Models in Microbial Ecology and Related Problems in Ordinary and Partial Differential Equations

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Abstract. In this article, we survey the mathematical models of severval microorganisms competing for a single-limited nutrient. Several mathematical models in the form of system of ordinary differential equations or partial differential equations are presented to explain the competitive exclusion and the coexistence of the species.

1 · Introduction

Human being cannot survive without microorganisms. It is important to understand the ecological behavior of microorganisms in their communities, for example, the competition, predation, mutualism, inhibition effects [FS]. In this paper, we shall restrict our attentitions to the mathematical models of several microorganisms competing for a single-limited nutrient. For the case of multiple nutrients, interested readers may consult [CHH], [WHH]. There are several reasons to study these mathematical models. Firstly, we have different ecological view points about the validity of the classical Lotka-Volterra two species competition model

$$\frac{dx_1}{dt} = \gamma_1 x_1(1 - \frac{x_1}{k_1}) - \alpha_1 x_1 x_2,$$

$$(1.1) \qquad \frac{dx_2}{dt} = \gamma_2 x_2(1 - \frac{x_2}{k_2}) - \alpha_2 x_1 x_2,$$

$$x_1(0) > 0, \qquad x_2(0) > 0$$

Gauss (1934) used bacteria to verify the validity of the model [H]. However the competition coefficients α_1, α_2 are not "physical" parameters which cannot be measured in advance of the experiments. In contrast to the model (1.1), we shall utilitize the resource-consumers

type interaction, coupled with the type 2 functional response, to construct our mathematical models where the parameters can be measured in advanced of the experiments. Secondly, the models are also related to the ecology of lakes and streams. We shall present the "chemostat" equation. Chemostat is an laboratory apparatus where both theory and experiments are tractable and match. Thirdly in the application to Chemical engineering, the industrial microorganisms are used to "eat" the industrial waste water [PC], [SFA]. Chemostat is also used to culture the useful industrial microorganisms. And finally, there are interesting mathematical problems in these mathematical models, for example, global stability, uniqueness of limit cycle, periodic solutions, persistent theory in dynamical system, strongly monotone flows.

2   Simple Chemostat (Well-Stirred)

The chemostat is a piece of laboratory apparatus used for culturing microorganisms. It has a constant nutrient source, containing all nutrients needed by the microorganisms in aboundance except one. The nutrient is pumped at a constant rate into a culture vessel. Constant volume is maintained in the culture vessel by allowing an overflow or by pumping the contents of vessel out at the same rate that nutrient is pumped in. The output of the culture vessel is collected in a receptacle. The culture vessel is charged with a quantity of a given type of microorganism and the collection vessel then contains both organisms and nutrient. This provides a continuous supply of microorganisms. For ecological purpose the chemostat is the laboratory realization of a very simple lake; the importance of the chemostat as an experimental vehicle is well documented [FS], [V], [P]. It is also of interesting in chemical engineering where it is a simplified model of the wastewater treatment process [PC].

Based on the experimental evidence [Mo], we assumed

(i) The growth rate of a microorganism species obeys the Michaelis-Menten kinetics, i.e.

$$\frac{1}{x}\frac{dx}{dt} = \frac{mS}{a+S}$$

Where S is the concentration of the nutrient, m is the maximum growth rate and a is the half-saturation constant.

(ii) The growth can be expressed in terms of the nutrient consumed by

$$[GROWTH] = y[CONSUMPTION]$$

Where y is a yield constant expressed as

$$y = \frac{\text{organism formed}}{\text{substrate used}}$$

For simplicity, we consider the case of two species. Then chemostat equation takes the form

(2.1)

$$\frac{dS}{dt} = (S^{(0)} - S)D - \frac{m_1}{y_1}\frac{S}{a_1 + S}x_1 - \frac{m_2}{y_2}\frac{S}{a_2 + S}x_2$$

$$\frac{dx_1}{dt} = (\frac{m_1 S}{a_1 + S} - D)x_1$$

$$\frac{dx_2}{dt} = (\frac{m_2 S}{a_2 + S} - D)x_2$$

$$S(0) \geq 0, \quad x_1(0) > 0, \quad x_2(0) > 0$$

where

S(t) = the concentration of nutrient at time t

x_i(t) = the concentration of i-th micororganism at time t

$S^{(0)}$ = input concentration of the nutrient

q = flow rate

V = volume of the vessel

D = $\frac{q}{V}$ = dilution rate

We note that the parameters $S^{(0)}$ and q are controlled by the experimenter.

