boundedness and the family** BC_X. Let $f\colon D \to S_X$, $w \in D$, and $0 < r \leq 1$. Set

$$T_X(w,f,r) = \pi^{-1}\int_0^r t^{-1}A_X(f,\Delta(w,t))dt$$

and set $T_X(w,f) = T_X(w,f,1)$. Recall that $T_E(0,f,r)$ is the so-called Shimizu-Ahlfors characteristic function of $f\colon D \to \mathbb{C}^*$: see [N, p. 177], where the term "sphärische Normalform der Charakteristik" is used. One can observe that $T_X(w,f,r) = T_X(0,f_w,r)$, where $f_w(z) = f\Big((z + w)/(1 + \bar{w}z)\Big)$, $z \in D$. We can show that $T_X(w,f)$ is either constantly $+\infty$ or the Green potential of the measure $(\partial_X f)^2(z)dxdy$ in D, namely,

$$T_X(w,f) = \iint_D g_D(z,w)(\partial_X f)^2(z)dxdy < +\infty,$$

at each $w \in D$ with $g_D(z,w) = -\log \delta_H(z,w)$, the Green function of D with its pole at w.

A meromorphic function $f\colon D \to S_X$ is called X-bounded, $f \in BC_X$ in notation, if there exists $w \in D$ such that $T_X(w,f) < +\infty$. It is easily proved that if $f \in BC_X$, then $T_X(w,f) < +\infty$ at each $w \in D$. Furthermore, $BC_H \subset B_H$ is trivial. There is no inclusion relation between BC_X and B_X for $X = E$, P.

The family BC of functions of bounded Nevanlinna charcteristic in D [N, p. 185 *et seq.*] is just BC_E. The Hardy class H^2 in D is just BC_P; see [Le] and [Y29]. Finally, the hyperbolic Hardy class H^1_σ (or H^1_h) [Y3, 8, 11, 36] is nothing but BC_H.

The celebrated F. Riesz theorem shows that if a subharmonic function F in D has a harmonic function h as a majorant, that is, $F \le h$ in D, then the least harmonic majorant $F^\#$ of F, the smallest among all the harmonic majorants of F in D exists and $F^\# - F$ is a Green potential with a suitable measure in D.

The problem is therefore whether there exists a subharmonic function $F_{X,f}$ relating to f: $D \to S_X$ such that $f \in BC_X$ if and only if $F_{X,f}$ has a harmonic majorant in D and furthermore, $F^\#_{X,f}(w) - F_{X,f}(w) = T_X(w,f)$ everywhere in D. The answer is in the positive.

<u>Case</u> $X = E$: Each meromophic function f in D can be expressed as a quotient $f = f_1/f_2$ in D where f_1 and f_2 are single-valued holomorphic functions without common zero in D. We choose *one* pair f_1, f_2, and set $F_{E,f} = 2^{-1}\log\left(|f_1|^2 + |f_2|^2\right)$. Thus the difference $F^\#_{E,f} - F_{E,f}$ is independent of the particular choice of a pair.

<u>Case</u> $X = P$: Let $F_{P,f} = 2^{-1}|f|^2$.

<u>Case</u> $X = H$: Let $F_{H,f} = -\ 2^{-1}\log\left(1 - |f|^2\right)$.

It now follows from the Green formula, together with $\Delta F_{X,f} = 2(\partial_X f)^2$ (in particular, $F_{X,f}$ is subharmonic in D), that

$$2A_X(f,\Delta(0,t)) = \iint_{\Delta(0,t)} \Delta F_{X,f}(z)\,dxdy =$$

$$\int_0^{2\pi}\left((\partial/\partial t)F_{X,f}(te^{i\theta})\right)td\theta = t(d/dt)\int_0^{2\pi}F_{X,f}(te^{i\theta})d\theta.$$

Hence $T_X(0,f) = F^{\#}_{X,f}(0) - F_{X,f}(0)$ for $f\colon D \to S_X$. It is not difficulat to observe that $F^{\#}_{X,f_w}(0) = F^{\#}_{X,f}(w)$ and $F_{X,f_w}(0) = F_{X,f}(w)$, so that $T_X(w,f) = T_X(0,f_w) = F^{\#}_{X,f}(w) - F_{X,f}(w)$, $w \in D$.

To state area criteria we let $G \neq \emptyset$ be a subdomain of D and let $r\theta_G(r)$ be the length of the intersection $G \cap \{|z| = r\}$, $0 < r \leq 1$. We denote by $\mathcal{G}$ the family of subdomains $G \neq \emptyset$ such that 1 is on the boundary of G and $\theta_G(r)/(1 - r)$ is bounded and bounded away from zero as $r \to 1$. namely,

$$0 < \liminf_{r \to 1} \theta_G(r)/(1 - r), \quad \limsup_{r \to 1} \theta_G(r)/(1 - r) < +\infty.$$

Let $\mathcal{T}$ be the family of trianglular domains Δ at 1, that is, Δ is the interior of a triangle $\subset D$ with one vertex at 1. Then $\mathcal{T} \subset \mathcal{G}$. Let $G_\zeta = \{\zeta z;\ z \in G\}$ be the rotation of $G \in \mathcal{G}$, $\zeta \in \partial D$.

