数理解析研究所講究録
第 826 巻 1993 年 27-41

Associated Varieties and Gelfand-Kirillov Dimensions for the Discrete Series of a Semisimple Lie Group

To the memory of Mr. Atsushi Yamaguchi

BY

Hiroshi YAMASHITA
(京大・理　山下 博)

Introduction.

Let G be a connected semisimple Lie group with finite center, and K be a maximal compact subgroup of G. The corresponding complexified Lie algebras are denoted respectively by $\mathfrak{g}$ and $\mathfrak{k}$. We assume Harish-Chandra's rank condition $\operatorname{rank} G = \operatorname{rank} K$, which is necessary and sufficient for G to have a non-empty discrete series, consisting of square-integrable irreducible unitary representations of G ([4]).

Concrete geometric realizations of discrete series representations have been obtained in several ways (see e.g., the survey article [3] and the papers cited there). Among others, Hotta and Parthasarathy [6] realize such representations on the kernel spaces of certain G-invariant differential operators $\mathcal{D}_\lambda$ of gradient-type, defined on vector bundles over the symmetric space G/K, by using some elementary differential calculas on G/K (see §5). Here λ denotes the lowest highest weight of corresponding discrete series. As we have shown in [12], the operators $\mathcal{D}_\lambda$ allow us to determine the embeddings of discrete series into various important induced G-modules.

In this paper, we describe the associated varieties of Harish-Chandra $(\mathfrak{g}, K)$-modules of discrete series, by quite an elementary method based on the above work of Hotta-Parthasarathy. Our description is as in

Theorem. (Theorem 3.1) *If H_Λ is the $(\mathfrak{g}, K)$-module of discrete series with Harish-Chanda parameter $\Lambda = \lambda + \rho_c - \rho_n$ (see §2), then its associated variety $\mathcal{V}(H_\Lambda) \subset \mathfrak{g}$ (see §1 for the definition) coincides with the (Zariski) closure of the nilpotent cone $K_C\mathfrak{p}_-$. Here K_C is the analytic subgroup of adjoint group $G_C := \operatorname{Int}(\mathfrak{g})$ of $\mathfrak{g}$, with Lie algebra $\mathfrak{k}$, and $\mathfrak{p}_-$ denotes the sum of root subspaces of $\mathfrak{g}$ corresponding to the non-compact roots which are negative with respect to Λ (see (3.1)).*

This theorem enables us to deduce that the variety $\mathcal{V}(U(\mathfrak{g})/I_\Lambda)$ associated to primitive ideal $I_\Lambda := \operatorname{Ann}_{U(\mathfrak{g})}(H_\Lambda)$ in the enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ is just the closure of the cone $G_C\mathfrak{p}_-$ (Theorem 3.2). We further derive an explicit (recursion) formula for the Gelfand-Kirillov dimensions $d(H_\Lambda) := \dim \mathcal{V}(H_\Lambda)$ of discrete series in the case of unitary groups $G = SU(p,q)$ (Theorem 8.1 and Corollary 8.1).

To prove the above Theorem 3.1, we pass to the space of coefficients of Taylor expansions of analytic sections in $\operatorname{Ker} \mathcal{D}_\lambda$. This space of coefficients admits a natural $S(\mathfrak{g})$-module structure, where $S(\mathfrak{g}) \simeq \operatorname{gr} U(\mathfrak{g})$ denotes the symmetric algebra of $\mathfrak{g}$. By using Theorem 1 and Lemma 5.2 of [6], we can show that, if the parameter λ is sufficiently regular, the corresponding annihilator ideal in $S(\mathfrak{g})$ defines the associated variety of discrete series as the set of common zero points. With in mind the Zuckerman translation principle, Theorem 3.1 follows by examining this annihilator a little more closely.

One may say that the above description of $\mathcal{V}(H_\Lambda)$ is known among the specialists of D-module theory. This is because: (a) the associated variety of a Harish-Chandra module is gained, through the moment map, as the image of characteristic variety of corresponding D-module over the complexified flag variety X of G (see [2, III]), and (b) the characteristic variety of a discrete series D-module can be specified as a conormal bundle on X.

However, we can not find good and self-contained references for Theorem 3.1. Moreover these (a) and (b) rely on several deep results about the classification of irreducible G-representations through D-modules, K_C-orbit structure of the variety X, etc., although the associated variety is a very simple object defined for each finitely generated $U(\mathfrak{g})$-module in a purely algebraic context (see §1).

From this reason, we make here a short-cut and describe directly our variety $\mathcal{V}(H_\Lambda)$ only by using some basic facts on realization of discrete series. Here are placed our motivation and emphasis of this presentation.

The organization of this paper is as follows. We begin with introducing in §§1-2 three principal objects of our concern: the associated variety, Gelfand-Kirillov dimension for $U(\mathfrak{g})$-module; and the discrete series for G. In §3, our main theorem for the variety $\mathcal{V}(H_\Lambda)$ is given as Theorem 3.1, and then we deduce from it two important consequences (Theorem 3.2 and Proposition 3.2). The succeeding four sections, §§4-7, are devoted to proving Theorem 3.1, where we are based on the excellent work [6]. The last section, §8, gives an explicit formula for the Gelfand-Kirillov dimensions $d(H_\Lambda)$. We concentrate on the groups $G = SU(p,q)$, where p and q range over non-negative integers such that $(p,q) \neq (0,0)$. Our formula obtained in Theorem 8.1 is recursive with respect to the parameter $n = p + q$.

An enlarged version of this article, with complete proofs, will appear elsewhere.

ACKNOWLEDGEMENTS. The author is grateful to Professor Y.Benoist for the communication concerning the above mentioned facts (a) and (b).

1. Associated varieties for $U(\mathfrak{g})$-modules.

Let $\mathfrak{g}$ be a finite-dimensional complex Lie algebra, and $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. We begin with introducing two important invariants: the associated variety and Gelfand-Kirillov dimension, for finitely generated $U(\mathfrak{g})$-modules.

Denote by $(U_k(\mathfrak{g}))_{k=0,1,..}$ the natural increasing filtration of $U(\mathfrak{g})$, where $U_k(\mathfrak{g})$ is the subspace of $U(\mathfrak{g})$ generated by elements $X_1 \cdots X_m$ $(m \leq k)$ with $X_j \in \mathfrak{g}$ $(1 \leq j \leq m)$. By the Poincaré-Birkhoff-Witt theorem, we can and do identify the associated graded ring

$$\operatorname{gr} U(\mathfrak{g}) = \bigoplus_{k \geq 0} U_k(\mathfrak{g})/U_{k-1}(\mathfrak{g}) \quad (U_{-1}(\mathfrak{g}) := (0))$$

with the symmetric algebra $S(\mathfrak{g}) = \bigoplus_k S^k(\mathfrak{g})$ of $\mathfrak{g}$ in the canonical way. Here $S^k(\mathfrak{g})$ denotes the homogeneous component of $S(\mathfrak{g})$ of degree k.