For system (2.1) our basic assumption is

(H) $0 < \lambda_1 < \lambda_2 < S^{(0)}$

where

$$\lambda_i = \frac{a_i D}{m_i - D}$$ is the "break-even" concentration for i-th species

<u>Theorem</u> [HHW1], [Hsu1]

Let (H) hold. Then the solution of (2.1) satisfy

$$\lim_{t \to \infty} S(t) = \lambda_1, \quad \lim_{t \to \infty} x_1(t) = x_1^* > 0, \quad \lim_{t \to \infty} x_2(t) = 0$$

<u>Remark</u> : Species with smallest λ wins the competition. Smaller half-saturationor larger maximum growth rate implies smaller λ.

<u>Sketch of the Proof</u> :

Construct the following Liapunov function

$$V(S, x_1, x_2) = \int_{\lambda_1}^{S} \frac{\xi - \lambda_1}{\xi} d\xi + c_1 \int_{x_1^*}^{x_1} \frac{\xi - x_1^*}{\xi} d\xi + c_2 x_2$$

for some $c_1 > 0$, $c_2 > 0$. Applying LaSalle's invariance principle completes the proof.

<u>Remark</u> : Recently Wolkowicz & Lu [WL] construct a new Liapunov function for general functional responses.

The best known series of laboratory experiments preformed for the purpose of testing the validity of the chemostat equation were carried out by Hansen and Hubbell [HH]. A summary of their experiments is given in Table below. The parameters of the model were measured by growing each of the competitors separately on the growth-limiting nutrient tryptophan and assuming Michaelis-Menten functional responses. In Experiment 1, C-8 is a particular strain of *Escherichia coli* and PAO283 is a strain of *Psuedomonas aeruginosa*. In Experiment 2, the competition was between two variants of C-8, one which is resistant to the inhibitor naladixic acid but susceptible to spectinomycin, and the other the reverse. In both of these experiments, the species predicted to win by the model did indeed win even though it was originally inoculated into the growth vessel in a much smaller amount than the predictd loser. In the third experiment, the C-8 variants were used again, but naladixic acid was added to the culture medium in the proper amount so that the parameters $\lambda_i = P_i^{-1}(D), i = 1, 2$ were very close where $P_i(S) = \frac{m_i S}{a_i + S}$. It was found that in this case, the competitiors could

coexist in the growth vessel for as long as the experiment was run (120 hours).

Exp.	Bact. Strain	$S^{(0)}$ (g./liter)	D (hr.$^{-1}$)	$P_i(S^{(0)})$ (hr.$^{-1}$)	$\lambda_i = P_i^{-1}(D)$ (g./liter)	Winner
1	C-8	1×10^{-4}	6×10^{-2}	7.86×10^{-1}	2.4×10^{-7}	x
	PAO283			2.22×10^{-1}	2.19×10^{-5}	
2	C-8nal^{r}spec^S	5×10^{-6}	7.5×10^{-2}	5.15×10^{-1}	1.98×10^{-7}	
	C-8nal^{S}spec^r			7.27×10^{-1}	1.36×10^{-7}	x
3	C-8nal^{r}spec^S	5×10^{-6}	7.5×10^{-2}	5.15×10^{-1}	1.98×10^{-7}	
	C-8nal^{S}spec^r			3.47×10^{-1}	2.01×10^{-7}	

Table 5.1 Summary of Hansen-Hubbell experiments

Inspired by the Hansen-Hubbell experiment, Lenski and Hattingh [LH] construct the following model describing the competition in a chemostat with external inhibitor. Let P(t) be the concentration of external inhibitor, for example, pollutant or antibody. We assume the species 1 is susceptible to the inhibitor while the species 2 is resistant. Only species 2 consumes the inhibitor. Then the equations takes the form

$$\frac{dS}{dt} = (S^{(0)} - S)D - \frac{m_1 S}{a_1 + S}x_1 e^{-\lambda P} - \frac{m_2 S}{a_2 + S}x_2$$

$$\frac{dx_1}{dt} = (\frac{m_1 S}{a_1 + S}e^{-\lambda P} - D)x_1$$

(2.2)
$$\frac{dx_2}{dt} = (\frac{m_2 S}{a_2 + S} - D)x_2$$

$$\frac{dP}{dt} = (P^{(0)} - P)D - \frac{\delta x_2 P}{K + P}$$

$$S(0) \geq 0, \quad x_1(0) > 0, \quad x_2(0) > 0, \quad P(0) \geq 0,$$

Where $P^{(0)}$ is the input concentration of the inhibitor.