Theorem III. *Let $f\colon D \to S_X$ be meromorphic. Then $f \in BC_X$ if and only if one of the following holds:*

(4.1) *For each $G \in \mathcal{G}$,*

$$\int_0^{2\pi} A_X(f,G_\zeta)d \arg \zeta < +\infty.$$

(4.2) *There exists $G \in \mathcal{G}$ such that*

$$\int_0^{2\pi} A_X(f,G_\zeta)\,d\arg\zeta < +\infty.$$

The case where $X = E$ and G is essentially resricted to be in $\mathcal{T}$ is due to V. I. Gavrilov [Gv, Theorem 2]. The case where $X = P$ and G is essentially retricted to be in $\mathcal{T}$ is partially due to N. N. Lusin [Lu, pp. 146-147]($f \in H^2 \Rightarrow$ (4.1) for each $G = \Delta \in \mathcal{T}$) and Gavrilov [Gv, Theorem 3] for the converse. The proof of the general form including the case $X = H$ as above is immediate in view of the paper [Y3]. See also [Y12] for $X = P$.

5. Uniform X-boundedness and the family UBC_X. We call a meromorphic f: $D \to S_X$ uniformly X-bounded, $f \in UBC_X$ in notation, if

$$\sup_{w\in D} T_X(w,f) < +\infty.$$

A potential-theoretic characterization is : $f \in UBC_X$ if and only if $F^{\#}_{X,f}$ exists (this is the case if and only if $f \in BC_X$) and the Green potential $F^{\#}_{X,f}(w) - F_{X,f}(w) = T_X(w,f)$ is bounded in D.

Case $X = E$: The family $UBC = UBC_E$ is introduced in [Y16]; see also [Y19, 23, 24, 25, 26, 27, 32, 39, 40].

Case $X = H$: The family $BMOA_\sigma = UBC_H$ is introduced in [Y31]; see also [Y39].

Case $X =$ P: The family $BMOA = UBC_P$ is known as the family of holomorphic functions of bounded mean oscillation in D. Let $|J| > 0$ be the length of an open subarc $J \subset \partial D$. For a Lebesgue integrable, complex-valued function φ on ∂D we then set $J(\varphi) = |J|^{-1}\int_J \varphi(\zeta)|d\zeta|$, the mean of φ on J. Then φ is called of bounded mean oscillation, or $\varphi \in BMOA(\partial D)$ in notation, if $J(|\varphi - J(\varphi)|)$, the mean oscillation of φ on J, remains bounded as J ranges over all open subarcs of ∂D. A holomorphic function f in D is called $BMOA$ if (1) f has a radial limit $\hat{f}(\zeta) = \lim_{r\to 1-0} f(r\zeta) \in \mathbb{C}$ at almost every point $\zeta \in \partial D$; (2) $\hat{f}$ is of bounded mean oscillation on ∂D; ; and (3) f is the Poisson integral of $\hat{f}$, namely.

$$f(z) = (2\pi)^{-1}\int_0^{2\pi}\hat{f}(\zeta)\left(1 - |z|^2\right)/|\zeta - z|^2 d \arg \zeta \quad \left(\zeta \in \partial D\right)$$

at all $z \in D$. One can prove that a holomorphic function f is in $BMOA$ if and only if

$$\sup_{w\in D} \left(|f_w - f(w)|^2\right)^{\#}(0) < +\infty,$$

Thus,

$$\left(|f_w - f(w)|^2\right)^{\#}(0) = (2\pi)^{-1}\int_0^{2\pi}\left|\hat{f}\left((\zeta + w)/(1 + w\zeta)\right) - f(w)\right|^2 d \arg \zeta.$$

Ignoring the priority of the term UBC in [Y16] some authors try to rename it $BMOM$ (M: meromorphic). However, for a meromorphic function f in D to be in UBC its boundary value does not play a decisive role. There exists a Blaschke products b_1 and b_2 such that $f = b_1/b_2$ is not in UBC; see [Y16, 25], yet, obviously $|\hat{f}| = 1$ a.e. Trivially $f \in UBC$ cannot, in general, be reproduced as the Poisson integral of $\hat{f}$. It would be interesting if one can show that each $f \in UBC$ has "bounded mean oscillation" in a reasonable sense.

As to $UBC_H = BMOA_\sigma$ we have the following [Y31]: A holomorphic function $f\colon D \to D$ is in $BMOA_\sigma$ if and only if

$$\sup_{w\in D} \left(d_H(f_w, f(w))\right)^{\#}(0) < +\infty,$$

Let $E(f,k)$ be the set of points $a \in \mathbb{C}^*$ such that the equation $f(z) = a$ for a meromorphic $f\colon D \to S_E$ has at most $k \geq 0$ roots in D, counted according to multiplicities. It is well known that if the logarithmic capacity of $E(f,k)$ is positive for a $k \geq 0$, then $f \in BC$ [N, p. 213]. We can further show that $f \in UBC$ [Y19].

We propose here relations among three families, UBC_X, B_X, and BC_X for $X = P$, E, respectively.