Let H be a finitely generated $U(\mathfrak{g})$-module. Take a finite-dimensional subspace H_0 of H such that $H = U(\mathfrak{g})H_0$. Setting $H_k = U_k(\mathfrak{g})H_0$ $(k = 1, 2, ...)$, one gets an increasing filtration $(H_k)_k$ of H and correspondingly a finitely generated, graded $S(\mathfrak{g})$-module

$$M = \mathrm{gr}(H; H_0) := \bigoplus_{k \geq 0} M_k \quad \text{with } M_k = H_k/H_{k-1}. \tag{1.1}$$

The annihilator $\mathrm{Ann}_{S(\mathfrak{g})}M := \{D \in S(\mathfrak{g}) |\ Dv = 0\ (\forall v \in M)\}$ of M is a graded ideal of $S(\mathfrak{g})$, and it defines an algebraic cone in the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$:

$$\mathcal{V}(M) := \{\lambda \in \mathfrak{g}^* |\ f(\lambda) = 0\ (\forall f \in \mathrm{Ann}_{S(\mathfrak{g})}M)\},$$

as the set of common zeros of elements of $\mathrm{Ann}_{S(\mathfrak{g})}M$. Here $S(\mathfrak{g})$ is looked upon as the polynomial ring over $\mathfrak{g}^*$ in the canonical way. It is then easily seen that the variety $\mathcal{V}(M)$ does not depend on the choice of a generating subspace H_0. So, hereafter we write $\mathcal{V}(H)$ for this invariant $\mathcal{V}(M)$ of H.

Definition. (Cf. [9], [13]) For a finitely generated $U(\mathfrak{g})$-module H, the variety $\mathcal{V}(H) \subset \mathfrak{g}^*$ and its dimension $d(H) := \dim \mathcal{V}(H)$ are called respectively the *associated variety* and the *Gelfand-Kirillov dimension* of H.

It should be noticed that, by the Hilbert-Serre theorem (cf. [13, Th.1.1]), the map $k \to \dim H_k$ coincides with a polynomial in k of degree $d(H)$ for sufficiently large k.

2. Discrete series for a semisimple Lie group.

Let G be a connected semisimple Lie group with finite center, and K be a maximal compact subgroup of G. The corresponding Lie algebras are denoted respectively by $\mathfrak{g}_0$ and $\mathfrak{k}_0$. Then one has a Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ of $\mathfrak{g}_0$. We always assume the rank condition $\mathrm{rank}\, G = \mathrm{rank}\, K$, which is necessary and sufficient for G to have a non-empty discrete series. In this section we collect some basic facts and fix notations on the discrete series representations of G.

Take a maximal abelian subalgebra $\mathfrak{t}_0$ of $\mathfrak{k}_0$, which is, by the above assumption on G, a compact Cartan subalgebra of $\mathfrak{g}_0$. Let $\mathfrak{g}$ denote the complexification of $\mathfrak{g}_0$, and we write $\mathfrak{h} \subset \mathfrak{g}$ for the complexification of a real vector subspace $\mathfrak{h}_0$ of $\mathfrak{g}_0$ by dropping the subscript '0'. If $\alpha \in \mathfrak{t}^*$ is a root of $\mathfrak{g}$ with respect to $\mathfrak{t}$, the corresponding root subspace

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} |\ [H, X] = \alpha(H)X \quad (\forall H \in \mathfrak{t})\}$$

is contained either in $\mathfrak{k}$ or in $\mathfrak{p}$. A root α is said to be compact or non-compact according as $\mathfrak{g}_\alpha \subset \mathfrak{k}$ or $\mathfrak{g}_\alpha \subset \mathfrak{p}$. We denote the totality of roots (resp. compact roots, non-compact roots) by Δ (resp. Δ_c, Δ_n).

Now fix a positive system Δ_c^+ of Δ_c. Let Ξ be the set of linear forms Λ on $\mathfrak{t}$ satisfying the following three conditions:

(2.1) $(\Lambda, \alpha) \geq 0$ for any $\alpha \in \Delta_c^+$, i.e., Λ is Δ_c^+-dominant.

(2.2) $(\Lambda, \alpha) \neq 0$ for any $\alpha \in \Delta$, i.e., Λ is Δ-regular,

(2.3) the map $H \rightarrow \exp < \Lambda + \rho, H >$ $(H \in \mathfrak{t}_0)$ defines a well-defined unitary character of the Cartan subgroup $T := \exp \mathfrak{t}_0$, i.e., $\Lambda + \rho$ is T-integral.

Here $(\cdot, \cdot)$ denotes the bilinear form on $\mathfrak{t}^*$ induced canonically from the Killing form of $\mathfrak{g}$ restricted to $\mathfrak{t}$, and ρ is half the sum of positive roots in Δ with respect to any fixed positive system of Δ. Notice that the condition (2.3) does not depend on the choice of positive system which defines ρ.

By Harish-Chandra, there exists a bijective correspondance, say $\Lambda \rightarrow \pi_\Lambda$, from Ξ onto the set of (equivalence classes) of discrete series representations of G (see e.g., [12, I, Prop.1.1]). We say that the discrete series representation π_Λ has *Harish-Chandra parameter* Λ.

What is more important in this article is however the lowest K-type property which characterizes the discrete series π_Λ. To be precise, for a Δ_c^+-dominant, T-integral linear form $\mu \in \mathfrak{t}^*$, let (τ_μ, V_μ) denote the finite-dimensional irreducible K-module with highest weight μ. We set for a $\Lambda \in \Xi$,

$$\lambda := \Lambda - \rho_c + \rho_n = (\Lambda - 2\rho_c) + \rho = (\Lambda + 2\rho_n) - \rho, \tag{2.4}$$

where half the sum ρ of positive roots is defined by the positive system $\Delta^+ := \{\alpha \in \Delta |\ (\Lambda, \alpha) > 0\}$, and $\rho_c := (1/2) \sum_{\alpha \in \Delta_c^+} \alpha$, $\rho_n := \rho - \rho_c$.

Proposition 2.1. (See e.g., [3]) (i) *The discrete series representation π_Λ, looked upon as a K-module, has lowest K-type τ_λ:*

(a) *π_Λ contains τ_λ with multiplicity one,*

(b) *the highest weight of any irreducible K-representation occuring in π_Λ is of the form*

$$\lambda + \sum_{\alpha \in \Delta^+} n_\alpha \alpha$$

with non-negative integers n_α.

(ii) *Conversely, if an irreducible unitary representation π of G satisfies* (a) *and* (b), *then π is unitarily equivalent to π_Λ.*

Suggested by this proposition, we call $\lambda = \Lambda - \rho_c + \rho_n$ the *lowest highest weight* (or the *Blattner parameter*) of π_Λ.

3. Description of the associated varieties for discrete series.

We now present the main result (Theorem 3.1) of this paper and deduce from it two important consequences (Theorem 3.2 and Proposition 3.2), concerning the associated varieties and Gelfand-Kirillov dimensions for the discrete series.

3.1. Varieties $\mathcal{V}(H_\Lambda)$ and $\mathcal{V}(U(\mathfrak{g})/I_\Lambda)$. For a $\Lambda \in \Xi$, let H_Λ be the Harish-Chandra $(\mathfrak{g}, K)$-module corresponding to π_Λ, which is gained by passing to the K-finite part of π_Λ. It follows that H_Λ is irreducible as a $U(\mathfrak{g})$-module because of the irreducibility of the corresponding G-representation π_Λ. See e.g., [11, I, 2.4] for the definition and basic facts on Harish-Chandra $(\mathfrak{g}, K)$-modules.