In [HsW1] the authors observed that the solution of (2.2) satisfies

$$S(t) + x_1(t) + x_2(t) = S^{(0)} + O(e^{-Dt}) \quad \text{as} \quad t \to \infty$$

Thus we study the dynamics on the w-limit set of the solution of (2.2),

$$\frac{dx_1}{dt} = (\frac{m_1(S^{(0)} - x_1 - x_2)}{a_1 + (S^{(0)} - x_1 - x_2)}e^{-\lambda P} - D)x_1$$

$$\frac{dx_2}{dt} = (\frac{m_2(S^{(0)} - x_1 - x_2)}{a_2 + (S^{(0)} - x_1 - x_2)} - D)x_2$$

(2.3)

$$\frac{dP}{dt} = (P^{(0)} - P)D - \frac{\delta x_2 P}{K + P}$$

$$0 < x_1(0) + x_2(0) < S^{(0)}, \quad P(0) \geq 0.$$

Since (2.3) is a competitive system in R^3, from [Hir], [Sm3] we have Poincaré-Bendixson Theorem. When the interior equilibrium $E_c = (x_1^*, x_2^*, P^*)$ exists , we show that for large λ, δ and small K, E_c is unstable. Thus the coexistence occurs in the form of periodic solutions

3 Coexistence

In this section we shall search for possible reasons for coexistence which is aften observed in the nature.

I. Assume S is the prey which grows logistically. Consider the following "model" equation first studied by Koch [Ko] and then analyzed in [HHW3]

$$\frac{dS}{dt} = rS(1 - \frac{S}{K}) - \frac{m_1 S}{a_1 + S}x_1 - \frac{m_2 S}{a_2 + S}x_2$$

$$\frac{dx_1}{dt} = (\frac{m_1 S}{a_1 + S} - D_1)x_1$$

(3.1)

$$\frac{dx_2}{dt} = (\frac{m_2 S}{a_2 + S} - D_2)x_2$$

$$S(0) > 0, \quad x_1(0) > 0, \quad x_2(0) > 0$$

Our basic assumption is

(H1) $0 < \lambda_1 < K$, $\lambda_i = \dfrac{a_i D_i}{m_i - D_i}$, $i = 1, 2$

<u>Theorem 3.1</u> : (Extinction) Let (H1) hold and $b_i = \dfrac{m_i}{D_i}$, $i = 1, 2$. If

(3.2) $a_1 < a_2$, $b_1 > b_2$ or

(3.3) $a_1 < a_2$, $b_1 < b_2$, $K < \dfrac{b_1 a_2 - b_2 a_1}{b_2 - b_1}$

then $\lim_{t \to \infty} x_2(t) = 0$

To understand the dynamics of (3.1), we need to study the two-dimensional Predator-Prey system.

$$\frac{dS}{dt} = \gamma S(1 - \frac{S}{K}) - \frac{mS}{a+S}x$$

(3.4) $\dfrac{dx}{dt} = (\dfrac{mS}{a+S} - D)x$

$$S(0) > 0, \quad x(0) > 0$$

<u>Theorem 3.2</u> : Let $0 < \lambda < K$, $\lambda = \dfrac{aD}{m - D}$.

(i) If $\dfrac{K - a}{2} \leq \lambda$ then the solution S(t), x(t) of (3.4) satisfy

$$\lim_{t \to \infty} S(t) = \lambda, \quad \lim_{t \to \infty} x(t) = x^* > 0$$

(ii) If $\dfrac{K - a}{2} > \lambda$, then there exists a unique limit cycle.

Insterested reader may find the proof of (i) in [HHW3] and that of (ii) in [Ch]. Condiser our one prey-two predators system (3.1), we have the following extinction results.