Theorem IV. *Let $X = E$, P. Then UBC_X is strictly contained in the intersection $B_X \cap BC_X$.*

The case $X = P$ is in [CCS]. For the case $X = E$ we have a holomorphic function f in D such that (1) $f \in N$; (2) f is in all the Hardy class H^p, $0 < p < +\infty$; (3) f is not in UBC [Y19].

For $w \in D$, $w \neq 0$, we consider the annular trapezium:

$$\mathfrak{R}(w) = \{z;\ |w| < |z| < 1 \text{ and } |\arg(z/w)| < \pi(1 - |w|)\}$$

and $\mathfrak{R}(0) = D$. Then $|\partial D \cap \partial\mathfrak{R}(w)| = 2\pi(1 - |w|)$. A measure m (≥ 0) in D is then called a Carleson measure if

$$\sup_{w\in D} m\left(\mathfrak{R}(w)\right)/\left[2\pi(1 - |w|)\right] < +\infty.$$

We then have

Theorem V. *Let f: $D \to S_X$ be meromorphic. Then $f \in UBC_X$ if and only if the measure* $\left(1 - |z|^2\right)(\partial_X f)(z)^2 dxdy$ *($z = x + iy$; in the differential form) is a Carleson measure.*

Case $X = P$: See [Gr, p. 240], for example. Case $X = H$: [Y31, Theorem 1]. Case $X = E$: "only if" [Y24, Theorem 2]. Depending on some results of the present author, Ž. Pavićević [P, Theorem 3] found a proof of the "if" part; see [Y27, Problem (VI)].

6. **Relation between ∂_X-boundedness and local X-boundedness.** Let $D(w,r)$ $\{z;\ |z - w| < r(1 - |w|)\}$, $w \in D$, $0 < r \leq 1$, and set

$$T_X^*(w,f,r) = \pi^{-1}\int_0^r t^{-1} A_X(f,D(w,t))dt$$

for f: $D \to S_X$. This is just $T_X(0,g,r)$ for $g(z) = f\left((1 - |w|)z + w\right)$, $z \in D$.

Theorem VI. [Y29] *Let f: $D \to S_X$ be meromorphic, $X = E$, P. Then $f \in B_X$ if and only if one of the following holds:*

(6.1) *For each $c \in (0,1)$,*

$$\sup_{c \leq |w| < 1} T_X^*(w,f,1) < +\infty.$$

(6.2) *There exist c and r in $(0,1)$ such that*

$$\sup_{c \le |w| < 1} T_X^*(w,f,r) < +\infty.$$

Thus a meromorphic function f in D is normal if and only if f is of BC "uniformly" in each disk $D(w,1)$ when w is near ∂D. We have the similar characterization in terms of H^2 for a holomorphic f to be Bloch, $f \in B$. How is the situation in case $X = H$? If $f\colon D \to D$ is holomorphic, then for each $c \in (0,1)$ we have

$$\sup_{c \le |w| < 1} T_H^*(w,f,1) \le c^{-1/2}(1 + c)^{-1}.$$

Thus f is in the hyperbolic Hardy class H_σ^1 "uniformly" in each $D(w,1)$ when w is near ∂D. The right-hand-side constant appears to be not sharp.

7. Table. We summarize the quantities and families we have proposed in the

TABLE

X	E	P	H
∂_X-bdd	N	B	SPL
X-bdd	BC	H^2	H_σ^1
un X-bdd	UBC	$BMOA$	$BMOA_\sigma$
$\partial_X f$	$\|f'\|/\left(1 + \|f\|^2\right)$	$\|f'\|$	$\|f'\|/\left(1 - \|f\|^2\right)$
$2F_{X,f}$	$\log\left(\|f_1\|^2 + \|f_2\|^2\right)$	$\|f\|^2$	$-\log\left(1 - \|f\|^2\right)$

There are analogues H_σ^p of H^p in the hyperbolic case, yet there is essentially no analogue of H^p in the elliptic case [Y9].

8. **Functional-analytic research.** Functional-analytic research, or "Algebra + Analysis" research in the field of Complex Analysis has succeeded in obtaining many results in many parts. However, among the families significant in Complex Analysis, even restricted to the ones we have hitherto proposed, *which is a linear space over* $\mathbb{C}$? The families B_H (trivially), $B_E = N$, $BC_H = H_\sigma^1$, $UBC_E = UBC$ and $UBC_H = BMOA_\sigma$ all are *not* closed even for the usual sum $f + g$. As a decisive example one can obviously propose the most important family of univalent meromorphic (or holomorphic) functions in D, which is not closed for the summation. (The functions $f(z) = z(z - 2)$ and $g(z) = 2z$ are univalent in D, yet $f(z) + g(z) = z^2$ is not.) Thus, analysts have been storing many purely analytic methods in Complex Analysis. P. Montel's normal family research is typical. "Non-modern" or "really analytic" methods are still alive and main roles are, of course, played by them.

The present survey article partially depends on the author's lecture, held on May 21, 1987, at Department of Mathematics, University of Helsinki, sponsored by the Finnish Mathematical Society. He would like to express his gratitude to all the staffs who organized the lecture.

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