Now we put

$$\mathfrak{p}_\pm := \bigoplus_{\alpha \in \Delta_n^+} \mathfrak{g}_{\pm\alpha}, \tag{3.1}$$

where $\Delta_n^+ = \{\alpha \in \Delta_n | \ (\Lambda, \alpha) > 0\}$ denotes the set of non-compact positive roots with respect to Λ. Notice that the subspaces $\mathfrak{p}_\pm$ depend only on the chamber in which the Harish-Chandra parameter Λ lives. Let G_C be the adjoint group of $\mathfrak{g}$, and K_C be the analytic subgroup of G_C corresponding to the Lie subalgebra $\mathfrak{k}$.

We can describe the associated variety $\mathcal{V}(H_\Lambda)$ of H_Λ by means of the subspace $\mathfrak{p}_-$, as in

Theorem 3.1. *The associated variety $\mathcal{V}(H_\Lambda)$ of discrete series Harish-Chandra module H_Λ coincides with the Zariski closure of the nilpotent cone $K_C\mathfrak{p}_-$. Here $\mathcal{V}(H_\Lambda)$ is regarded as a variety in $\mathfrak{g}$ by identifying $\mathfrak{g}^*$ with $\mathfrak{g}$ through the Killing form of $\mathfrak{g}$.*

We will prove this theorem in the succeeding sections, §§4-7, by using the gradient-type differential operators on G/K whose kernels realize the discrete series representations of G (cf. [6]).

The above theorem allows us to describe also the variety $\mathcal{V}(U(\mathfrak{g})/I_\Lambda)$ associated to the primitive ideal $I_\Lambda := \mathrm{Ann}_{U(\mathfrak{g})}H_\Lambda$, as follows.

Theorem 3.2. *One has the equality $\mathcal{V}(U(\mathfrak{g})/I_\Lambda) = \overline{G_C\mathfrak{p}-}$, where $\overline{A}$ denotes the Zariski closure of a subset A of $\mathfrak{g}$, and $U(\mathfrak{g})$ acts on $U(\mathfrak{g})/I_\Lambda$ by left multiplication.*

This theorem is a direct consequence of Theorem 3.1 together with the following proposition.

Proposition 3.1. *Let H be an irreducible $(\mathfrak{g}, K)$-module and $I = \mathrm{Ann}_{U(\mathfrak{g})}H$ be the corresponding primitive ideal of $U(\mathfrak{g})$. Then variety $\mathcal{V}_I := \mathcal{V}(U(\mathfrak{g})/I)$ is related to the associated variety $\mathcal{V}(H)$ of H as*

$$\mathcal{V}_I = \overline{G_C\mathcal{V}(H)}. \tag{3.2}$$

3.2. The proof of Proposition 3.1 requires four fundamental facts concerning the nilpotent G_C- or K_C-orbits, associated varieties and primitive ideals, which we are going to list up.

Lemma 3.1. (Cf. [13, Lemma 3.1]) *Let $\mathcal{N}$ be the variety of all nilpotent elements of $\mathfrak{g}$, and put $\mathcal{N}(\mathfrak{p}) := \mathcal{N} \cap \mathfrak{p}$. If H and $I = \mathrm{Ann}_{U(\mathfrak{g})}H$ are as in Proposition 3.1, the variety $\mathcal{V}_I$ (resp. $\mathcal{V}(H)$) is a G_C-stable (resp. K_C-stable) cone contained in $\mathcal{N}$ (resp. in $\mathcal{V}_I \cap \mathfrak{p} \subset \mathcal{N}(\mathfrak{p})$).*

Lemma 3.2. (Joseph, cf. [8, Th.3.1]) *For the above H and I, one has the equality* $\dim \mathcal{V}_I = 2\dim \mathcal{V}(H)$.

Lemma 3.3. (See e.g., [2, III]) *The variety $\mathcal{V}_I$ associated to a primitive ideal $I = \mathrm{Ann}_{U(\mathfrak{g})}H \subset U(\mathfrak{g})$ is the closure of a single nilpotent G_C-orbit $\mathcal{O}_1$ in $\mathfrak{g}$: $\mathcal{V}_I = \overline{\mathcal{O}_1}$.*

Lemma 3.4. *If $\mathcal{O}$ is a nilpotent K_C-orbit in $\mathfrak{p}$, the dimension of G_C-orbit $\mathcal{O}_1 := G_C\mathcal{O}$ containing $\mathcal{O}$, equals $2\dim\mathcal{O}$.*

Remark. The varieties $\mathcal{V}(H)$, $\mathcal{V}_I$ are closely related to the asymptotic support and wave front set of the distribution character of H ([1]; see also [10]).

Proof of Proposition 3.1. The inclusion $\overline{G_C\mathcal{V}(H)} \subset \mathcal{V}_I$ in (3.2) is clear from Lemma 3.1. To show the converse inclusion, take a nilpotent K_C-orbit $\mathcal{O}$ in $\mathfrak{p}$ such that $\dim\mathcal{V}(H) = \dim\mathcal{O}$. Such an $\mathcal{O}$ actually exists since the number of nilpotent K_C-orbits in $\mathfrak{p}$ is finite (see [5, Chap.III, Th.4.8]). Set $\mathcal{O}_1 = G_C\mathcal{O}$ ($\subset \mathcal{V}_I$). Then it follows from Lemmas 3.2 and 3.4 that

$$\dim\mathcal{O}_1 = 2\dim\mathcal{O} = 2\dim\mathcal{V}(H) = \dim\mathcal{V}_I.$$

Hence $\mathcal{O}_1$ is an open subset of $\mathcal{V}_I$. By virtue of Lemma 3.3, we conclude that $\mathcal{V}_I = \overline{\mathcal{O}_1} \subset \overline{G_C\mathcal{V}(H)}$. This completes the proof of Proposition 3.1. Q.E.D.

3.3. Theorem 3.2, combined with Lemmas 3.2 and 3.3, gives the following proposition, which will be useful for computing explicitly the Gelfand-Kirillov dimensions for the discrete series (see §8).

Proposition 3.2. *For a $\Lambda \in \Xi$, define a subspace $\mathfrak{p}_- \subset \mathcal{N}(\mathfrak{p})$ as in* (3.1).

(i) *If $\Omega_{\mathfrak{p}_-}$ denotes the set of nilpotent G_C-orbits $\mathcal{O}_1$ in $\mathfrak{g}$ such that $\mathcal{O}_1 \cap \mathfrak{p}_- \neq \emptyset$, there exists a unique orbit $\mathcal{O}_{\mathfrak{p}_-} \in \Omega_{\mathfrak{p}_-}$ for which $\overline{\mathcal{O}_{\mathfrak{p}_-}} \supset \mathcal{O}_1$ holds for any $\mathcal{O}_1 \in \Omega_{\mathfrak{p}_-}$.*

(ii) *The Gelfand-Kirillov dimension $d(H_\Lambda)$ of discrete series $U(\mathfrak{g})$-module H_Λ coincides with* $(1/2)\dim\mathcal{O}_{\mathfrak{p}_-}$.