<u>Theorem 3.2</u>: Let (H1) hold and either (3.2) or (3.3) hold. Then the solution $(S(t), x_1(t), x_2(t))$ of (3.1) satisfies.

(i) If $\dfrac{K - a_1}{2} \leq \lambda_1$, then

$$\lim_{t\to\infty} S(t) = \lambda_1$$

$$\lim_{t\to\infty} x_1(t) = x_1^* > 0$$

$$\lim_{t\to\infty} x_2(t) = 0$$

(ii) If $\dfrac{K - a_1}{2} > \lambda_1$ then the trajectory $(S(t) \cdot x_1(t), x_2(t))$ approach the unique limit cycle Γ_1 in $S - x_1$ plane except the one dimensional stable manifold of $(\lambda_1, x_1^*, 0)$

In [HHW2], the numerical studies shows that under the assumption $0 < \lambda_1 < \lambda_2 < K$, $a_1 < a_2$, $b_1 < b_2$, $K > \frac{b_1 a_2 - b_2 a_2}{b_2 - b_1}$, varying K from $\frac{b_1 a_2 - b_2 a_2}{b_2 - b_1}$ to infinity produces a family of positive periodic solutions emerging from Γ_1 and decending to Γ_2. Recently [MR] Murator assume the parameter γ is large and apply the singular pertubation technique to justify the phenomena. Butler and Waltman [BW] apply the result in [Ch] to show that when Γ_1 becomes unstable, there is a family of positive periodic solutions bifurcating from Γ_1. Smith [Sm2] also shows the existence of positive solution by Hopf bufurcation. Next we consider one nutrient-one prey-two predators in the chemostat. The equations take the following form.

(3.5)

$$\frac{dS}{dt} = (S^{(0)} - S)D - \frac{m_1 S}{a_1 + S} x$$

$$\frac{dx}{dt} = \left(\frac{m_1 S}{a_1 + S} - D - \frac{m_2 y}{a_2 + x} - \frac{m_3 z}{a_3 + x} \right) x$$

$$\frac{dy}{dt} = \left(\frac{m_2 x}{a_2 + x} - D \right) y$$

$$\frac{dz}{dt} = \left(\frac{m_3 x}{a_3 + x} - D \right) z$$

$$S(0) \geq 0, \quad x(0) > 0, \quad y(0) > 0, \quad z(0) > 0$$

The behavior of solutions of (3.5) is similar to that of (3.1). Interested reader may consult [BHW1]

II. Periodic input and periodic washout rate when the imput concentration is a periodic function of time, the chemostat equation takes the form

$$\frac{dS}{dt} = (\varphi(t) - S) - \frac{m_1 S}{a_1 + S}x_1 - \frac{m_2 S}{a_2 + S}x_2$$

$$\frac{dx_1}{dt} = \left(\frac{m_1 S}{a_1 + S} - D\right)x_1$$

(3.6)

$$\frac{dx_2}{dt} = \left(\frac{m_2 S}{a_2 + S} - D\right)x_2$$

$$S(0) > 0, \quad x_1(0) > 0, \quad x_2(0) > 0$$

In [Hsu2], we study the extinction, persistence of the solutions of (3.6) for the special case $\varphi(t) = S^{(0)} + b\sin\omega t$. A numerical study shows the coexistence is possible in the b-w parameter region. Smith [Sm1] shows the existence of $\frac{2\pi}{\omega}$-periodic solutions by Hopf bifurcation Hale and Somolinas [HaS] observe the relationship $S(t) + x_1(t) + x_2(t) = \Phi(t) + o(e^{-\alpha t})$, $\alpha > 0$, $\Phi(t + w) = \Phi(t)$ and reduce the dynamics of (3.6) to a competitive, periodic two dimensional system

$$\frac{dx_1}{dt} = \left(\frac{m_1\left(\Phi(t) - x_1 - x_2\right)}{a_1 + \left(\Phi(t) - x_1 - x_2\right)} - D\right)x_1$$

(3.7)

$$\frac{dx_2}{dt} = \left(\frac{m_2\left(\Phi(t) - x_1 - x_2\right)}{a_2 + \left(\Phi(t) - x_1 - x_2\right)} - D\right)x_2$$

and apply the results obtained by de Mottoni and Schiaffino [MS] which states any solution of two-dimensional, periodic, competitive system approaches to a periodic solution. For the appliction to industrial waste water in Chemical Engineering, the dilution rate is periodic. Thus we have [BHW2]

