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On character identities in some enlarged L-packets for SU(2,2)

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Introduction.

Let $\mathbf{G}$ be a connected reductive linear algebraic group defined over $\mathbf{R}$ and $G = \mathbf{G}(\mathbf{R})$ the group of $\mathbf{R}$-rational points on $\mathbf{G}$. For an irreducible representation π of G, we denote by Θ_π its character. In the Langlands classification of irreducible admissible representations, they (to be more precise, their equivalence classes) are partitioned into finite sets, called L-packets. Then an L-packet $\hat{\Pi}$ consists of only tempered representations or only non-tempered ones. When $\hat{\Pi}$ is a tempered type, the sum $\sum_{\pi\in\hat{\Pi}}\Theta_\pi$ is a stable tempered invariant eigendistribution. Moreover Shelstad defined the operation 'lifting' for such eigendistributions and established functoriality with respect to L-groups.

In connection with her theory, we obtained the following theorem for $G = Sp(n, \mathbf{R})$ or $SU(p, q)$ in [6] and [7].

Theorem. Let T_s and T_c be a maximally $\mathbf{R}$-split and a compact Cartan subgroup respectively. Put $\Theta = \sum c_\pi\Theta_\pi$ ($\pi \in \hat{\Pi}$, $c_\pi \in \mathbf{C}$), and suppose that Θ has a regular integral infinitesimal character. Then the following two conditions are equivalent:

1) Θ is identically zero on $T_s \cap G'$,

2) Θ satisfies the property (P) on $T_c \cap G'$.

Here G' denotes the set of all regular elements of G. (For the definition of the property (P), see §3.) Furthermore, character identities of type 1) are essentially exausted by what Shelstad obtained in [9].

Now we turn the topic into non-tempered cases. Then the situation is quite different. For example, a non-tempered regular character Θ_π is not completely determined by the restriction on its highest Cartan subgroup. Furthermore, the sum $\sum_{\pi\in\hat{\Pi}}\Theta_\pi$ is not stable in general. But stableness is very important to extend our theorem to non-tempered cases. In [1], Adams and Johnson constructed an enlarged L-packet Π such that $\sum_{\pi\in\Pi}\epsilon_\pi\Theta_\pi$ is stable where the sign $\epsilon_\pi = \pm 1$ is determined explicitly by π. (They also defined lifting for such sums.) Therefore we start studying character identities of type 1) in the enlarged L-pachet Π. For groups of $\mathbf{R}$-rank 1, the problem is automatically reduced to tempered cases. So we treat the cases $G = Sp(2,\mathbf{R})$ and $SU(2,2)$ of $\mathbf{R}$-rank 2 as a starter and we get our main therem for the enlarged L-packet Π (see §3).

Theorem. Put $\Theta = \sum c_\pi \Theta_\pi$ ($\pi \in \Pi$). Then Θ is identically zero on $T_s \cap G'$ if and only if Θ satisfies the property (P) on any Cartan subgroups not conjugate to T_s.

In this note, we describe only the case $G = SU(2,2)$, but in exactly the same way, we can obtain similar results for $Sp(2,\mathbf{R})$.

To the proof of this theorem, Propositions 3.1 and 3.2 are essential. The former is proved for $SU(p,p)$ ($p \geq 1$). The latter states character identities among discrete series for $SU(p,q)$, and this ia a part of the results in [7]. Here we remark that results for tempered invariant eigendistributions play an important role for non-tempered ones.

§1. Cohomological parabolic induction and a $(\mathfrak{g}, K)$-module $A_\mathfrak{q}(\lambda)$

In this section, we review some definitions and properties about $(\mathfrak{g}$,K)-modules and cohomological parabolic induction.

1.1. Construction of a $(\mathfrak{g}, K)$-module $A_\mathfrak{q}(\lambda)$. Let $\mathbf{G}$ be a connected reductive linear algebraic group defined over $\mathbf{R}$ and $G = \mathbf{G}(\mathbf{R})$. We assume that G is connected and contains a compact Cartan subgroup T. We fix K a maximal compact subgroup such that $K \supseteq T$. Let $\mathfrak{g}_0$ be the Lie algebra of G and $\mathfrak{g}$ its complexification. In what follows, we will denote a Lie group with roman upper case letters and its Lie algebra with corresponding German lower case letters and will use analogous notations to distinguish the real Lie algebra and its complexification. For an element $\lambda_0 \in \sqrt{-1}\mathfrak{t}_0{}^*$, we put

$$L = L(\lambda_0) = \{g \in G; \mathrm{Ad}(g)^* \lambda_0 = \lambda_0\}. \tag{1.1}$$

Obviously, L is a reductive Lie group and contains T as its compact Cartan subgroup. Now denote by $\Delta(\mathfrak{g}, \mathfrak{t})$ the root system of $(\mathfrak{g}, \mathfrak{t})$. Then

$$\mathfrak{l} = \mathfrak{l}(\lambda_0) = \mathfrak{t} + \sum_{(\lambda_0, \alpha)=0} \mathfrak{g}^\alpha, \tag{1.2}$$

where $\mathfrak{g}^\alpha$ is the root space for α. Put

$$\mathfrak{u} = \mathfrak{u}(\lambda_0) = \sum_{(\lambda_0, \alpha)>0} \mathfrak{g}^\alpha, \tag{1.3}$$

then $\mathfrak{q} = \mathfrak{q}(\lambda_0) = \mathfrak{l} + \mathfrak{u}$ is a parabolic subalgebra of $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$ and we denote the corresponding Cartan involution by θ. Then we get $\theta\mathfrak{q} = \mathfrak{q}$, $\theta\mathfrak{l} = \mathfrak{l}$, $\bar{\mathfrak{l}} = \mathfrak{l}$, $\bar{\mathfrak{q}} = \mathfrak{l} + \bar{\mathfrak{u}}$, $\bar{\mathfrak{u}} = \sum_{(\lambda_0,\alpha)<0} \mathfrak{g}^{\alpha}$. By the upper bar we indicate the complex conjugation in $\mathfrak{g}$ with respect to $\mathfrak{g}_0$. Apparently, $\bar{\mathfrak{q}}$ is the parabolic subalgebra of $\mathfrak{g}$ opposite to $\mathfrak{q}$.

Let π be a one-dimensional representation of L. By differentiating the representation $\pi_{|T}$ (restriction of π to T), we get an element $\lambda \in \sqrt{-1}\mathfrak{t}_0{}^*$. We canonically view π as a one-dimensional $(\mathfrak{l}, L \cap K)$-module. Then we get a $(\mathfrak{g}, K)$-module by the method of cohomological parabolic induction:

$$A_{\mathfrak{q}}(\lambda) = (\mathcal{R}_{\mathfrak{q}}^{\mathfrak{g}})^i(\pi), \tag{1.4}$$

where $i = \dim(\mathfrak{u} \cap \mathfrak{k})$. We write $\mathcal{R}_{\mathfrak{q}}^i(\pi)$ instead of $(\mathcal{R}_{\mathfrak{q}}^{\mathfrak{g}})^i(\pi)$ when it is clear that we consider $(\mathfrak{g}, K)$-modules.

Here we state a brief explanation of cohomological parabolic induction. For more precise definitions, see [10]. The functor $\mathcal{R}$ is composed of two steps. The first one is as follows. For a Lie algebra $\mathfrak{g}$, we denote its universal enveloping algebra by $U(\mathfrak{g})$ as usual. Then $U(\mathfrak{g})$ turns out to be a $U(\mathfrak{q})$-module by left multiplication. Let W be a $(\mathfrak{l}, L\cap K)$-module. Making $\mathfrak{u}$ operate trivially, we regard the $(\mathfrak{l}, L\cap K)$-module $W \otimes \bigwedge^{\dim \mathfrak{u}}\mathfrak{u}$ as a $U(\mathfrak{q})$-module. Then we get a $(\mathfrak{g}, L \cap K)$-module $\mathrm{pro}(W)$ in the following way:

$$\mathrm{pro}(W) = \mathrm{Hom}_{U(\mathfrak{q})}(U(\mathfrak{g}), W \otimes \bigwedge\nolimits^{\dim \mathfrak{u}} \mathfrak{u})_{L\cap K\text{-finite}}. \tag{1.5}$$

The $(\mathfrak{g}, L \cap K)$-module structure of $\mathrm{pro}(W)$ is given by

$$\begin{aligned} (X \cdot f)(Y) &= f(YX), \\ (x \cdot f)(Y) &= x \cdot (f(\mathrm{Ad}(x^{-1})Y)), \end{aligned} \tag{1.6}$$

where $X \in \mathfrak{g}, Y \in U(\mathfrak{g})$ and $x \in L \cap K$. We also require that f satisfies $L \cap K$-finiteness condition. That is, the elements $x \cdot f$ for all $x \in L \cap K$ span a finite-dimensional subspace.

The second step is an induction from $(\mathfrak{g}, L \cap K)$-modules to $(\mathfrak{g}, K)$-modules. For brevity, we describe it only for the case that K is connected. For a $(\mathfrak{g}, L\cap K)$-module V, put

$$\Gamma_0(V) = \{v \in V; \dim U(\mathfrak{k}) \cdot v < +\infty\}. \tag{1.7}$$

Let $\tilde{K}$ be the universal covering group of K and p its covering map. Set $Z = \{z \in \tilde{K};\ p(z) = 1\}$. Then $\Gamma_0(V)$ becomes a $(\mathfrak{g}, \tilde{K})$-module by lifting the action of $\mathfrak{k}$ up to $\tilde{K}$. Put

$$\Gamma(V) = \Gamma_0(V)^Z = \{v \in \Gamma_0(V); zv = v \text{ for any } z \in Z\}. \tag{1.8}$$

Thus we get a $(\mathfrak{g}, K)$-module $\Gamma(V)$, and Γ becomes a functor from the category of $(\mathfrak{g}, L\cap K)$-modules to that of $(\mathfrak{g}, K)$-modules. Clearly, Γ is a left exact functor and we denote its j-th derived functor by Γ^j. After these preparations, we can describe the Zuckerman functor or cohomological parabolic induction as follows.