4. Associated varieties and realization of Harish-Chandra modules on G/K.

For a finite-dimensional representation (τ, V_τ) of K, let $\mathcal{A}(\tau)$ be the space of real analytic functions $f : G \to V_\tau$ satisfying

$$f(gk) = \tau(k)^{-1}f(g) \quad (g \in G, k \in K). \tag{4.1}$$

The group G acts on $\mathcal{A}(\tau)$ by left translation, and $\mathcal{A}(\tau)$ admits a $U(\mathfrak{g})$-module structure through differentiation. We call $\mathcal{A}(\tau)$ the G- and $U(\mathfrak{g})$-module analytically induced from τ.

This section develops a general method for describing the associated variety $\mathcal{V}(H)$ of a Harish-Chandra module H in relation with a realization of its K-finite dual module H^* in $\mathcal{A}(\tau)$. This is a preliminary step for the proof of Theorem 3.1.

4.1. **$(S(\mathfrak{g}), K)$-module** $\mathrm{Gr}\,\mathcal{A}(\tau)$. At first, we define subspaces $\mathcal{A}_{(k)}$ $(k \in \mathbf{Z})$ of $\mathcal{A}(\tau)$ by

$$(4.2) \qquad \mathcal{A}_{(k)} := \{f \in \mathcal{A}(\tau) | \ (X^m f)(1) = (0) \ \ (\forall X \in \mathfrak{p}, \ 0 \leq \forall m \leq k)\}$$

for $k \geq 0$, and $\mathcal{A}_{(k)} := \mathcal{A}(\tau)$ for $k < 0$, where 1 denotes the identity element of G. Then $(\mathcal{A}_{(k)})_{k \in \mathbf{Z}}$ is a decreasing filtration of $\mathcal{A}(\tau)$ such that

$$(4.3) \qquad \text{each } \mathcal{A}_{(k)} \text{ is a } K\text{-stable subspace of } \mathcal{A}(\tau),$$

$$(4.4) \qquad \dim \mathcal{A}(\tau)/\mathcal{A}_{(k)} < \infty \quad \text{and} \quad \bigcap_k \mathcal{A}_{(k)} = (0),$$

$$(4.5) \qquad U_m(\mathfrak{g})\mathcal{A}_{(k)} \subset \mathcal{A}_{(k-m)} \quad \text{for all integers } k, m \geq 0.$$

Correspondingly, one obtains a graded $S(\mathfrak{g})$-module

$$(4.6) \qquad \mathrm{Gr}\,\mathcal{A}(\tau) := \bigoplus_k \mathcal{A}_{(k)}/\mathcal{A}_{(k+1)},$$

which admits by (4.3) a K-module structure, compatible with the $S(\mathfrak{g})$-action.

It is not difficult to analyze this $(S(\mathfrak{g}), K)$-module. To do this, let $(X_i)_{i=0}^s$ and $(X_i^*)_{i=0}^s$ be two bases of the vector space $\mathfrak{p}$ such that $B(X_i, X_j^*) = \delta_{ij}$ (the Kronecker δ) for the Killing form B of $\mathfrak{g}$. We put

$$(4.7) \qquad \iota_k(f) := \sum_{|\alpha|=k+1} \frac{1}{\alpha!}(X^*)^\alpha \otimes (X^\alpha f)(1) \in S^{k+1}(\mathfrak{p}) \otimes V_\tau \quad (f \in \mathcal{A}_{(k)}),$$

where $X^\alpha := X_1^{\alpha_1} \cdots X_s^{\alpha_s}$ and $(X^*)^\alpha := (X_1^*)^{\alpha_1} \cdots (X_s^*)^{\alpha_s}$ for multi-indices $\alpha = (\alpha_1, \ldots, \alpha_s)$ of length $|\alpha| := \alpha_1 + \cdots + \alpha_s = k+1$. Observe that the assignment $\mathcal{A}_{(k)} \ni f \to \iota_k(f) \in S^{k+1}(\mathfrak{p}) \otimes V_\tau$ is independent of the choice of of dual bases $(X_i)_i$ and $(X_i^*)_i$, and ι_k naturally gives rise to a K-isomorphism:

$$(4.8) \qquad \tilde{\iota}_k : \mathcal{A}_{(k)}/\mathcal{A}_{(k+1)} \simeq S^{k+1}(\mathfrak{p}) \otimes V_\tau,$$

where $S^{k+1}(\mathfrak{p})$ is looked upon as a K-module by the adjoint action.

Through the Killing form B, we identify the symmetric algebra $S(\mathfrak{p}) = \bigoplus_k S^k(\mathfrak{p})$ of $\mathfrak{p}$ with the ring of polynomial functions on $\mathfrak{g}$ which vanish identically on $\mathfrak{k}$. Let $S(\mathfrak{g})$ act on $S(\mathfrak{p})$ canonically as the ring of constant coefficient differential operators on the vector space $\mathfrak{g}$.

Summing up the isomorphisms $\tilde{\iota}_k$ $(k \in \mathbf{Z})$ in (4.8), one obtains the following lemma which describes the structure of $\mathrm{Gr}\,\mathcal{A}(\tau)$ in a simpler way.

Lemma 4.1. *The map* $\tilde{\iota} := \oplus_k \tilde{\iota}_k$ *gives a graded* $(S(\mathfrak{g}), K)$*-module isomorphism from* $\mathrm{Gr}\,\mathcal{A}(\tau)$ *onto the tensor product* $S(\mathfrak{p}) \otimes V_\tau$*, where* $S(\mathfrak{g})$ *acts on* V_τ *trivially.*

4.2. Variety $\mathcal{V}(H)$ in relation with $\mathrm{Gr}_\gamma(H^*)$. Now let H be an irreducible $(\mathfrak{g}, K)$-module. Then the full dual space H' of H, consisting of all linear forms on H, has a $(\mathfrak{g}, K)$-module structure contragredient to H. The K-finite part of H', say H^*, is an irreducible $(\mathfrak{g}, K)$-submodule of H'.

If (τ, V_τ) is a finite-dimensional K-module occuring in H^*, there exists, by a reciprocity theorem of Frobenius type, a $(\mathfrak{g}, K)$-module embedding γ from H^* into the analytically induced module $\mathcal{A}(\tau)$. Setting

$$H^*_{(k),\gamma} := \gamma(H^*) \cap \mathcal{A}_{(k)} \quad (k \in \mathbf{Z}) \tag{4.9}$$

with $\mathcal{A}_{(k)} \subset \mathcal{A}(\tau)$ in (4.2), we get a decreasing filtration $(H^*_{(k),\gamma})_k$ of $\gamma(H^*) \simeq H^*$ with properties (4.3)-(4.5). Write $\mathrm{Gr}_\gamma(H^*)$ for the corresponding $(S(\mathfrak{g}), K)$-module:

$$\bigoplus_k H^*_{(k),\gamma}/H^*_{(k+1),\gamma} \subset \mathrm{Gr}\,\mathcal{A}(\tau). \tag{4.10}$$