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$$\frac{dS}{dt} = (S^{(0)} - S)D(t) - \frac{m_1 S}{a_1 + S}x_1 - \frac{m_2 S}{a_2 + S}x_2$$

$$\frac{dx_1}{dt} = \left(\frac{m_1 S}{a_1 + S} - D(t)\right)x_1$$

(3.8)

$$\frac{dx_2}{dt} = \left(\frac{m_2 S}{a_2 + S} - D(t)\right)x_2$$

$$S(0) \geq 0, \quad x_1(0) > 0, \quad x_2(0) > 0$$

Where $D(t) = q(t)/V$, $q(t)$ is the periodic flow rate with periodic w. Use the relationship $S(t) + x_1(t) + x_2(t) = S^{(0)} + O(e^{-dt})$, $\alpha > 0$. We reduce the dynamics of (3.8) to

$$\frac{dx_1}{dt} = \left(\frac{m_1\left(S^{(0)} - x_1 - x_2\right)}{a_1 + \left(S^{(0)} - x_1 - x_2\right)} - D(t)\right)x_1$$

(3.9)

$$\frac{dx_2}{dt} = \left(\frac{m_2\left(S^{(0)} - x_1 - x_2\right)}{a_2 + \left(S^{(0)} - x_1 - x_2\right)} - D(t)\right)x_2$$

$$x_1(0) > 0, \quad x_2(0) > 0, \quad x_1(0) + x_2(0) < S^{(0)}$$

As in (3.7), (3.9) is also a two dimensional, periodic, competitive system. The solution of (3.9) approaches a periodic solution. In the following, we state the results of coexistence in [BHW2].

Theorem 3.4 : Let m_2 be a bifurcation parameter. There exists a continuous one-parameter family of positive w-periodic solutions connecting E_1, E_2 where E_i is the unique positive w-periodic solution on x_i-axis.

III. Gradostat and unstirred chemostat

Gradostat is a concatenation of chemostats, designed by Lovitt and Wimpenny [LW] to achieve nutrient gradient in experiments. (See Fig. 1, for linear-chained vessels)
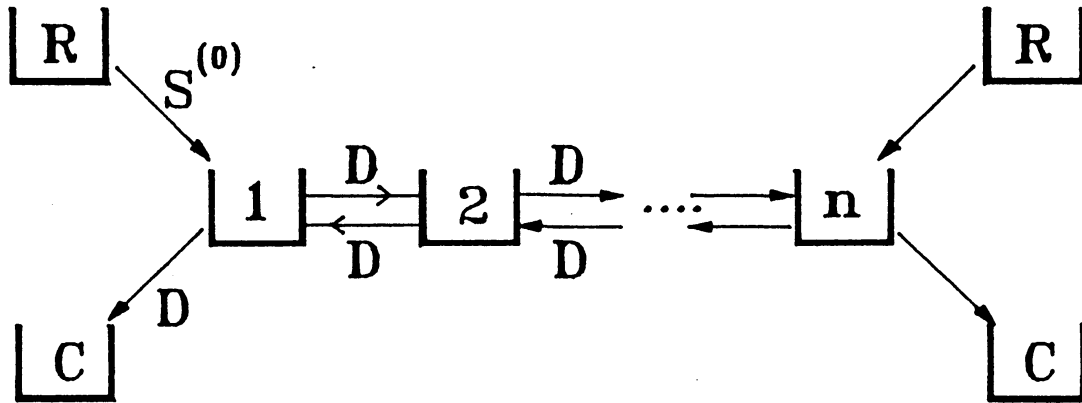
Fig. 1