For a $(\mathfrak{l}, L\cap K)$-module W, put

$$\mathcal{R}_{\mathfrak{q}}^{\mathfrak{g}}(W) = \Gamma(\mathrm{pro}(W)). \tag{1.9}$$

Since the functor pro is exact, we get that $(\mathcal{R}_{\mathfrak{q}}^{\mathfrak{g}})^j(W) = \Gamma^j(\mathrm{pro}(W))$. Put $i = \dim(\mathfrak{u}\cap\mathfrak{k})$. Replacing W by π, we obtain the $(\mathfrak{g}, K)$-module $A_{\mathfrak{q}}(\lambda)$.

Now we fix a positive system $\Delta^+(\mathfrak{l})$ of $\Delta(\mathfrak{l}, \mathfrak{t})$ and put

$$\Delta(\mathfrak{u}) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{t}); \mathfrak{g}^\alpha \subseteq \mathfrak{u}\}, \tag{1.10}$$

$$\Delta^+(\mathfrak{g}, \mathfrak{t}) = \Delta^+(\mathfrak{l}) \cup \Delta(\mathfrak{u}).$$

Obviously, $\Delta^+(\mathfrak{g}, \mathfrak{t})$ is a positive root system of $\Delta(\mathfrak{g}, \mathfrak{t})$ and we define $\rho(\Delta^+(\mathfrak{l}))$,$\rho(\mathfrak{u})$ and $\rho(\mathfrak{q})$ as follows:

$$\rho(\Delta^+(\mathfrak{l})) = \frac{1}{2}\sum_{\alpha\in\Delta^+(\mathfrak{l})} \alpha\ ,\ \rho(\mathfrak{u}) = \frac{1}{2}\sum_{\alpha\in\Delta(\mathfrak{u})} \alpha, \tag{1.11}$$

$$\rho(\mathfrak{q}) = \rho(\Delta^+(\mathfrak{g}, \mathfrak{t})) = \rho(\Delta^+(\mathfrak{l})) + \rho(\mathfrak{u}).$$

Then the following proposition holds (cf.[10]).

Proposition 1.1. *Let $A_{\mathfrak{q}}(\lambda)$ be a $(\mathfrak{g}, K)$-module obtained as above. Then it has infinitesimal character $\lambda + \rho(\mathfrak{q}) \in \mathfrak{t}^*$.*

1.2. Enlarged L-packets. Next we will define an enlarged L-packet. Denote by $W(\mathfrak{g}, \mathfrak{t})$ the Weyl group of $\Delta(\mathfrak{g}, \mathfrak{t})$. For any $w \in W$, $w\lambda_0$ also belongs to $\sqrt{-1}\mathfrak{t}_0^*$. So we can construct a θ-stable parabolic subalgebra $\mathfrak{q}_w$ just in the same way as $\mathfrak{q}$. That is, we put

$$L_w = L(w\lambda_0) = \{g \in G; Ad(g)^* w\lambda_0 = w\lambda_0\}. \tag{1.12}$$

Then its complexified Lie algebra and the nilpotent radical $\mathfrak{u}_w$ of $\mathfrak{q}_w$ are expressed as follows:

$$
(1.13) \qquad
\begin{aligned}
\mathfrak{l}_w &= \mathfrak{l}(w\lambda_0) = \mathfrak{t} + \sum_{(w\lambda_0,\alpha)=0} \mathfrak{g}^{\alpha} = \mathfrak{t} + \sum_{(\lambda_0,\alpha)=0} \mathfrak{g}^{w\alpha}, \\
\mathfrak{u}_w &= \mathfrak{u}(w\lambda_0) = \sum_{(w\lambda_0,\alpha)>0} \mathfrak{g}^{\alpha} = \sum_{(\lambda_0,\alpha)=0} \mathfrak{g}^{w\alpha}, \\
\mathfrak{q}_w &= \mathfrak{q}(w\lambda_0) = \mathfrak{l}(w\lambda_0) + \mathfrak{u}(w\lambda_0).
\end{aligned}
$$

In [1], Adams and Johnson proved that there exists a one-dimensional representation π_w of L_w such that $w\lambda$ coincides with the differential representation of $\pi_{w|T}$. (They showed that this proposition holds true for not necessarily connected group G.) So we can construct a $(\mathfrak{g}, K)$-module $A_{\mathfrak{q}_w}(\pi_w)$ which is induced from $(\mathfrak{l}_w, L_w \cap K)$-module π_w. (In the sequel of this note, we also denote this $(\mathfrak{g}, K)$-module by $A(w\lambda, \pi_w)$.)

Definition. An element $\lambda \in \mathfrak{t}^*$ is called $\mathfrak{u}$-admissible when it satisfies the following two conditions:

1) There exists a one-dimensional unitary representation π of L such that λ is the differential of $\pi_{|T}$;

2) $(\lambda, \alpha) \geq 0$ for all $\alpha \in \Delta(\mathfrak{u})$.

Put $W_G(T) = N_G(T)/T$, where $N_G(T)$ denotes the normalizer of T in G. We will consider $W_G(T)$ as a subgroup of $W(\mathfrak{g}, \mathfrak{t})$. Vogan proved the next proposition (cf. [10],[11]).

Proposition 1.2.

1) *The* $(\mathfrak{g}, K)$*-module* $A(w\lambda, \pi_w)$ *is irreducible and unitary when* λ *is* $\mathfrak{u}$*-admissible.*

2) *For* $w, w' \in W(\mathfrak{g}, \mathfrak{t}), A(w\lambda, \pi_w) = A(w'\lambda, \pi_{w'})$ *if and only if* $W_G(T)wW(\mathfrak{l}, \mathfrak{t}) = W_G(T)w'W(\mathfrak{l}, \mathfrak{t})$.

Thus it makes sense to write $A(w\lambda, \pi_w)$ for $w \in W_G(T)\backslash W(\mathfrak{g}, \mathfrak{t})/W(\mathfrak{l}, \mathfrak{t})$. Put $\Pi = \{A(w\lambda, \pi_w); w \in W_G(T)\backslash W(\mathfrak{g}, \mathfrak{t})/W(\mathfrak{l}, \mathfrak{t})\}$, and we call it an enlarged L-packet.

We remark that when $L = T$, Π is nothing but an L-packet consisting of discrete series representations with a same infinitesimal character. In this note, we will study character identities in certain enlarged L-packets for $SU(2,2)$.

1.3. Before doing that, it is necessary to explain some properties of cohomological parabolic induction.

At first, we review how discrete series representations are related to cohomological induction. Let G be a reductive Lie group with a compact Cartan subgroup T. Take a regular element $\mu \in \mathfrak{t}^*$ such that $\mu - \rho$ is integral. Here ρ is half the sum of positive roots for certain positive system of $\Delta(\mathfrak{g}, \mathfrak{t})$. Put $\Delta_\mu^+ = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{t}); (\alpha, \mu) > 0\}$. Then Δ_μ^+ is a positive root system of $\Delta(\mathfrak{g}, \mathfrak{t})$ and we denote by $\mathfrak{b}_\mu$ the Borel subalgebra of $\mathfrak{g}$ corresponding to Δ_μ^+. That is,

$$\mathfrak{b}_\mu = \mathfrak{t} + \sum_{\alpha \in \Delta_\mu^+} \mathfrak{g}^\alpha. \tag{1.14}$$

Obviously, $\mathfrak{u}_\mu = \sum_{\alpha \in \Delta_\mu^+} \mathfrak{g}^\alpha$ is its nilpotent radical. We denote by ρ_μ instead of $\rho(\mathfrak{b}_\mu) = \rho(\Delta_\mu^+)$. Since $\mu - \rho_\mu$ is integral, we regard $\mathbf{C}$ as a $(\mathfrak{b}_\mu, T)$-module in the following way:

$$\begin{aligned} (X+Y)z &= (\mu - \rho_\mu)(X)z, \qquad X \in \mathfrak{t}, Y \in \mathfrak{u}_\mu, \\ t \cdot z &= \exp(\mu - \rho_\mu)(\log t)z, \quad t \in T, z \in \mathbf{C}. \end{aligned} \tag{1.15}$$

We write $\mathbf{C}_{\mu-\rho_\mu}$ for this one-dimensional $(\mathfrak{b}_\mu, T)$-module. Then we get a $(\mathfrak{g}, K)$-module $\mathcal{R}^i{}_{\mathfrak{b}_\mu}(\mathbf{C}_{\mu-\rho_\mu})$. Here $i = \dim(\mathfrak{u}_\mu \cap \mathfrak{k})$ and this is equal to the number of positive compact roots. This module has infinitesimal character μ and Theorem 6.3.12 in [10] tells us its lowest K-type. Thus we get that $\mathcal{R}^i_{\mathfrak{b}_\mu}(\mathbf{C}_{\mu-\rho_\mu})$ is equal to Harish-Chandra module of discrete series representation $\Theta^G(\mu, C)$. Here C is a unique Weyl chamber in $\sqrt{-1}\mathfrak{t}_0{}^*$ with respect to which μ is dominant.

Secondly, we introduce a lemma on induction by stages (cf.[10],Lemma 6.3.6).

Lemma 1.3. *Suppose we are given two θ-stable parabolic subalgebras $\mathfrak{q}^i = \mathfrak{l}^i + \mathfrak{u}^i (i = 1,2)$ as in (1.2) and (1.3). We assume that $\mathfrak{q}^1 \subseteq \mathfrak{q}^2, \mathfrak{l}^1 \subseteq \mathfrak{l}^2, \mathfrak{u}^1 \supseteq \mathfrak{u}^2$ and $L^1 \cap K \subseteq L^2 \cap K$. Put $\mathfrak{u}^0 = \mathfrak{u}^1 \cap \mathfrak{l}^2$ and $\mathfrak{q}^0 = \mathfrak{l}^1 + \mathfrak{u}^0$. Then $\mathfrak{q}^0$ is a θ-stable parabolic subalgebra of $\mathfrak{l}^2$ and $\mathfrak{l}^1$ is its Levi part. For an $(\mathfrak{l}^1, L^1 \cap K)$-module W, we assume $(\mathcal{R}^{\mathfrak{l}^2}_{\mathfrak{q}^0})^q(W) = 0$ unless $q = q_0$. Then*

$$(\mathcal{R}^{\mathfrak{g}}_{\mathfrak{q}^2})^p (\mathcal{R}^{\mathfrak{l}^2}_{\mathfrak{q}^0})^{q_0}(W) = (\mathcal{R}^{\mathfrak{g}}_{\mathfrak{q}^1})^{p+q_0}(W). \tag{1.16}$$