On the other hand, the filtration $(H^*_{(k),\gamma})_k$ of H^* gives rise to an increasing filtration $(H_{k,\gamma})_k$ of H with

$$H_{k,\gamma} := \{v \in H | \ < w^*, v >= 0 \ \ (\forall w^* \in H^*_{(k),\gamma})\}, \tag{4.11}$$

by passing to the orthogonal in H. If

$$\mathrm{gr}_\gamma(H) := \bigoplus_k H_{k+1,\gamma}/H_{k,\gamma} \tag{4.12}$$

denotes the corresponding graded $(S(\mathfrak{g}), K)$-module, the dual pairing $< \cdot, \cdot >$ on $H^* \times H$ naturally induces a non-degenerate $(S(\mathfrak{g}), K)$-invariant pairing on $\mathrm{Gr}_\gamma(H^*) \times \mathrm{gr}_\gamma(H)$. By using the latter pairing, one easily finds that

$$\mathrm{Ann}_{S(\mathfrak{g})}\mathrm{Gr}_\gamma(H^*) = \mathrm{Ann}_{S(\mathfrak{g})}\mathrm{gr}_\gamma(H), \tag{4.13}$$

and that

$$\mathrm{gr}_\gamma(H) = \mathrm{gr}(H; H_{0,\gamma}) \quad (\text{see } (1.1)). \tag{4.14}$$

We have thus obtained the following proposition, which enables us to describe the associated variety $\mathcal{V}(H)$ of Harish-Chandra module H by means of the annihilator of $\mathrm{Gr}_\gamma(H^*)$.

Proposition 4.1. *Under the above notation one has the equality*

$$\mathcal{V}(H) = \{X \in \mathfrak{g} | \ \ f(X) = 0 \text{ for all } f \in \mathrm{Ann}_{S(\mathfrak{g})}(\mathrm{Gr}_\gamma(H^*))\}. \tag{4.15}$$

5. Graded modules Gr H_Λ and differential operators $\mathcal{D}_\lambda$ of gradient-type.

Let H_Λ be the $(\mathfrak{g}, K)$-module of discrete series π_Λ with Harish-Chandra parameter $\Lambda \in \Xi$. Since the lowest K-type $(\tau_\lambda, V_\lambda)$, $\lambda = \Lambda - \rho_c + \rho_n$, appears in H_Λ with multiplicity one (see Proposition 1.1), there exists a unique ,up to scalar multiples, $(\mathfrak{g}, K)$-module embedding γ_λ from H_Λ into the analytically induced module $\mathcal{A}(\tau_\lambda)$.

This section interprets after Hotta-Parthasarathy [6], the $(S(\mathfrak{g}), K)$-module $\mathrm{Gr}\, H_\Lambda := \mathrm{Gr}_{\gamma_\lambda}(H_\Lambda)$ defined in 4.2, by means of the gradient-type differential operator $\mathcal{D}_\lambda$ whose kernel realizes π_Λ. Here we treat H_Λ itself instead of its dual $(\mathfrak{g}, K)$-module H_Λ^*, by noting that

$$H_\Lambda^* \simeq H_{-w_0\Lambda} \quad \text{as } (\mathfrak{g}, K)\text{-modules}, \tag{5.1}$$

for the longest element w_0 of the Weyl group of Δ_c.

5.1. Operator $\mathcal{D}_\lambda$ and realization of discrete series.

Let $(X_i)_{i=1}^s$ and $(X_i^*)_{i=1}^s$ be dual basis of $\mathfrak{p}$ as in 4.1. We set for $f \in \mathcal{A}(\tau_\lambda)$,

$$\nabla_\lambda f(g) := \sum_{i=1}^{s} R_{X_i} f(g) \otimes X_i^* \quad (g \in G), \tag{5.2}$$

where R_X denotes the left G-invariant vector field on G defined by

$$R_X f(g) := \frac{d}{dt}(f(g \exp tY) + \sqrt{-1} f(g \exp tZ))_{|t=0}$$

for $X = Y + \sqrt{-1}Z$ with $Y, Z \in \mathfrak{g}_0$. It is then easy to see that ∇_λ is independent of the choice of dual bases and that it defines a first order, left G-invariant differential operator from $\mathcal{A}(\tau_\lambda)$ to $\mathcal{A}(\tau_\lambda \otimes \mathrm{Ad}_\mathfrak{p})$. Here $\mathrm{Ad}_\mathfrak{p}$ denotes the adjoint representation of K on $\mathfrak{p}$.

Notice that the tensor product K-representation $\tau_\lambda \otimes \mathrm{Ad}_\mathfrak{p}$ decomposes into irreducibles as

$$\tau_\lambda \otimes \mathrm{Ad}_\mathfrak{p} \simeq \bigoplus_{\beta \in \Delta_n} [m_\beta] \cdot \tau_{\lambda+\beta}, \tag{5.3}$$

and that the multiplicity m_β of $\tau_{\lambda+\beta}$ is either 1 or 0 for every $\beta \in \Delta_n$. Let $(\tau_\lambda^\pm, V_\lambda^\pm)$ be the subrepresentations of $\tau_\lambda \otimes \mathrm{Ad}_\mathfrak{p}$ such that $\tau_\lambda^\pm \simeq \bigoplus_{\beta \in \Delta_n^+} [m_\beta] \cdot \tau_{\lambda \pm \beta}$, and $P_\lambda : V_\lambda \to V_\lambda^-$ be the projection along the decomposition $V_\lambda = V_\lambda^- \oplus V_\lambda^+$.

We now put

$$\mathcal{D}_\lambda f(g) := P_\lambda(\nabla_\lambda f(g)) \quad (f \in \mathcal{A}(\tau_\lambda)). \tag{5.4}$$

Then $\mathcal{D}_\lambda$ gives a G-invariant differential operator from $\mathcal{A}(\tau_\lambda)$ to $\mathcal{A}(\tau_\lambda^-)$.

It follows immediately from the lowest K-type property of H_Λ that

$$\gamma_\lambda(H_\Lambda) \subset \mathrm{Ker}\, \mathcal{D}_\lambda. \tag{5.5}$$

Moreover, the following result, due to Hotta-Parthasarathy, Schmid and Wallach, says that the L^2-kernel of $\mathcal{D}_\lambda$ realizes the discrete series π_Λ.

Proposition 5.1. (Cf. [12, I, Th.1.5]) *For any $\Lambda \in \Xi$, the $(\mathfrak{g}, K)$-module $\gamma_\lambda(H_\Lambda)$, isomorphic to H_Λ, consists exactly of all functions $f \in \mathrm{Ker}\mathcal{D}_\lambda$ which are left K-finite and square-integrable on G.*

5.2. A result of Hotta-Parthasarathy. Let $(\mathcal{A}_{(k)})_{k\in\mathbf{Z}}$ (resp. $(\mathcal{A}^-_{(k)})_{k\in\mathbf{Z}}$) be the decreasing filtration of $\mathcal{A}(\tau_\lambda)$ (resp. $\mathcal{A}(\tau_\lambda^-)$), defined by (4.2). Since $\mathcal{D}_\lambda$ sends $\mathcal{A}_{(k)}$ into $\mathcal{A}^-_{(k-1)}$, the operator $\mathcal{D}_\lambda$ induces an $(S(\mathfrak{g}), K)$-homomorphism, say $\mathrm{Gr}[\mathcal{D}_\lambda]$, from $\mathrm{Gr}\,\mathcal{A}(\tau_\lambda)$ to $\mathrm{Gr}\,\mathcal{A}(\tau_\lambda^-)$. Through the isomorphism $\tilde{\iota}$ in Lemma 4.1, we regard this homomorphism as a map

$$\mathrm{Gr}[\mathcal{D}_\lambda] : \ S(\mathfrak{p}) \otimes V_\lambda \to S(\mathfrak{p}) \otimes V_\lambda^- . \tag{5.6}$$

Observe that $\mathrm{Gr}[\mathcal{D}_\lambda]$ is given as

$$(\mathrm{Gr}[\mathcal{D}_\lambda]f)(Y) = P_\lambda(\sum_i (X_i f)(Y) \otimes X_i^*) \quad (Y \in \mathfrak{g}) \tag{5.7}$$

for $f \in S(\mathfrak{p}) \otimes V_\lambda$. Here $S(\mathfrak{p}) \otimes V$, $V = V_\lambda$ or V_λ^-, is identified in the canonical way with the space of V-valued polynomial functions on $\mathfrak{g}$, vanishing identically on $\mathfrak{k}$.