R = Reservoir
C = Collecting Vessel

The mathematical analysis for the growth of a population in a gradostat was given by Tang [T]. Competition of two population in the two vessels case was studied by Jäger at al [JTSW] and a complete classification of limiting behavior was given including the case of coexistence. The equations of two species competition in a linear-chained n-vessel gradostat take the form:

$$\frac{dS_i}{dt} = (S_{i-1} - 2S_i + S_{i+1})D - U_i f_u(S_i) - V_i f_v(S_i)$$

$$\frac{dU_i}{dt} = (U_{i-1} - 2U_i + U_{i+1})D + U_i f_u(S_i)$$

$$\frac{dV_i}{dt} = (V_{i-1} - 2V_i + V_{i+1})D + V_i f_v(S_i)$$

(3.10)

$$S_i(0) \geq 0, \quad U_i(0) > 0, \quad V_i(0) > 0, \quad i = 1, \ldots n$$

$$S_0 = S^{(0)}, \quad U_0 = V_0 = 0, \quad S_{n+1} = U_{n+1} = V_{n+1} = 0$$

$$f_u(S) = \frac{m_u S}{a_u + S}, \quad f_v(S) = \frac{m_v S}{a_v + S}$$

Where $S_i(t)$, $U_i(t)$, $V_i(t)$ are the concentration of nutrient, u-species, v-species at i-th vessel at time t respectively.

In [STW] Smith et. consider a gereral gradostat equation for possible combinations of vessels

including (3.10) as a special case. They classify all possible cases by the sets of equilibria. Sufficient conditions for two species to coexist in the gradostat are derived using the theory of monotone dynamical systems and global bifurcation theory. Numerical computations required to verify the hypotheses of the coexistence results suggested the coexistence is more likely as the number of vessels increases.

When n becomes larger, it is harder to analyze (3.10). In [HsW2] we remove the "well-stirred" hypothesis in chemostat and construct the model for two species competition in unstirred chemostat.

$$\frac{\partial S}{\partial t} = d\frac{\partial^2 S}{\partial x^2} - \frac{m_1 S}{a_1 + S}U - \frac{m_2 S}{a_2 + S}V$$

$$\frac{\partial U}{\partial t} = d\frac{\partial^2 U}{\partial x^2} - \frac{m_1 S}{a_1 + S}U \qquad 0 < x < 1,\ t > 0$$

$$\frac{\partial V}{\partial t} = d\frac{\partial^2 V}{\partial x^2} - \frac{m_1 S}{a_1 + S}V$$

(3.11)

$$\frac{\partial S}{\partial x}(t,0) = -S^{(0)} \qquad , \ \frac{\partial S}{\partial x}(t,1) + \gamma S(t,1) = 0$$

$$\frac{\partial U}{\partial x}(t,0) = 0 \qquad , \ \frac{\partial U}{\partial x}(t,1) + \gamma U(t,1) = 0$$

$$\frac{\partial V}{\partial x}(t,0) = 0 \qquad , \ \frac{\partial V}{\partial x}(t,1) + \gamma V(t,1) = 0$$

$$S(0,x) = S_0(x) \geq 0, \quad U(0,x) = U_0(x) \geq 0, \quad V(0,x) = V_0(x) \geq 0.$$

Here we assume the equal diffusions for nutrient and species for mathematical reasons. With the equal diffusions, we have

$$S(t,\cdot) + U(t,\cdot) + V(t,\cdot) = \varphi(\cdot) + O(e^{-\alpha t})$$

as $t \to \infty$ for some $\alpha > 0$, where

$$\varphi(x) = S^{(0)} = \left(\frac{\gamma + 1}{\gamma} - x\right), 0 < x < 1$$

Thus we reduce (3.11) to

$$\frac{\partial U}{\partial t} = d\frac{\partial^2 U}{\partial x^2} + \frac{m_1(\varphi(x) - U - V)}{a_1(\varphi(x) - U - V)}U$$

$$\frac{\partial V}{\partial t} = d\frac{\partial^2 V}{\partial x^2} + \frac{m_2(\varphi(x) - U - V)}{a_2(\varphi(x) - U - V)}V$$

(3.12)
$$\frac{\partial U}{\partial x}(t,0) = 0 \quad , \quad \frac{\partial U}{\partial x}(t,1) + \gamma U(t,1) = 0$$

$$\frac{\partial V}{\partial x}(t,0) = 0 \quad , \quad \frac{\partial V}{\partial x}(t,1) + \gamma V(t,1) = 0$$

$$U(0,x) = U_0(x) \geq 0, \qquad V(0,x) = V_0(x) \geq 0$$

$$U_0(x) + V_0(x) \leq \varphi(x) \quad for \quad 0 < x < 1$$

The system (3.12) generates a monotone flow. We apply the persistent theory in infinite dimensional space [Ha W] to obtain the coexistence results.

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