Now we consider the following case that $\mathfrak{q}^2 = \mathfrak{q} = \mathfrak{q}(\lambda_0) = \mathfrak{l} + \mathfrak{u}^2$, and $\mathfrak{q}^1 = \mathfrak{b}$ $= \mathfrak{t} + \sum_{\alpha \in \Delta^+(\mathfrak{g},\mathfrak{t})} \mathfrak{g}^\alpha$ is a Borel subalgebra contained in $\mathfrak{q}^2$. In this case, $\mathfrak{u}^0 = \mathfrak{l} \cap \mathfrak{u}_{\rho(\mathfrak{b})}$ and $\mathfrak{q}^0 = \mathfrak{t} + \mathfrak{u}^0$. Let μ be an integral element in $\mathfrak{t}^*$ such that $(\mu, \alpha) \geq 0$ for any $\alpha \in$

$\Delta^+(\mathfrak{g}, \mathfrak{t})$. Put $W = \mathbf{C}_\mu$, $\Delta^+(\mathfrak{l}, \mathfrak{t}) = \Delta^+(\mathfrak{g}, \mathfrak{t}) \cap \Delta(\mathfrak{l}, \mathfrak{t})$, and $q_0 = \dim(\mathfrak{l} \cap \mathfrak{u}_{\rho(\mathfrak{b})} \cap \mathfrak{k})$. Then as mentioned above, we get that

$$(\mathcal{R}^{\mathfrak{l}}_{\mathfrak{q}^0})^{q_0}(W) = \Theta^L(\mu + \rho(\mathfrak{l}), C^L),$$

where $\rho(\mathfrak{l}) = \frac{1}{2}\sum_{\alpha \in \Delta^+(\mathfrak{l})} \alpha$ and C^L is the Weyl chamber in $\sqrt{-1}\mathfrak{t}_0^*$ for $\mathfrak{l}$ with respect to which $\mu + \rho(\mathfrak{l})$ is dominant. Moreover $(\mathcal{R}^{\mathfrak{l}}_{\mathfrak{q}^0})^q(\mathbf{C}_\mu) = 0$ unless $q = q_0$. Therefore we can apply Lemma 1.3 to this case. Hence we conclude that

$$\begin{aligned}(1.17) \qquad (\mathcal{R}^{\mathfrak{g}}_{\mathfrak{q}})^p(\Theta^L(\mu + \rho(\mathfrak{l}), C^L)) &= (\mathcal{R}^{\mathfrak{g}}_{\mathfrak{q}})^p(\mathcal{R}^{\mathfrak{l}}_{\mathfrak{q}^0})^{q_0}(\mathbf{C}_\mu) \\ &= (\mathcal{R}^{\mathfrak{g}}_{\mathfrak{b}_\mu})^{p+q_0}(\mathbf{C}_\mu).\end{aligned}$$

Put $p = p_0 = \dim(\mathfrak{u} \cap \mathfrak{k})$, then $p_0 + q_0 = \dim(\mathfrak{u}_{\rho(\mathfrak{b})} \cap \mathfrak{k})$. Then we have the next proposition.

Proposition 1.4. *In the above setting, let C be the Weyl chamber in $\sqrt{-1}\mathfrak{t}_0^*$ for $\mathfrak{g}$ with respect to which $\mu + \rho(\mathfrak{b})$ is dominant. Then,*

$$(\mathcal{R}^{\mathfrak{g}}_{\mathfrak{q}})^{p_0}(\Theta^L(\mu + \rho(\mathfrak{l}), C^L)) = \Theta^G(\mu + \rho(\mathfrak{b}), C).$$

§2. A resolution of $A_{\mathfrak{q}}(\lambda)$ for $SU(2,2)$

In this section, we study a resolution of $A_{\mathfrak{q}}(\lambda)$ by standard modules for $SU(2,2)$.

2.1. Cartan subgroups of $SU(p,q)$. Let $SU(p,q)$ be the group of matrices g in $SL(p+q, \mathbf{C})$ satisfying ${}^t\bar{g}I_{p,q}g = I_{p,q}$, where $I_{p,q} = \begin{pmatrix} 1_p & 0 \\ 0 & -1_q \end{pmatrix}$ and 1_p is the identity matrix of order p. Then the Lie algebra $\mathfrak{g}_0$ of $G = SU(p,q)$ is as follows:

$$\mathfrak{g}_0 = \{X \in \mathfrak{sl}(p+q, \mathbf{C});\ {}^t\overline{X}I_{p,q} + I_{p,q}X = 0\}.$$

We assume $p \geq q$ and put $n = p + q$. For any k such that $0 \leq k \leq q$, put $T_k = T_k^- T_k^+$,where T_k^- and T_k^+ are subgroups consisting of all matrices of the following forms respectively:

$$(2.1) \quad T_k^- = \left\{ \operatorname{diag}(e^{i\varphi_1}, \ldots, e^{i\varphi_{p-k}}, e^{i\theta_k}, \ldots, e^{i\theta_1}, e^{i\theta_1}, \ldots, e^{i\theta_k}, e^{i\psi_{q-k}}, \ldots, e^{i\psi_1})\right\},$$

(2.1)'

$$T_k^+ = \left\{ \begin{pmatrix} 1_{p-k} & & & & & & & & & 0 \\ & \mathrm{ch}t_k & 0 & \cdots & \cdots & \cdots & \cdots & 0 & \mathrm{sh}t_k & \\ & 0 & \mathrm{ch}t_{k-1} & \cdots & \cdots & \cdots & \cdots & \mathrm{sh}t_{k-1} & 0 & \\ & \vdots & \vdots & \ddots & & & & \vdots & \vdots & \\ & \vdots & \vdots & & \mathrm{ch}t_1 & \mathrm{sh}t_1 & & \vdots & \vdots & \\ & \vdots & \vdots & & \mathrm{sh}t_1 & \mathrm{ch}t_1 & & \vdots & \vdots & \\ & \vdots & \vdots & & & & \ddots & \vdots & \vdots & \\ & 0 & \mathrm{sh}t_{k-1} & \cdots & \cdots & \cdots & \cdots & \mathrm{ch}t_{k-1} & 0 & \\ & \mathrm{sh}t_k & 0 & \cdots & \cdots & \cdots & \cdots & 0 & \mathrm{ch}t_k & \\ 0 & & & & & & & & & 1_{q-k} \end{pmatrix} \right\},$$

where the blank spaces of matrix (2.1)' must be filled with 0's. Then T_j's are not conjugate to each other under G and any Cartan subgroup of G is conjugate to one of them. We fix a maximal compact subgroup K of G as follows:

$$K = \left\{ g = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in SU(p,q); \quad A \in U(p),\ B \in U(q) \right\}.$$

Then its Lie algebra $\mathfrak{k}_0$ is given by

$$\mathfrak{k}_0 = \left\{ X = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}; \quad {}^t\overline{X_i} = -X_i \ (i = 1,2), \quad \sum \mathrm{tr}(X_i) = 0 \right\}.$$

Then the mapping θ: $X \to I_{p,q} X I_{p,q}$ is the Cartan involution. It is obvious that $T_0 \subseteq K$. In the rest of this section, we denote this compact Cartan subgroup by T instead of T_0. Then the Lie algebra of T and its complexification are as follows:

$$\begin{aligned} \mathfrak{t}_0 &= \{X = \mathrm{diag}(\sqrt{-1}y_1, \cdots, \sqrt{-1}y_n);\, y_j \in \mathbf{R}, \quad \textstyle\sum y_j = 0\} \\ \mathfrak{t} = (\mathfrak{t}_0)_c &= \{X = \mathrm{diag}(x_1, \cdots, x_n);\, x_j \in \mathbf{C}, \quad \textstyle\sum x_j = 0\}. \end{aligned}$$

We define an element $e_i \in \mathfrak{t}^*$ by $e_i(X) = x_i$. Then the root system $\Delta(\mathfrak{g}, \mathfrak{t})$ is given by

$$\Delta(\mathfrak{g}, \mathfrak{t}) = \{\pm(e_i - e_j); \quad 1 \leq i < j \leq n\}.$$

2.2. The reductive subgroup L for $SU(2,2)$. In the rest of this section, we put $p = q = 2$, and in this subsection we construct the reductive group L explicitly. For $G = SU(2,2)$, the set $\{T = T_0, T_1 \text{ and } T_2\}$ is a complete system of its Cartan subgroups. At first, put $\lambda_0 = e_1 - e_4 \in \mathfrak{t}^*$. We write L and $\mathfrak{q}$ instead of $L(\lambda_0)$ (in (1.2)) and $\mathfrak{q}(\lambda_0)$ (in (1.3)) respectively.

It is easy to see that only the roots $\pm(e_2 - e_3)$ are perpendicular to λ_0. Also,

$$\begin{aligned}\Delta(\mathfrak{u}) &= \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{t})\,;\, (\alpha, \lambda_0) > 0\} \\ &= \{e_1 - e_j\,;\, (2 \le j \le 4),\quad e_i - e_4\,;\, (2 \le i \le 3)\}.\end{aligned}$$

So we have that

$$(2.2)\qquad \begin{aligned} L &= \left\{ g = \begin{pmatrix} e^{i\varphi_1} & & \\ & g_1 & \\ & & e^{i\psi_1} \end{pmatrix} \in SU(2,2),\ g_1 \in U(1,1),\ \varphi_1,\ \psi_1 \in \mathbf{R} \right\}, \\ \mathfrak{l} &= \mathfrak{t} + \mathfrak{g}^{e_2-e_3} + \mathfrak{g}^{e_3-e_2}, \qquad \mathfrak{u} = \sum_{\alpha \in \Delta(\mathfrak{u})} \mathfrak{g}^{\alpha}, \\ \mathfrak{q} &= \mathfrak{l} + \mathfrak{u}. \end{aligned}$$

Obviously L is isomorphic to the direct product of $U(1,1)$ and $\mathbf{T}^1 = \{z \in \mathbf{C};\ |z| = 1.\}$

Let π be a one-dimensional representation of L defined as

$$(2.3)\qquad \pi(g) = e^{im\varphi_1} (\det g_1)^n.$$

Then $\lambda \in \mathfrak{t}^*$, the differential representation of $\pi_{|T}$, is given by

$$(2.4)\qquad \begin{aligned}\lambda &= m(e_1 - e_2) + (m+n)(e_2 - e_3) + (m+2n)(e_3 - e_4) \\ &= me_1 + ne_2 + ne_3 - (m+2n)e_4.\end{aligned}$$

We also denote this λ by $(m, n, n, -(m+2n))$ for brevity. Now fix a positive system $\Delta^+(\mathfrak{l}, \mathfrak{t}) = \{e_2 - e_3\}$, and assume that λ is $\mathfrak{u}$-admissible. With our parametrization, this condition is equivalent to $m \ge n \ge -\frac{m}{3}$.