By virtue of (5.5), one can easily deduce the inclusion

$$\mathrm{Gr}\, H_\Lambda = \mathrm{Gr}_{\gamma_\lambda}(H_\Lambda) \subset \mathrm{Ker}(\mathrm{Gr}[\mathcal{D}_\lambda]) \tag{5.8}$$

for every Harish-Chandra module H_Λ of discrete series. Furthermore, Theorem 1 of [6] combined with the Blattner multiplicity formula (cf. [12, I, Prop.1.2]) gives immediately the following theorem.

Theorem 5.1. (Hotta-Parthasarathy) *The equality* $\mathrm{Gr}\, H_\Lambda = \mathrm{Ker}(\mathrm{Gr}[\mathcal{D}_\lambda])$ *holds in* (5.8) *provided that the lowest hightest weight* $\lambda = \Lambda - \rho_c + \rho_n$ *of* H_Λ *is far from the walls:*

$$\lambda - \sum_{\beta\in Q} \beta \textit{ is } \Delta_c^+\textit{-dominant for any subset } Q \textit{ of } \Delta_n^+. \tag{5.9}$$

Combining this theorem with Proposition 4.1, we make an essential step forward the proof of Theorem 3.1, as in

Theorem 5.2. *Let* H_Λ $(\Lambda \in \Xi)$ *be a Harish-Chandra module of discrete series, and* $H_\Lambda^* \simeq H_{-w_0\Lambda}$ *(see* (5.1)*) be its dual* $(\mathfrak{g}, K)$*-module. If* $\lambda = \Lambda - \rho_c + \rho_n$ *is far from the walls, the associated variety* $\mathcal{V}(H_\Lambda^*)$ *of discrete series* H_Λ^* *is determined by the annihilator of operator* $\mathrm{Gr}[\mathcal{D}_\lambda]$ *in* (5.6)*:*

$$\mathcal{V}(H_\Lambda^*) = \{X \in \mathfrak{g} |\ f(X) = 0 \quad \forall f \in \mathrm{Ann}_{S(\mathfrak{g})}\mathrm{Ker}(\mathrm{Gr}[\mathcal{D}_\lambda])\}. \tag{5.10}$$

Remark. By (5.8) and Proposition 4.1, the inclusion $\subset$ is always true in (5.10) without any assumption on the regulality of λ.

6. $(S(\mathfrak{g}), K)$-modules $\mathrm{Ker}(\mathrm{Gr}[\mathcal{D}_\lambda])$ and the corresponding annihilator ideals.

We now go into more detailed structure of graded $(S(\mathfrak{g}), K)$-modules $\mathrm{Ker}(\mathrm{Gr}[\mathcal{D}_\lambda]) \subset S(\mathfrak{p}) \otimes V_\lambda$ defined in 5.2, and their annihilators $\mathrm{Ann}_{S(\mathfrak{g})}(\mathrm{Ker}(\mathrm{Gr}[\mathcal{D}_\lambda])) \subset S(\mathfrak{g})$.

6.1. Generating subspace of $\mathrm{Ker}(\mathrm{Gr}[\mathcal{D}_\lambda])$ as a K-module. Let $f = X^m \otimes v$ be an element of $S(\mathfrak{p}) \otimes V_\lambda$ with $X \in \mathfrak{p}$, $v \in V_\lambda$ and an integer $m \geq 0$. In view of (5.7) one can compute $\mathrm{Gr}[\mathcal{D}_\lambda]f \in S(\mathfrak{p}) \otimes V_\lambda^-$ as

$$\mathrm{Gr}[\mathcal{D}_\lambda]f = mX^{m-1} \otimes P_\lambda(v \otimes X), \tag{6.1}$$

where P_λ is, as in 5.1, the projection from $V_\lambda = V_\lambda^+ \oplus V_\lambda^-$ onto V_λ^-. This implies that f lies in $\mathrm{Ker}(\mathrm{Gr}[\mathcal{D}_\lambda])$ if and only if $v \otimes X \in V_\lambda^+$. Notice that, if v_λ is a non-zero highest weight vector of V_λ, the vector $v_\lambda \otimes X_+$ belongs to V_λ^+ for every $X_+ \in \mathfrak{p}_+ = \sum_{\alpha \in \Delta_n^+} \mathfrak{g}_\alpha$.

This discussion leads us immediately to

Proposition 6.1. *The kernel* $\mathrm{Ker}(\mathrm{Gr}[\mathcal{D}_\lambda])$ *contains the K-submodule* $\{S(\mathfrak{p}_+) \otimes v_\lambda\}_K$ *of* $S(\mathfrak{p}) \otimes V_\lambda$ *generated by subspace* $S(\mathfrak{p}_+) \otimes v_\lambda$.

Conversely, we can prove, by using Lemma 5.2 of [6], that $\{S(\mathfrak{p}_+) \otimes v_\lambda\}_K$ exhausts $\mathrm{Ker}(\mathrm{Gr}[\mathcal{D}_\lambda])$ in the following sense.

Theorem 6.1. *For each integer $m \geq 0$, there exists a constant $c_m > 0$ such that*

$$\mathrm{Ker}^m(\mathrm{Gr}[\mathcal{D}_\lambda]) = \{S^m(\mathfrak{p}_+) \otimes v_\lambda\}_K \tag{6.2}$$

holds if the lowest highest weight λ satisfies the condition

$$(\lambda, \alpha) \geq c_m \quad \textit{for all } \alpha \in \Delta_c^+. \tag{6.3}$$

Here $\mathrm{Ker}^m(\mathrm{Gr}[\mathcal{D}_\lambda]) := \mathrm{Ker}(\mathrm{Gr}[\mathcal{D}_\lambda]) \cap (S^m(\mathfrak{p}) \otimes V_\lambda)$ *denotes the homogeneous component of* $\mathrm{Ker}(\mathrm{Gr}[\mathcal{D}_\lambda])$ *of degree m.*

This theorem plays a definitive role in proving Theorem 3.1.