Next we consider the case that $\lambda_1 = e_1 - e_2$. Choose such an element $w \in W(\mathfrak{g}, \mathfrak{t})$ that $w\lambda_0 = \lambda_1$. Then we easily get that

$$\begin{aligned} L_w = L(\lambda_1) &= \left\{ g = \begin{pmatrix} g_1 & & \\ & e^{i\psi_2} & \\ & & e^{i\psi_1} \end{pmatrix} \in SU(2,2),\quad g_1 \in U(2),\ \psi_1,\ \psi_2 \in \mathbf{R} \right\}, \\ \mathfrak{l}_w &= \mathfrak{t} + \mathfrak{g}^{e_1-e_2} + \mathfrak{g}^{e_2-e_1}, \qquad \mathfrak{u}_w = \sum_{\alpha \in \Delta(\mathfrak{u})} \mathfrak{g}^{w\alpha}, \\ \mathfrak{q}_w &= \mathfrak{l}_w + \mathfrak{u}_w. \end{aligned}$$

As mentioned in §1, there exists a unique one-dimensional representation π_w of L_w such that $w\lambda = (n, n, m, -(m+2n))$ is equal to the differential representation of $\pi_w|T$. In fact the explicit expression of π_w is given as

$$\pi_w(g) = (\det g_1)^m e^{in\psi_2}, \quad g = \begin{pmatrix} g_1 & & \\ & e^{i\psi_2} & \\ & & e^{i\psi_1} \end{pmatrix}, \; g_1 \in U(2), \psi_1, \; \psi_2 \in \mathbf{R}.$$

We remark that L_w becomes compact when $w\lambda_0$ is a compact root. In our setting, these two cases are typical ones and for elements in the W-orbit of $\lambda_0 = e_1 - e_4$, the following lemma holds.

Lemma 2.1. *Let $w \in W = W(\mathfrak{g}, \mathfrak{t})$ and $\lambda_0 = e_1 - e_4 \in \mathfrak{t}^*$. Then $L_w \cong U(2) \times \mathbf{T}^1$ if $w\lambda_0 \in \Delta(\mathfrak{k}, \mathfrak{t})$ and $L_w \cong U(1,1) \times \mathbf{T}^1$ if $w\lambda_0 \in \Delta(\mathfrak{g}, \mathfrak{t}) \backslash \Delta(\mathfrak{k}, \mathfrak{t})$.*

Proof: Let $\{ i, j\}$ be a subset of $\{ 1,2,3,4 \}$ such that $w\lambda_0 = e_i - e_j$. It is easy to see that $\Delta(\mathfrak{l}_w, \mathfrak{t}) = \{\pm(e_k - e_l)\}$, where $\{k, l\}$ is the complement of $\{i, j\}$ in $\{1,2,3,4 \}$. Since $\Delta(\mathfrak{k}, \mathfrak{t}) = \{\pm(e_1 - e_2), \pm(e_3 - e_4)\}$, we get the conclusion.

2.3. Now we will proceed to construct a $(\mathfrak{g}, K)$-module $A(w\lambda, \pi_w) = \mathcal{R}_{\mathfrak{q}_w}(\pi_w)$ concretely. At first we treat the case that L_w is not compact. To study $A(w\lambda, \pi_w)$ for these w's, it is sufficient to consider the case $w = 1$,that is, $\lambda_0 = w\lambda_0 = e_1 - e_4$ and $w\lambda = \lambda$. Put

$$S = \left\{ \tilde{g} = \begin{pmatrix} 1 & & \\ & g_1 & \\ & & 1 \end{pmatrix} \in L(\lambda_0), g_1 \in SU(1,1) \right\},$$
$$T_S = K \cap S = \left\{ t_\theta = \operatorname{diag}(1, e^{i\theta}, e^{-i\theta}, 1) \right\}.$$

By identifying $\tilde{g}$ with g_1, we view $SU(1,1)$ as a subgroup of $L(\lambda_0)$. It is apparent that T_S is a maximal compact subgroup of S as well as a compact Cartan subgroup of S. We set

$$A = \left\{ a_t = \begin{pmatrix} \operatorname{ch}t & \operatorname{sh}t \\ \operatorname{sh}t & \operatorname{ch}t \end{pmatrix} \right\} (\subset S). \tag{2.5}$$

Then A is the vector subgroup of a maximally $\mathbf{R}$-split Cartan subgroup of $L(\lambda_0)$. Let M_S be the centralizer of A in T_S and $P = M_S AN$ a minimal parabolic subgroup of S. Here N is chosen such that it satisfies the following condition. Denote by α the unique positive (restricted) root in $\Delta(\mathfrak{s}, \mathfrak{a})$ corresponding to N. Then $\rho_P = \frac{\alpha}{2}$ can be lifted up to a character of A, which is denoted also by ρ_P. In our setting $\rho_P(a_t)$ is assumed to be expressed as $\rho_P(a_t) = e^t$.

Define a one-dimensional representation of $P = M_SAN$ as $(1 \otimes (-\rho_P) \otimes 1)(ma_tn) = e^{-t}$ and induce it up to a (non-unitary principal series) representation of S. As is well known, $\mathrm{Ind}_P^S(1 \otimes (-\rho_P) \otimes 1)$ contains the trivial representation $\mathbf{1}$ of S as a subrepresentation.

On the other hand, the root system $\Delta(\mathfrak{s}, \mathfrak{t}_\mathfrak{s})$ consists of two elements $\pm\beta$ and we will identify β with $e_2 - e_3 \in \Delta(\mathfrak{g}, \mathfrak{t})$. Put $\mu_0 = \frac{\beta}{2}$. Then we get the following exact sequence:

$$
\begin{aligned}
(2.6) \qquad & 0 \longrightarrow \mathbf{1} \longrightarrow \mathrm{Ind}_P^S(1 \otimes -\rho_P \otimes 1) \\
& \longrightarrow \Theta^S(\mu_0; C^S) \otimes \Theta^S(w_0\mu_0; w_0C^S) \longrightarrow 0.
\end{aligned}
$$

Here $w_0 = s_\beta$ denotes the reflection with respect to the hyperplane defined by $\beta(X) = 0$ and C^S is the Weyl chamber in $\sqrt{-1}(\mathfrak{t}_\mathfrak{s})_0^*$ with respect to which β is dominant regular. Let us recall that discrete series repersentations $\Theta^S(\mu_0; C^S)$ and $\Theta^S(w_0\mu_0; w_0C^S)$ have the same infinitesimal character with the trivial representation of S. Denote by D the center of $L = L(\lambda_0)$ and put $\chi = \lambda_{|D}$. Since $L = S \cdot D$, π can be expressed as $\pi_w = \mathbf{1} \otimes \chi$. Multiplying the character χ of D to the sequence (2.6), we get the following exact sequence of representations of L.

$$
\begin{aligned}
(2.7) \qquad & 0 \longrightarrow \pi \longrightarrow \mathrm{Ind}_P^S(1 \otimes (-\rho_P) \otimes 1) \otimes \chi \\
& \longrightarrow \Theta^S(\mu_0; C^S) \otimes \chi \oplus \Theta^S(w_0\mu_0; w_0C^S) \otimes \chi \longrightarrow 0.
\end{aligned}
$$

Put $P_L = (DM_S)AN$, then it is a minimal parabolic subgroup of L. Let us denote by C^L the Weyl chamber in $\sqrt{-1}\mathfrak{t}_0^*$ for $\mathfrak{l}$ determined in the same way as C^S. Since $\mathrm{Ind}_P^S(1 \otimes (-\rho_P) \otimes 1) \otimes \chi \cong \mathrm{Ind}_{P_L}^L(\chi \otimes (-\rho_P) \otimes 1)$ and $\Theta^S(\mu_0; C^S) \otimes \chi \cong \Theta^L(\lambda + \mu_0; C^L)$, the sequence (2.7) is rewritten as

$$
\begin{aligned}
& 0 \longrightarrow \pi \longrightarrow \mathrm{Ind}_{P_L}^L(\chi \otimes (-\rho_P) \otimes 1) \\
& \longrightarrow \Theta^L(\lambda + \mu_0, C^L) \oplus \Theta^L(\lambda + w_0\mu_0, w_0C^L) \longrightarrow 0.
\end{aligned}
$$

We regard each of these representations as $(\mathfrak{l}, L \cap K)$-modules, and apply the functor $\mathcal{R}$ to this sequence. Then we obtain the next long exact sequence:

$$
\begin{aligned}
(2.8) \qquad & \cdots \longrightarrow \mathcal{R}_\mathfrak{q}^{j-1}\left(\Theta^L(\lambda + \mu_0, C^L)\right) \oplus \mathcal{R}_\mathfrak{q}^{j-1}\left(\Theta^L(\lambda + w_0\mu_0, w_0C^L)\right) \\
& \longrightarrow \mathcal{R}_\mathfrak{q}^j(\pi) \longrightarrow \mathcal{R}_\mathfrak{q}^j\left(\mathrm{Ind}\,_{P_L}^L(\chi \otimes (-\rho_P) \otimes 1)\right) \\
& \longrightarrow \mathcal{R}_\mathfrak{q}^j\left(\Theta^L(\lambda + \mu_0, C^L)\right) \oplus \mathcal{R}_\mathfrak{q}^j\left(\Theta^L(\lambda + w_0\mu_0, w_0C^L)\right) \\
& \longrightarrow \mathcal{R}_\mathfrak{q}^{j+1}(\pi) \longrightarrow \cdots
\end{aligned}
$$

Now we put $i = \dim(\mathfrak{u} \cap \mathfrak{k})$. Vogan showed that for any $j > i$ and any $(\mathfrak{l}, L \cap K)$-module W, $\mathcal{R}^j_{\mathfrak{q}}(W) = 0$ ([10], Cor.6.3.21). On the other hand, by virtue of Theorem 6.3.12 in [10] and Proposition 1.4 in §1, we have that

$$
\begin{aligned}
(2.9)\qquad & \mathcal{R}^{i-1}_{\mathfrak{q}}\left(\Theta^L(\lambda+\mu_0, C^L)\right) = 0,\\
& \mathcal{R}^{i}_{\mathfrak{q}}\left(\Theta^L(\lambda+\mu_0, C^L)\right) = \left(\mathcal{R}^{\mathfrak{g}}_{\mathfrak{q}}\right)^i \left(\mathcal{R}^{\mathfrak{l}}_{\mathfrak{q}^0}\right)^0 (\mathrm{C}_\lambda) = \left(\mathcal{R}^{\mathfrak{g}}_{\mathfrak{b}_1}\right)^i (\mathrm{C}_\lambda)\\
& \qquad = \Theta^G(\lambda+\rho_1, C).
\end{aligned}
$$

Here $\mathfrak{q}^0 = \mathfrak{t} + \mathfrak{g}^\beta$, $\quad \mathfrak{b}_1 = \mathfrak{t} + \mathfrak{g}^\beta + \mathfrak{u}$ and $\rho_1 = \frac{1}{2}\left(\beta + \sum_{\alpha\in\Delta(\mathfrak{u})}\alpha\right)$. We choose the Weyl chamber C in $\sqrt{-1}\mathfrak{t}_0^*$ for $\mathfrak{g}$ with respect to which ρ_1 is dominant.

Similarly, we get that

$$
\begin{aligned}
(2.10)\qquad & \\
& \mathcal{R}^{i-1}_{\mathfrak{q}}\left(\Theta^L(\lambda+w_0\mu_0, w_0C^L)\right) = 0,\\
& \mathcal{R}^{i}_{\mathfrak{q}}\left(\Theta^L(\lambda+w_0\mu_0, w_0C^L)\right) = \left(\mathcal{R}^{\mathfrak{g}}_{\mathfrak{b}_2}\right)^i (\mathrm{C}_{w_0\lambda}) = \Theta^G(\lambda+\rho_2, w_0C).
\end{aligned}
$$

Here $\mathfrak{b}_2 = \mathfrak{t} + \mathfrak{g}^{-\beta} + \mathfrak{u}$, $\rho_2 = \frac{1}{2}\left(-\beta + \sum_{\alpha\in\Delta(\mathfrak{u})}\alpha\right)$ and let us recall $w_0 = s_\beta$.

Combining these relations, we obtain the following short exact sequence:

$$
\begin{aligned}
(2.11)\qquad & 0 \longrightarrow A_{\mathfrak{q}}(\lambda) \longrightarrow \mathcal{R}^i_{\mathfrak{q}}\left(\mathrm{Ind}\,^L_{P_L}(\chi \otimes (-\rho_P) \otimes 1)\right)\\
& \qquad \longrightarrow \Theta^G(\lambda+\rho_1, C) \oplus \Theta^G(\lambda+\rho_2, w_0C) \longrightarrow 0.
\end{aligned}
$$

2.4. Finally, we will state the relation between cohomological parabolic inductions and (usual) parabolic inductions. In order that, we introduce some notations. We assume that $L = L(\tilde{\lambda})$ is quasi-split and fix a θ-stable maximally $\mathbf{R}$-split Cartan subgroup H of L. Then H is decomposed as $H = T_L A_L$ so that T_L is contained in K and A_L is a vector subgroup. Put $M_G A_L = Z_G(A_L) = \{g \in G;\, ga = ag \text{ for any } a \in A_L\}$. Let us denote by $\widehat{T}_L$ the totality of characters of T_L, and take a $\delta \in \widehat{T}_L$ which is fine with respect to L. (For the definition of 'fine', see [10],p.173. In our case every $\delta \in \widehat{T}_L$ is fine because L is split.) We fix a $\nu \in \widehat{A}_L \cong \mathfrak{a}_L^*$ and choose a cuspidal parabolic subgroup $P_G = M_G A_L N$ of G such that ν is negative for the roots of $\mathfrak{a}_L$ in $\mathfrak{n}$. Pick up $N_L \subseteq N$ as explained in §2.3, then $P_L = T_L A_L N_L$ is a minimal parabolic subgroup of L.

Lemma 2.2. (Vogan [10]) *In the above setting, there exists a discrete series representation π_d of M_G such that the following two $(\mathfrak{g}, K)$-modules are equivalent:*

$$
\mathcal{R}^i_{\mathfrak{q}}\left(\mathrm{Ind}\,^L_{P_L}(\delta \otimes \nu \otimes 1)\right) \cong \mathrm{Ind}\,^G_{M_G A_L N}(\pi_d \otimes \nu \otimes 1).
$$

It is explicitly known how the discrete series π_d is parametrized. But we omit an explanation of it because it is not necessary in the following consideration.

Now we return to the case that $G = SU(2,2), L = L(\lambda_0)$ and $\lambda_0 = e_1 - e_4$. We choose T_1 as a maximally $\mathbf{R}$-split Cartan subgroup H of L. That is, $T_L = T_1 \cap K = \left\{\mathrm{diag}(e^{i\varphi_1}, e^{i\theta_1}, e^{i\theta_1}, e^{i\psi_1}) \in SU(2,2)\right\}$ and $A_L = A$ as in (2.5). Since $\rho(\mathfrak{u}) = (\frac{3}{2}, 0, 0, -\frac{3}{2})$, it is easy to see that

$$\begin{aligned}
\rho_1 &= \left(\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\right), \ \rho_2 = \left(\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{3}{2}\right), \\
\lambda + \rho_1 &= \left(m + \frac{3}{2}, n + \frac{1}{2}, n - \frac{1}{2}, -(m + 2n + \frac{3}{2})\right), \\
\lambda + \rho_2 &= \left(m + \frac{3}{2}, n - \frac{1}{2}, n + \frac{1}{2}, -(m + 2n + \frac{3}{2})\right) = s_\beta(\lambda + \rho_1).
\end{aligned}$$

Let us apply Lemma 2.2 to $\mathcal{R}^i_{\mathfrak{q}}\left(\mathrm{Ind}^L_{P_L}(\chi \otimes (-\rho_P) \otimes 1)\right)$ in (2.11). Then the exact sequence (2.11) is rewritten as follows:

$$\begin{aligned}
(2.12) \qquad 0 \longrightarrow A_{\mathfrak{q}}(\lambda) &\longrightarrow \mathrm{Ind}^G_{M_G A_L N}(\pi_d \otimes (-\rho_P) \otimes 1) \\
&\longrightarrow \Theta^G(\lambda + \rho_1, C) \oplus \Theta^G(\lambda + \rho_2, w_0 C) \longrightarrow 0.
\end{aligned}$$

Therefore the caluculation of the character of $A_{\mathfrak{q}}(\lambda)$ is reduced to that for standard modules and discrete series. Since $\mathrm{Ind}^G_{M_G A_L N}(\pi_d \otimes (-\rho_P) \otimes 1)$ is not tempered, neither is $A_{\mathfrak{q}}(\lambda)$. When $L(w\lambda_0)$ is isomorphic to $U(1,1) \times \mathbf{T}^1$, $A(w\lambda, \pi_w)$ has the same structure as $A_{\mathfrak{q}}(\lambda)$.

On the contrary, when $L(w\lambda_0)$ is isomorphic to $U(2) \times \mathbf{T}^1$, $A(w\lambda, \pi_w) = A_{\mathfrak{q}_w}(\pi_w)$ corresponds to a discrete series representation of G which has infinitesimal character $\lambda + \rho(\mathfrak{q}_w)$ (see [1], p.281).

In the next section, we will study character identities in the enlarged L-packet $\Pi = \{A(w\lambda, \pi_w); w \in W_G(T)\backslash W(\mathfrak{g}, \mathfrak{t})/W(\mathfrak{l}, \mathfrak{t})\}$. We note here again that Π consists of both tempered $(\mathfrak{g}, K)$-modules and non-tempered ones.

Remark. Johnson constructed a resolution of $A_{\mathfrak{q}}(\lambda)$ by standard modules in [5], and the sequence (2.12) is a special case of his resolution. But in our case, $L(\lambda_0)$ has only two types of Cartan subgroups, so the length of the resolution is at most three. For this reason, we drew out the sequence (2.12) directly using the properties of the functor $\mathcal{R}$.

§3. Character identities in the enlarged L-packet

3.1. Analytic functions κ^t and $\tilde{\kappa}^t$. In this subsection, we review some general theory about invariant eigendistributions. Let G be a connected semisimple

Lie group with finite center and Θ an invariant eigendistribution on G. We denote by G' the set of all regular elements in G. Then Θ is not only a locally summable function on G but a real analytic one on G', which we denote by the same letter Θ.

Let T be a Cartan subgroup of G. For a root $\alpha \in \Delta(\mathfrak{g}, \mathfrak{t})$, we choose a root vector X_α in $\mathfrak{g}^\alpha$ and define a character ξ_α on T as

$$\xi_\alpha(t)X_\alpha = \mathrm{Ad}(t)X_\alpha \qquad (t \in T) \tag{3.1}$$

We fix a positive root system $\Delta^+(\mathfrak{g}, \mathfrak{t})$ and put $\rho = \frac{1}{2}\sum_{\alpha \in \Delta^+(\mathfrak{g},\mathfrak{t})} \alpha$. Under the assumption that G is acceptable (cf. [2], p.33), there exists a character ξ_ρ on T such that its differential is equal to $\rho \in \mathfrak{t}^*$. Now let us define the following functions on $T \cap G'$ as

$$\Delta^{\mathfrak{t}}(t) = \xi_\rho(t) \prod_{\alpha \in \Delta^+(\mathfrak{g},\mathfrak{t})} \left(1 - \xi_\alpha(t)^{-1}\right),$$

$$\varepsilon_R^{\mathfrak{t}}(t) = \mathrm{sgn}\left(\prod_{\alpha \in \Delta_R^+(\mathfrak{g},\mathfrak{t})} (1 - \xi_\alpha(t)^{-1})\right) \qquad (t \in T \cap G').$$

Here $\Delta_R^+(\mathfrak{g}, \mathfrak{t})$ denotes the set of all real positive roots. For each Cartan subgroup T and a given invariant eigendistribution Θ, we put

$$\begin{aligned} \tilde{\kappa}^{\mathfrak{t}}(t) &= \Delta^{\mathfrak{t}}(t)\Theta(t), \\ \kappa^{\mathfrak{t}}(t) &= \varepsilon_R^{\mathfrak{t}}(t)\, \Delta^{\mathfrak{t}}(t)\Theta(t) \qquad (t \in T \cap G'). \end{aligned} \tag{3.2}$$

Since Θ is analytic on $T \cap G'$, so are $\tilde{\kappa}^{\mathfrak{t}}$ and $\kappa^{\mathfrak{t}}$. Furthermore, they can be extended to analytic functions on $T'(\mathbf{R})$, where $T'(\mathbf{R}) = \{t \in T\,;\, \xi_\alpha(t) \neq 1\,, \forall\alpha \in \Delta_R^+(\mathfrak{g}, \mathfrak{t})\}$. Now we list up their properties.