6.2. Annihilator $\mathrm{Ann}_{S(\mathfrak{g})}\mathrm{Ker}(\mathrm{Gr}[\mathcal{D}_\lambda])$. For a subset A of $\mathfrak{g}$, let $\mathcal{I}(A)$ denote the ideal of $S(\mathfrak{g})$ determined by A:

$$\mathcal{I}(A) := \{f \in S(\mathfrak{g}) |\ f(X) = 0 \ \ \forall X \in A\}. \tag{6.4}$$

Two results in 6.1 allow us to establish the following

Theorem 6.2. *Let $\lambda = \Lambda - \rho_c + \rho_n$ be the lowest highest weight of discrete series H_Λ. Then one has*

$$\mathrm{Ann}_{S(\mathfrak{g})}\mathrm{Ker}(\mathrm{Gr}[\mathcal{D}_\lambda]) \subset \mathcal{I}(K_C\mathfrak{p}_+). \tag{6.5}$$

Moreover there exists a positive constant c such that the equality holds in (6.5) *provided that $(\lambda, \alpha) \geq c$ for all $\alpha \in \Delta_c^+$.*

This theorem together with Theorem 5.1 immediately yields

Corollary 6.1. *If the lowest highest weight λ is sufficiently Δ_c^+-regular, the annihilator ideal of graded $S(\mathfrak{g})$-module $\mathrm{Gr}\, H_\Lambda$ (see 5.1) coincides with its radical.*

Proof of Theorem 6.2. The inclusion (6.5) follows immediately from Proposition 6.1. To prove the second assertion, note at first that $\mathcal{I}(K_C\mathfrak{p}_+)$ is a graded ideal of $S(\mathfrak{g})$ containing $\mathfrak{k}S(\mathfrak{g})$. Since $S(\mathfrak{g})$ is a Noetherian ring, there exists a finite number of homogeneous elements $D_j \in S(\mathfrak{p})$ $(1 \leq j \leq r)$ such that

$$\mathcal{I}(K_C\mathfrak{p}_+) = \mathfrak{k}S(\mathfrak{g}) + S(\mathfrak{g})D_1 + \cdots + S(\mathfrak{g})D_r.$$

Let c_j be the positive constants in Theorem 6.1 associated to $d_j := \deg D_j$ $(1 \leq j \leq r)$, and put $c := \max_j(c_j)$. Then (6.2) tells us that, if $(\lambda, \alpha) \geq c$ $(\forall \alpha \in \Delta_c^+)$, then each D_j is identically zero on $\mathrm{Ker}^{d_j}(\mathrm{Gr}[\mathcal{D}_\lambda])$. One easily sees from this fact that D_j annihilates all the vectors in $\mathrm{Ker}(\mathrm{Gr}[\mathcal{D}_\lambda])$. We thus conclude $\mathcal{I}(K_C\mathfrak{p}_+) = \mathrm{Ann}_{S(\mathfrak{g})}\mathrm{Ker}(\mathrm{Gr}[\mathcal{D}_\lambda])$ as desired. Q.E.D.

7. Completion of the proof of Theorem 3.1.

By virtue of Theorems 5.2 and 6.2, we find that

$$\mathcal{V}(H_\Lambda^*) = \overline{K_C\mathfrak{p}_+}, \tag{7.1}$$

if the corresponding lowest highest weight λ is sufficiently Δ_c^+-regular. A standard argument of Zuckerman's translation principle (cf. [12, I, 3.4]) shows that (7.1) is always true for any $\Lambda \in \Xi$. In view of (5.1), our theorem is now completely proved. Q.E.D.

8. A recursion formula for the Gelfand-Kirillov dimensions of discrete series.

We finish this article with giving an explicit formula for the Gelfand-Kirillov dimensions $d(H_\Lambda) = \dim \mathcal{V}(H_\Lambda)$ of discrete series. Proposition 3.2 gives us a method for computing $d(H_\Lambda)$. We concentrate here on the case of unitary groups $G = SU(p,q)$ with integers $p, q \geq 0$, $(p,q) \neq (0,0)$. Our formula for $d(H_\Lambda)$ is recursive with respect to the parameter $n := p + q$.

8.1. The function GKD. Realize our group G as

$$G = \{g \in SL(n, \mathbf{C}) \mid {}^t\overline{g} I_{p,q} g = I_{p,q}\} \quad (n = p + q) \tag{8.1}$$

with

$$I_{p,q} = \begin{pmatrix} I_p & 0 \\ O & -I_q \end{pmatrix} \quad (I_r \text{ the identitiy matrix of degree } r),$$

where ${}^tg$ (resp. $\overline{g}$) denotes the transposed (resp. the complex conjugate) of a matrix g. Then the Lie algebras $\mathfrak{g}, \mathfrak{k}, \mathfrak{t}$ and subspace $\mathfrak{p}$ can be written as follows.

$$\mathfrak{g} = \mathfrak{sl}(n, \mathbf{C}) := \{X \in M(n,n) \mid \mathrm{tr}\, X = 0\}, \tag{8.2}$$

$$(8.3) \qquad \mathfrak{k} = \left\{ \begin{pmatrix} Y & 0 \\ 0 & Z \end{pmatrix} \in \mathfrak{g} \;\middle|\; Y \in M(p,p),\ Z \in M(q,q) \right\},$$

$$(8.4) \qquad \mathfrak{t} = \{H = \mathrm{diag}(t_1, \ldots, t_n) |\ t_i \in \mathbf{C},\ \mathrm{tr}\, H = 0\},$$

$$(8.5) \qquad \mathfrak{p} = \left\{ \begin{pmatrix} 0 & V \\ W & 0 \end{pmatrix} \in \mathfrak{g} \;\middle|\; V \in M(p,q),\ W \in M(q,p) \right\}.$$

Here $M(p,q)$ denotes the space of complex matrices of size $p \times q$. The root system Δ (resp. $\Delta_c \subset \Delta$) of $\mathfrak{g}$ (resp. $\mathfrak{k}$) with respect to $\mathfrak{t}$ is of type A_{n-1} (resp. $A_{p-1} \times A_{q-1}$), and it is described respectively as

$$(8.6) \quad \Delta = \{e_{ij} | 1 \le i,j \le n, i \neq j\}, \quad \Delta_c = \{e_{ij} \in \Delta | 1 \le i,j \le p \text{ or } p < i,j \le n\}$$

with $e_{ij}(H) := t_i - t_j$ $(H \in \mathfrak{t})$. We fix as in §2 a positive system of Δ_c:

$$(8.7) \qquad \Delta_c^+ := \{e_{ij} \in \Delta_c |\ i < j\}.$$

Let $\Pi_{p,q}$ be the totality of maps h from $F(n) := \{1, 2, \ldots, n\}$ to the set $\{a, b\}$ of two elements a and b, such that

$$\#(h^{-1}(a)) = p, \quad \text{and} \quad \#(h^{-1}(b)) = q,$$

where $\#(S)$ denotes the cardinal number of a set S. For an $h \in \Pi_{p,q}$, arrange the elements of $h^{-1}(a)$ and $h^{-1}(b)$ respectively as

$$(w_1, w_2, \ldots, w_p) \quad \text{with} \quad w_1 < w_2 < \ldots < w_p,$$

$$(w_{p+1}, w_{p+2}, \ldots, w_n) \quad \text{with} \quad w_{p+1} < w_{p+2} < \ldots < w_n,$$

and we put

$$(8.8) \qquad \Delta^+(h) := \{e_{ij} |\ w_i < w_j\}.$$

It is then elementary to verify

Lemma 8.1. *The assignment $h \to \Delta^+(h)$ gives a bijective correspondance from $\Pi_{p,q}$ to the totality of positive systems of Δ including Δ_c^+ in* (8.7).