1) Let F be a connected component of $T'(\mathbf{R})$ and take an element a_0 in $Cl\,(F)$, the closure of F. We choose an element $\mu \in \mathfrak{t}^*$ which corresponds to the infinitesimal character of Θ through Harish-Chandra isomorphism. Then $\tilde{\kappa}^{\mathfrak{t}}$ is expressed as:

$$\tilde{\kappa}^{\mathfrak{t}}(a_0 \exp X) = \sum_{w \in W(\mathfrak{g},\mathfrak{t})} p_w(X, F) \exp(w\mu, X), \tag{3.3}$$

for $a_0 \exp X \in F$ and $X \in \mathfrak{t}_0$. We say Θ is regular when $w\mu \neq \mu$ for any $w \neq 1$ in $W(\mathfrak{g}, \mathfrak{t})$. In general, $p_w(X, F)$ is a polynomial function, but when Θ is regular, it is a constant. In the following, we will treat only regular cases, so we write $p_w(F)$ instead of $p_w(X, F)$. The function $\kappa^{\mathfrak{t}}(t)$ has a similar expression as $\tilde{\kappa}^{\mathfrak{t}}$.

2) Put $W_G(F) = \{w \in W_G(T);\ w(F) \subseteq F\}$. For $w \in W_G(F)$ and $t \in F$, we define a function $\varepsilon(w,t)$ by $(\varepsilon_R^{\mathfrak{t}}\Delta^{\mathfrak{t}})(wt) = \varepsilon(w,t)(\varepsilon_R^{\mathfrak{t}}\Delta^{\mathfrak{t}})(t)$. Since Θ is invariant under inner automorphisms of G, $\kappa^{\mathfrak{t}}$ satisfies the same symmetry condition as $\varepsilon_R^{\mathfrak{t}}\Delta^{\mathfrak{t}}$, that is,

$$\kappa^{\mathfrak{t}}(wt) = \varepsilon(w,t)\kappa^{\mathfrak{t}}(t).$$

3) For a real root $\alpha \in \Delta(\mathfrak{g},\mathfrak{t})$, let us denote by ν_α the Cayley transformation with respect to α. (For definition, see [3], p.41.) Put $\mathfrak{s}_0 = \nu_\alpha(\mathfrak{t}) \cap \mathfrak{g}_0$. Then $\mathfrak{s}_0$ is another Cartan subalgebra of $\mathfrak{g}_0$ which is not conjugate to $\mathfrak{t}_0$ under G, and we denote by S the corresponding Cartan subgroup of G. For a root $\gamma \in \Delta(\mathfrak{g},\mathfrak{t})$, we define $\nu_\alpha\gamma$ by $(\nu_\alpha\gamma)(X) = \gamma\left(\nu_\alpha^{-1}(X)\right)$ for $X \in \mathfrak{s}$. Obviously, it is a root of $(\mathfrak{g},\mathfrak{s})$. We take $\nu_\alpha\left(\Delta^+(\mathfrak{g},\mathfrak{t})\right)$ as fixed positive system of $\Delta(\mathfrak{g},\mathfrak{s})$. Let H_γ be the element of $\mathfrak{t}$ such that $B(H_\gamma, H) = \gamma(H)$ for $H \in \mathfrak{t}$, where B is the Killing form of $\mathfrak{g}$. Note that H_γ belongs to $\mathfrak{t}_0$ or $\sqrt{-1}\mathfrak{t}_0$ according as γ is real or imaginary respectively.

We put $\beta = \nu_\alpha\alpha$, and regard H_α and H_β as differential operators in the following way.

$$\begin{aligned} &(3.4) \qquad H_\alpha\tilde{\kappa}^{\mathfrak{t}}(g) = \frac{d}{dt}\tilde{\kappa}^{\mathfrak{t}}(g\exp tH_\alpha)_{|t=0} \qquad (g \in T \cap G'), \\ &\qquad\qquad H_\beta\tilde{\kappa}^{\mathfrak{s}}(g) = \frac{1}{\sqrt{-1}}\frac{d}{dt}\tilde{\kappa}^{\mathfrak{s}}(g\exp\sqrt{-1}tH_\beta)_{|t=0} \qquad (g \in S \cap G'). \end{aligned}$$

Then for any semi-regular element $a \in T \cap S$, $\tilde{\kappa}^{\mathfrak{t}}$ and $\tilde{\kappa}^{\mathfrak{s}}$ satisfy the next boundary condition:

$$(3.5) \qquad (H_\alpha\tilde{\kappa}^{\mathfrak{t}})(a) = (H_\beta\tilde{\kappa}^{\mathfrak{s}})(a).$$

We remark that the both sides denote the limit values at a.

4) We assume that Θ is tempered. Then so is $\tilde{\kappa}^{\mathfrak{t}}$ on T. In particular, $\tilde{\kappa}^{\mathfrak{t}}$ is bounded if Θ is regular tempered.

3.2. Heredity of the property (P). In this subsection, we investigate the case $G = SU(p,p)$. Then the set $Car(G) = \{T = T_0, T_1, \cdots, T_p\}$ is a complete representative system of Cartan subgroups of G. We write $\tilde{\kappa}^j$ and κ^j instead of $\tilde{\kappa}^{\mathfrak{t}_j}$ and $\kappa^{\mathfrak{t}_j}$ respectively. As is easily seen, $\mathfrak{t}_{j-1} = \nu_{\alpha_j}(\mathfrak{t}_j)$, where α_j is a real root of $(\mathfrak{g},\mathfrak{t}_j)$ defined in the following way: Let $t = t^-t^+$ be an element in T_j such that $t^- \in T_j^-$ and $t^+ \in T_j^+$ are expressed as in (2.1),(2.1)' respectively. Then α_j is given by $\alpha_j(\log t) = 2t_j$.

We say that $T_i > T_j$ when $i < j$. For an invariant eigendistribution Θ, we put $Supp\,\Theta = \{T_j \in Car(G);\ \Theta_{|T_j\cap G'} \not\equiv 0\}$, and call the highest element in $Supp\,\Theta$ its height.

Let us denote by $\Delta_I(\mathfrak{g},\mathfrak{t}_j)$,$\Delta_c(\mathfrak{g},\mathfrak{t}_j)$ and $\Delta_s(\mathfrak{g},\mathfrak{t}_j)$ the set of all imaginary, compact imaginary and singular imaginary roots respectively. In the rest of this note, we fix a positive system $\Delta^+(\mathfrak{g},\mathfrak{t}_j)$ such that

$$s_\alpha\left(\Delta_s^+(\mathfrak{g},\mathfrak{t}_j)\right) \subseteq \Delta_s^+(\mathfrak{g},\mathfrak{t}_j) \qquad (\forall\alpha \in \Delta_c(\mathfrak{g},\mathfrak{t}_j)),$$

where $\Delta_s^+(\mathfrak{g},\mathfrak{t}_j) = \Delta^+(\mathfrak{g},\mathfrak{t}_j) \cap \Delta_s(\mathfrak{g},\mathfrak{t}_j)$. Let $W_I(\mathfrak{g},\mathfrak{t}_j)$ be the subgroup of $W(\mathfrak{g},\mathfrak{t}_j)$ generated by s_α's ($\alpha \in \Delta_I(\mathfrak{g},\mathfrak{t}_j)$). We denote by w_j the longest element in $W_I(\mathfrak{g},\mathfrak{t}_j)$ with respect to the above positive system. Then w_j acts on T_j by $w_j(\exp X) = \exp(w_j X)$ $(X \in (\mathfrak{t}_j)_0)$.

Definition. We say that Θ satisfies the property (P) on T_j if the following equation holds:

$$\Theta(w_j t) = -\Theta(t) \quad (t \in T_j \cap G').$$

Now we show a fundamental proposition about regular tempered invariant eigendistributions.

Proposition 3.1. *Let Θ be a regular tempered invariant eigendistribution on $SU(p,p)$. Suppose Θ satisfies the property (P) on T_j. If T_j is equal to or lower than the height of Θ, then Θ satisfies the property (P) on T_i for any $i \geq j$.*

Proof: We will show that Θ satisfies the property (P) on T_{j+1}. Let F^+ be the connected component of $T'_{j+1}(\mathbf{R})$ which is characterized as

$$F^+ = \{t \in T'_{j+1}(\mathbf{R})\,; \xi_\alpha(t) > 1 \text{ for any real positive root } \alpha\}.$$

Then for any connected component F of $T'_{j+1}(\mathbf{R})$, there exists a sequence of real roots $\alpha_1,\cdots,\alpha_r$ such that $s_{\alpha_1}\cdots s_{\alpha_r}F^+ = F$. Since s_{α_i} belongs to $W_G(T_{j+1})$ and $s_{\alpha_i}w_{j+1} = w_{j+1}s_{\alpha_i}$, we get that

$$\begin{aligned}\Theta(w_{j+1}s_{\alpha_1}\cdots s_{\alpha_r}t) &= \Theta(s_{\alpha_1}\cdots s_{\alpha_r}w_{j+1}t) = \Theta(w_{j+1}t),\\ \Theta(s_{\alpha_1}\cdots s_{\alpha_r}t) &= \Theta(t), \quad (t \in F^+ \cap G')\end{aligned}$$

Therefore it is sufficient to to show that $\Theta(w_{j+1}t) = -\Theta(t)$ for $t \in F^+ \cap G'$.

Put $\mathfrak{t}^+_{j+1} = \{X \in (\mathfrak{t}_{j+1})_0;\ \exp X \in F^+\}$. As mentioned in (3.3), $\tilde{\kappa}^{j+1}$ is expressed on F^+ as

$$\tilde{\kappa}^{j+1}(\exp X) = \sum_{w\in W(\mathfrak{g},\mathfrak{t}_{j+1})} p_w(F^+)\exp(w\mu, X) \qquad (X \in \mathfrak{t}^+_{j+1}),$$

with $a_0 = 1$. Here we can assume that μ satisfies the condition $(\mu,\alpha) \geq 0$ for any $\alpha \in \Delta_I^+(\mathfrak{g},\mathfrak{t}_{j+1})\,(= \Delta^+(\mathfrak{g},\mathfrak{t}_{j+1}) \cap \Delta_I(\mathfrak{g},\mathfrak{t}_{j+1}))$. Now we fix an element ω in

$W(\mathfrak{g}, \mathfrak{t}_{j+1})$. If $\Re(\omega\mu, H_\alpha) = 0$ for any real root $\alpha \in \Delta(\mathfrak{g}, \mathfrak{t}_{j+1})$, the height of Θ cannot exceed T_{j+1}. (For $z \in \mathbf{C}$, $\Re z$ denotes its real part.) Therefore we can choose a positive real root α such that $\Re(\omega\mu, H_\alpha) \neq 0$. Then $\nu_\alpha(\mathfrak{t}_{j+1}) \cap \mathfrak{g}_0$ is conjugate to $(\mathfrak{t}_j)_0$ under G.

We first consider the case $\Re(\omega\mu, H_\alpha) > 0$. Then $\exp(\omega\mu, X)$ is unbounded on $\mathfrak{t}^+_{j+1}$. Since the set $\{\exp(w\mu, X); w \in W(\mathfrak{g}, \mathfrak{t}_{j+1})\}$ is a family of linearly independent functions on $\mathfrak{t}^+_{j+1}$ and $\tilde{\kappa}^{j+1}$ is a bounded function, $p_\omega(F^+)$, the coefficient of $\exp(\omega\mu, X)$ in $\tilde{\kappa}^{j+1}$, must be zero. On the other hand, since

$$\begin{aligned}\Re(w_{j+1}\omega\mu, H_\alpha) &= \Re(\omega\mu, w_{j+1}H_\alpha)\\ &= \Re(\omega\mu, H_\alpha) > 0,\end{aligned}$$

the function $\exp(w_{j+1}\omega\mu, X)$ is unbounded on $\mathfrak{t}^+_{j+1}$. So $p_{w_{j+1}\omega}(F^+) = 0$. In this case,

$$p_\omega(F^+) = p_{w_{j+1}\omega}(F^+) = 0. \tag{3.7}$$

Next we consider the case $\Re(\omega\mu, H_\alpha) < 0$. Now we write down the boundary condition (3.5) in our case explicitly. Let $\hat{T}_j$ be a Cartan subgroup corresponding to $(\hat{\mathfrak{t}}_j)_0 = \nu_\alpha(\mathfrak{t}_{j+1}) \cap \mathfrak{g}_0$ and $\hat{F}^+$ the connected component of $\hat{T}'_j(\mathbf{R})$ just as F^+. We denote by A the totality of semi-regular elements in $F^+ \cap \hat{F}^+$. Let X be an element in $\mathfrak{t}_{j+1} \cap \hat{\mathfrak{t}}_j$ such that $\exp X \in A$. Then we get the following equation:

$$\begin{aligned}&\sum_{w\in W(\mathfrak{g},\mathfrak{t}_{j+1})} p_w(F^+)w\mu(H_\alpha)\exp(w\mu, X)\\ &= \sum_{w\in W(\mathfrak{g},\hat{\mathfrak{t}}_j)} p_w(\hat{F}^+)w\hat{\mu}(H_\beta)\exp(w\hat{\mu}, X),\end{aligned} \tag{3.8}$$

where $\beta = \nu_\alpha\alpha \in \Delta(\mathfrak{g}, \hat{\mathfrak{t}}_j)$ and $\hat{\mu} = \nu_\alpha\mu\nu_\alpha{}^{-1} \in \hat{\mathfrak{t}}_j{}^*$. Apparently, $\exp(w\mu, X) = \exp(s_\alpha w\mu, X)$ and $\exp(w\hat{\mu}, X) = \exp(s_\beta w\hat{\mu}, X)$. In addition, it is easy to see that under the identification of the preceeding pairs, the set $\{\exp w\mu; w \in W(\mathfrak{g}, \mathfrak{t}_{j+1})\}$ gives a family of linearly independent functions on A. Thus we get that

$$p_w(F^+) - p_{s_\alpha w}(F^+) = p_{\hat{w}}(\hat{F}^+) - p_{s_\beta\hat{w}}(\hat{F}^+). \tag{3.9}$$

Here the mapping $w \longrightarrow \hat{w}$ is an isomorphism from $W(\mathfrak{g}, \mathfrak{t}_{j+1})$ to $W(\mathfrak{g}, \hat{\mathfrak{t}}_j)$ determined by $\hat{s}_\gamma = s_\delta$ where $\delta = \nu_\alpha\gamma\nu_\alpha{}^{-1} \in \Delta(\mathfrak{g}, \hat{\mathfrak{t}}_j)$.

Since $\Re(s_\alpha\omega\mu, H_\alpha) > 0$, we get $p_{s_\alpha\omega}(F^+) = 0$ as proved above. Therefore by (3.9), we obtain

$$p_\omega(F^+) = p_{\hat\omega}(\hat F^+) - p_{s_\beta\hat\omega}(\hat F^+).$$

Furthermore, $\Re(s_\alpha w_{j+1}\omega\mu, H_\alpha) = -\Re(w_{j+1}\omega\mu, H_\alpha) = -\Re(\omega\mu, H_\alpha) > 0$, so we get $p_{s_\alpha w_{j+1}\omega}(F^+) = 0$ similarly. Since we choose $\Delta^+(\mathfrak{g}, \mathfrak{t}_j)$ compatibly for each j, we see that $\hat w_{j+1}s_\beta = s_\beta \hat w_{j+1} = w_j$, where w_j is the longest element in $W_I(\mathfrak{g}, \hat{\mathfrak{t}}_j)$. Combining (3.9) with this relation, we have

$$\begin{aligned}(3.10)\qquad p_{w_{j+1}\omega}(F^+) &= p_{\hat w_{j+1}\hat\omega_0}(\hat F^+) - p_{s_\beta\hat w_{j+1}\hat\omega}(\hat F^+)\\ &= -p_{w_j\hat\omega}(\hat F^+) + p_{s_\beta w_j\hat\omega}(\hat F^+),\\ &= -p_{w_j\hat\omega}(\hat F^+) + p_{w_j s_\beta\hat\omega}(\hat F^+).\end{aligned}$$

By the way, we assumed that Θ satisfies the property (P) on $\hat T_j'$: $\Theta(w_jt) = -\Theta(t)$ $(t \in T_j \cap G')$. We denote by $l(w)$ the length of w, then this equation easily can be transformed into

$$\tilde\kappa^j(w_jt) = (-1)^{l(w_j)+1}\tilde\kappa^j(t) \quad (t \in T_j \cap G').$$

So equalities $p_{w_j\hat w}(\hat F^+) = (-1)^{l(w_j)+1}p_{\hat w}(\hat F^+)$ hold for any $\hat w \in W(\mathfrak{g}, \hat{\mathfrak{t}}_j)$. Hence we get

$$\begin{aligned}(3.11)\qquad (-1)^{l(w_{j+1})+1}p_{w_{j+1}\omega}(F^+) &= (-1)^{l(w_{j+1})}\left\{p_{w_j\hat\omega}(\hat F^+) - p_{w_js_\beta\hat\omega}(\hat F^+)\right\}\\ &= (-1)^{l(w_{j+1})+l(w_j)+1}\left\{p_{\hat\omega}(\hat F^+) - p_{s_\beta\hat\omega}(\hat F^+)\right\}\\ &= p_\omega(F^+).\end{aligned}$$

Combining (3.7) and (3.11), we obtain

$$\tilde\kappa^{j+1}(w_{j+1}t) = (-1)^{l(w_{j+1})+1}\tilde\kappa^{j+1}(t) \qquad (t \in F^+ \cap G').$$

This means that Θ satisfies the property (P) on T_{j+1}.

We can repeat the above process as many times as necessary. So this completes the proof of Proposition 3.1.

3.3. Character identities among discrete series for $SU(p,q)$. In this subsection, we assume that Θ is a linear combination of the characters of discrete series representations of $G = SU(p,q)$ $(p \geq q)$. Let us recall that T_q is a maximally $\mathbf{R}$-split Cartan subgroup of G and T_0 a compact one. Then the next propsoition is proved in [7].

Proposition 3.2. *In the above setting, suppose that Θ is identically zero on $T_q \cap G'$. Then Θ satisfies the property(P) on T_0, that is,*

$$\Theta(w_0 t) = -\Theta(t) \qquad (t \in T_0 \cap G').$$

Here w_0 denotes the longest element in $W(\mathfrak{g}, \mathfrak{t}_0) = W_I(\mathfrak{g}, \mathfrak{t}_0)$.

In this paper, we use this proposition only for $G = SU(p,p)$ and $p = 1$ or 2. We review the case $p = 1$. We fix a root $\beta \in \Delta(\mathfrak{g}, \mathfrak{t}_0)$, then the complete list of discrete series representations are as follows:

$$\Theta^G\left(\frac{n\beta}{2}, C\right), \qquad \Theta^G\left(-\frac{n\beta}{2}, s_\beta C\right) \qquad (n = 1, 2, \cdots)$$

Here C is the Weyl chamber in $\sqrt{-1}\mathfrak{t}_0^*$ with respect to which β is dominant. To be more concrete, put $t_\theta = \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix}$ and choose β such that $\xi_\beta(t_\theta) = e^{2i\theta}$. Then

$$\left(\Delta^0 \Theta^G\left(\frac{n\beta}{2}, C\right)\right)(t_\theta) = e^{in\theta},$$

$$\left(\Delta^0 \Theta^G\left(-\frac{n\beta}{2}, s_\beta C\right)\right)(t_\theta) = -e^{-in\theta}$$

As is well known, on $T^1 \cap G'$, both $\Theta^G\left(\frac{n\beta}{2}, C\right)$ and $\Theta^G\left(-\frac{n\beta}{2}, s_\beta C\right)$ have the same expression.