Now let H_Λ be the discrete series module with Harish-Chandra parameter $\Lambda \in \Xi$. By definition this parameter set Ξ is written as a disjoint union of subsets $\Xi(h) := \{\Lambda \in \Xi |\ \Lambda \text{ is } \Delta^+(h)\text{-dominant}\}$ $(h \in \Pi_{p,q})$. Noting that the Gelfand-Kirillov dimension $d(H_\Lambda)$ is constant on each $\Xi(h)$ (cf. Theorem3.1), one can define a well-defined mapping:

$$(8.9) \qquad \mathrm{GKD}_{p,q} : \ \Pi_{p,q} \ni h \to d(H_\Lambda) \in \{0, 1, 2, \ldots\},$$

where $\Lambda \in \Xi(h)$. We call $\mathrm{GKD}_{p,q}$ the *Gelfand-Kirillov dimension map* for $G = SU(p,q)$.

Put $\Pi := \cup_{p,q} \Pi_{p,q}$ (disjoint union) by varying the non-negative integers p and q. Then $\mathrm{GKD}_{p,q}$ extends naturally to a function on Π with values in $\{0, 1, 2, \ldots\}$ which we denote by

$$(8.10) \qquad \mathrm{GKD} = \bigoplus_{p,q} \mathrm{GKD}_{p,q}.$$

It should be noticed that, for an integer $n > 0$, the subset $\Pi(n) := \bigoplus_{p+q=n} \Pi_{p,q} \subset \Pi$ consists of all mappings from $F(n)$ to $\{a, b\}$.

8.2. Recursion formula for GKD. We now define an assignment R on Π and describe the function GKD recursively, by means of R.

Let h be in $\Pi(n)$ with an integer $n > 0$. We say that two elements $i, j \in F(n)$ are connected with respect to h, or $i \sim j$ for short, if the function h is constant on the segment $[i, j] \subset F(n)$. This $\sim$ clearly gives an equivalence relation on $F(n)$. Each equivalence class of $(F(n), \sim)$, viewed as a subset of $F(n)$, is called an h-connected component of $F(n)$.

Take a complete system $J \subset F(n)$ of representatives of the set of h-connected components, and let ζ be the unique bijection

$$\zeta : \ F(n) \setminus J \to F(n - |h|), \tag{8.11}$$

characterized by

$$i < j \Leftrightarrow \zeta(i) < \zeta(j) \qquad \text{for } i, j \in F(n) \setminus J.$$

Here $|h|$ denotes the number of h-connected components.

We define $Rh \in \Pi(n - |h|)$ by

$$Rh := h \circ \zeta^{-1}. \tag{8.12}$$

Note that Rh is independent of the choice of a set of representatives J. Since $\Pi = \cup_{n>0}\Pi(n)$ (disjoint union), $R : \Pi(n) \to \Pi(n - |h|)$ naturally extends to an assingment defined on Π, which we denote by the same letter R.

Based on Proposition 3.2, we can derive the following explicit recursion formula for the Gelfand-Kirillov dimension map GKD by means of the above map R.

Theorem 8.1. *One has for $h \in \Pi(n) = \cup_{p+q=n}\Pi_{p,q}$ $(n > 0)$,*

$$\mathrm{GKD}(h) = \mathrm{GKD}(Rh) + (2n - |h|)(|h| - 1)/2, \tag{8.13}$$

where we set $\mathrm{GKD}(Rh) = 0$ for h's such that $|h| = n$.

Corollary 8.1. *The Gelfand-Kirillov dimension of an $h \in \Pi(n)$ is given as*

$$\mathrm{GKD}(h) = \frac{1}{2}\sum_{k=0}^{l}(2n_k(h) - |R^k(h)|)(|R^k(h)| - 1) \tag{8.14}$$

in terms of the finite sequences of positive integers: $(|R^k h|)_{1\le k\le l}$ and $(n_k(h))_{1\le k\le l}$ with $R^k(h) \in \Pi(n_k(h))$. Here $l > 0$ is the integer such that $|R^l(h)| = n_l(h)$.

Remarks. (i) An $h \in \Pi(n)$ satisfies the condition $|h| = n$ if and only if the Gelfand-Kirillov dimension $d(H_\Lambda)$ of corresponding discrete series equals $\#(\Delta_+)$, i.e., H_Λ is large in the sense of [8, §6].

(ii) The sequence $(R^k(h))_k$ in the above corollary gives a partition of n. It defines the nilpotent orbit $\mathcal{O}_{\mathfrak{p}\_}$ in Proposition 3.2 for the corresponding discrete series, as the G_C-orbit through the matrix

$$J(|h|) \oplus J(|Rh|) \oplus \cdots \oplus J(|R^l(h)|), \tag{8.15}$$

where $J(m)$ denotes the Jordan matrix of degree m.

References.

[1] D.Barbash and D.Vogan, The local structure of characters, *J.Funct.Anal.* **37** (1980), 27-55.

[2] W.Borho and J.-L.Brylinski, Differential operators on homogeneous spaces I, *Invent.math.* **69** (1982), 437-476;III *ibid.* **80** (1985), 1-68.

[3] M.Duflo, Représentations de carré intégrable des groupes semi-simples réels, Exposé 508, Séminaire Bourbaki, 1977/78, pp.22-40.

[4] Harish-Chandra, Discrete series for semisimple Lie groups, I, II, *Acta Math.*, **113** (1965), 241-318; *ibid.*, **116** (1966), 1-111.

[5] G.Helgason, "Groups and Geometric Analysis," Academic Press, 1984.

[6] R.Hotta and R.Parthasarathy, Multiplicity formulae for discrete series, *Invent.math.* **26** (1974), 133-178.

[7] A.Joseph, Gelfand-Kirillov dimension for the annihilators of simple quotients of Verma modules, *J.London Math.Soc.* **18** (1978), 50-60.

[8] D.A.Vogan, Gelfand-Kirillov dimension for Harish-Chandra modules, *Invent.math.* **48** (1978), 75-98.

[9] D.A.Vogan, Associated varieties and unipotent representations, *in* "Harmonic Analysis on Reductive Groups (W.Barker and P.Sally eds.)," Birkhauser, 1991, pp.315-388.

[10] A.Yamaguchi, On higher-order terms in asymptotic expansions for irreducible characters of semisimple Lie groups, Master thesis, Kyoto University, 1986.

[11] H.Yamashita, Finite multiplicity theorems for induced representations of semisimple Lie groups I, *J.Math.Kyoto Univ.* **28** (1988), 173-211; II, *ibid.* 383-444.

[12] H.Yamashita, Embeddings of discrete series into induced representations of semisimple Lie groups, I, *Japan.J.Math.(N.S.)* **16** (1990), 31-95; II, *J.Math.Kyoto Univ.* **31** (1991), 543-571.

[13] H.Yamashita, Criteria for the finiteness of restriction of $U(\mathfrak{g})$-modules to subalgebras and applications to Harish-Chandra modules, to appear. (The results have been announced in *Proc.Japan Acad.* **68** (1992), 316-321.)

Department of Mathematics
Faculty of Science
Kyoto University
606-01, Kyoto
Japan