Since only $\Theta^G\left(\frac{n\beta}{2}, C\right)$ and $\Theta^G\left(-\frac{n\beta}{2}, s_\beta C\right)$ have the same infinitesimal character $\frac{n\beta}{2}$ (or $s_\beta \frac{n\beta}{2} = -\frac{n\beta}{2}$), Θ is expressed as $\Theta = c_1 \Theta^G\left(\frac{n\beta}{2}, C\right) + c_2 \Theta^G\left(-\frac{n\beta}{2}, s_\beta C\right)$. Therefore, if Θ is identically zero on $T_1 \cap G'$, it follows that $c_1 = -c_2$ so $\Theta = c_1 \left\{\Theta^G\left(\frac{n\beta}{2}, C\right) - \Theta^G\left(-\frac{n\beta}{2}, s_\beta C\right)\right\}$. On the other hand, $\Theta^G\left(\frac{n\beta}{2}, C\right)(w_0 t) = \Theta^G\left(-\frac{n\beta}{2}, s_\beta C\right)(t)$ for $t \in T_0 \cap G'$. So we get that

$$\begin{aligned}
\Theta(w_0 t) &= c_1 \left\{\Theta^G\left(\frac{n\beta}{2}, C\right)(w_0 t) - \Theta^G\left(-\frac{n\beta}{2}, s_\beta C\right)(w_0 t)\right\} \\
&= c_1 \left\{\Theta^G\left(-\frac{n\beta}{2}, s_\beta C\right)(t) - \Theta^G\left(\frac{n\beta}{2}, C\right)(t)\right\} \\
&= -\Theta(t).
\end{aligned}$$

Hence Θ satisfies the property (P) on T_0.

In [7], we proved this proposition by induction on rank of G and we can apply this method naturally to the case $p = 2$. But when $p = 2$ we can also obtain this proposition directly from the explicit expression of characters for $SU(p,q)$ in [2].

3.4. Main theorem. Now we return to the non-tempered case considered in §2.4 for $G = SU(2,2)$. Let us recall $\lambda_0 = e_1 - e_4, \lambda = (m, n, n, -(m+2n))$ and the enlarged L-packet $\Pi = \{A(w\lambda, \pi_w); w \in W_G(T)\backslash W(\mathfrak{g}, \mathfrak{t})/W(\mathfrak{l}, \mathfrak{t})\}$. Denote by Θ_w the global character which corresponds to $A(w\lambda, \pi_w)$.

Now we state our main theorem.

Theorem. *Let $\Theta = \sum c_w \Theta_w$ be a linear combination of the characters of representations in the enlarged L-packet $\Pi = \{A(w\lambda, \pi_w); w \in W_G(T)\backslash W(\mathfrak{g}, \mathfrak{t})/W(\mathfrak{l}, \mathfrak{t})\}$. Then the following two conditions are equivalent:*

1) *Θ is identically zero on $T_2 \cap G'$,*

2) *Θ satisfies the property(P) on both T_0 and T_1.*

Before describing the proof, we need some preparations. Suppose $L_w = L(w\lambda_0)$ is not compact. In this paragraph, we omit subindex w in Θ_w. Then by (2.12), we easily see that Θ is decomposed as $\Theta = \Theta_0 + \Theta_1$. Here $-\Theta_0$ is a sum of the characters of discrete series and Θ_1 is the character of $\mathrm{Ind}_{P_G}^G(\chi \otimes (-\rho_P) \otimes 1)$ in the sequence (2.12). So the function $\tilde{\kappa}^i$ is also decomposed as $\tilde{\kappa}^i = \tilde{\kappa}_0^i + \tilde{\kappa}_1^i \quad (i = 0, 1, 2)$ according to the above decomposition. Furthermore, T_j is the height of Θ_j for $j = 0, 1$ respectively. Therefore on $T_1'(\mathbf{R})$, $\tilde{\kappa}_0^1$ is bounded while $\tilde{\kappa}_1^1$ is unbounded because $\mathrm{Ind}_{P_G}^G(\chi \otimes (-\rho_P) \otimes 1)$ is a non-tempered representation.

As for the behavior of κ on the height of Θ, Hirai proved the following proposition in [3].

Proposition 3.3. *Let Θ be an invariant eigendistribution and T a Cartan subgroup. Then the function κ^t can be extended to a continuous function on the whole T. In particular, if T is the height of Θ, this function becomes analytic on the whole T.*

3.5. Now we state the proof of our main theorem.

Proof: First suppose condition 1) holds. As noted above, Θ_w is decomposed as $\Theta_w = (\Theta_w)_0 + (\Theta_w)_1$. (When L_w is compact, $(\Theta_w)_1 = 0$ of course.) Put $\Theta_i = \sum c_w (\Theta_w)_i$ for $i = 0, 1$. Let F^+ be the connected component of $T_1'(\mathbf{R})$ determined as in (3.6). As mentioned in (3.3), the function $\tilde{\kappa}^1$ is expressed as

$$\tilde{\kappa}^1(\exp X) = \sum_{w \in W(\mathfrak{g}, \mathfrak{t})} p_w(F^+) \exp(w\mu, X) \qquad \text{for } \exp X \in F^+ \ (X \in \mathfrak{t}_0^1).$$

Let α be a real root in $\Delta(\mathfrak{g}, \mathfrak{t}_2)$ such that $\nu_\alpha(\mathfrak{t}_2) = \mathfrak{t}_1$ and put $\beta = \nu_\alpha \alpha$. Combining the boundary condition (3.9) with the assumption that $\tilde{\kappa}^2$ is identically zero, we obtain that

$$p_{s_\beta w}(F^+) = p_w(F^+)$$

for any $w \in W(\mathfrak{g}, \mathfrak{t}_1)$. Therefore we see that

$$\begin{aligned}\tilde{\kappa}^1(s_\beta \exp X) &= \sum_{w \in W(\mathfrak{g},\mathfrak{t}_1)} p_w(F^+) \exp(w\mu, s_\beta X) \\ &= \sum_{w \in W(\mathfrak{g},\mathfrak{t}_1)} p_{s_\beta w}(F^+) \exp(w\mu, X) \\ &= \sum_{w \in W(\mathfrak{g},\mathfrak{t}_1)} p_w(F^+) \exp(w\mu, X) \\ &= \tilde{\kappa}^1(\exp X).\end{aligned}$$

This means that Θ satisfies property (P) on T_1, because s_β is the longest element in $W_I(\mathfrak{g}, \mathfrak{t}_1)$. Since β is a singular imaginary root, the same equality holds for κ^1, that is,

$$\kappa^1(s_\beta \exp X) = \kappa^1(\exp X) \qquad (X \in (\mathfrak{t}_1)_0).$$

Therefore we get

$$\kappa_0^1(s_\beta \exp X) - \kappa_0^1(\exp X) = \kappa^1(\exp X) - \kappa_0^1(s_\beta \exp X). \tag{3.12}$$

Let us recall that κ_0^1 can be extended to a bounded continuous function on the whole T_1, whereas κ_1^1 can be extended to an analytic but not bounded function on it. In addition, Θ has regular infinitesimal character $\lambda + \rho(\mathfrak{q})$. So the both sides of (3.12) must be equal to zero. Hence the equation $\kappa_i^1(s_\beta \exp X) = \kappa_i^1(\exp X)$ holds for each i.

By definition, Θ_1 is a linear combination of the characters of induced representations from P_G in (2.12). Therefore $\tilde{\kappa}_1^1$ is expresed as

$$\tilde{\kappa}_1^1(t_L a_L) = \sum_k \tilde{\kappa}_{M,k}(t_L)\xi_{\rho_k}(a_L) \qquad (t_L \in T_L, \quad a_L \in A_L). \tag{3.13}$$

Here $\tilde{\kappa}_{M,k}$ denotes a function corresponds to a tempered invariant eigendistribution Θ_k on M_G and ξ_{ρ_k} belongs to $\hat{A}_L$. Furthermore we may assume that ξ_{ρ_k}'s are distinct from each other. Therefore it is easy to see that each Θ_k satisfies property(P) on T_L for L. Hirai gave the explicit expression of the characters of induced representations in [2] and [4]. And we also recall that T_L is a compact Catan subgroup of M_G. Hence combining his formula with Proposition 3.1, it follows that $\tilde{\kappa}_1^2$ is identically zero. Therefore $\tilde{\kappa}_0^2$ also becomes identically zero. Since Θ_0 is a linear combination of characters of discrete series, we can apply Proposition 3.2 to Θ_0. So we obtain that

$$\Theta_0(w_0 t) = -\Theta_0(t) \quad (t \in T_0 \cap G'),$$

where w_0 is the longest element in $W(\mathfrak{g}, \mathfrak{t}_0) = W_I(\mathfrak{g}, \mathfrak{t}_0)$. This proves that the condition 2) holds.

Next, suppose the condition 2) holds. Then we can apply Proposition 3.1 to Θ_0, because $\tilde{\kappa}^2 = \tilde{\kappa}_0^2$. Therefore we have $\tilde{\kappa}_2^2 \equiv 0$ and $\tilde{\kappa}_2^1(s_\beta \exp X) = \tilde{\kappa}_2^1(\exp X)$. So $\tilde{\kappa}_1^1 = \tilde{\kappa}^1 - \tilde{\kappa}_0^1$ satisfies the same condition on T_1, that is, Θ_1 satisfies the property(P) on T_1. In the same way as above, we obtain that $\tilde{\kappa}_1^2$ is identically zero on T_2. Hence $\tilde{\kappa}^2 = \tilde{\kappa}_1^2 + \tilde{\kappa}_0^2 \equiv 0$. This proves the condition 1).

Now we have completed the proof of our main theorem.

Remark. In this note, we treated only the case that $\lambda_0 = e_1 - e_4$. For other λ_0's the situation is quite similar. When λ_0 is regular, then $L(\lambda_0) = T$. So Π is nothing but a tempered L-packet of discrete series with a same infinitesimal character. When $\lambda_1 = (1,1,1,-3)$, for example, $L(\lambda_1)$ is isomorphic to $U(2,1) \times \mathbf{T}^1$. But $L(\lambda_1)$ also has the same types of Cartan subgroups as $L(\lambda_0)$ considered in §2. Consequently, in a resolution of $A(w\lambda, \pi_w)$, only similar members as we considered in this note appear. When $\lambda_1 = (1,-1,1,-1)$, for example, $L(\lambda_1)$ is of $\mathbf{R}$-rank 2. But since we consider only an invariant eigendistribution which is identically zero on T_2, non-unitary principal series representations of G do not effect our process. So we also get similar results for these cases.

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