数理解析研究所講究録
第 829 巻 1993 年 13-18

Order Three May Imply 'Order'

岡山大学経済学部: 藤本喬雄 (Takao Fujimoto)

Abstract. A nonlinear difference equation which diverges cyclically may be stabilized when time-lag is introduced in a certain way. It is shown by using a numerical example that while stabilization is impossible when the resulting difference equation is of order two, it can be made stable when the order is not less than three. One application is made to stabilize the examples of global instability of competitive equilibrium by Scarf. Some suggestions are made concerning the computation of a fixed point, and the shrinkage of chaotic areas.

1. Introduction

Let us consider a linear differential equation in the n-dimensional real Euclidean space, $dx/dt = Ax$, where x is an n-column vector and A is a given n by n real matrix. We suppose that each eigenvalue of A has a negative or zero real part, and at least one pair of eigenvalues are pure imaginary numbers, thus in general exhibiting cyclical movements. When this equation is discretized, we may have

$$x(t+1) = x(t) + aAx(t), \tag{1.1}$$

where a is a positive scalar, and may be called the *speed of adjustment*. However small a may be, (1.1) remains **unstable.**

Now we introduce time-lags into (1.1) as

$$x(t+1) = (kx(t) + (1-k)x(t-1)) + aAx(t), \tag{1.2}$$

where k is a positive scalar such that $0 \leq k \leq 1$. Unfortunately, this device does not work in most cases. So, we generalize (1.2) as

$$x(t+1) = \left(\sum_{i=0}^{p}\left(k_i x(t-i)\right)\right) + aAx(t), \tag{1.3}$$

where k_i's are positive scalars such that $\sum_{i=0}^{p} k_i = 1$, and $r \equiv p+1$ shows the order of eq.(1.3). When r is not less than three, (1.3) may be stable with a suitable choice of k_i.

In section 2, we present a simple numerical example which cannot be stabilized in the form (1.2),i.e.,of order two, but can be stable in the form (1.3) with $r=3$. A proof is given that the example cannot be stable when the order is two. In section 3, our method is applied to nonlinear difference equations. Specifically, we take up the discrete version of unstable examples by Scarf[2]. In the final section other possible applications are suggested.

2. An Example

Consider the Euclidean space of dimension two, R^2, and let A be

$$A \equiv \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Then,

$$x(t+1) = (kx(t) + \ell x(t-1) + mx(t-2)) + aAx(t) \tag{2.1}$$

can be stable with a suitable choice of k, ℓ, m, and a. This is to be confirmed by way of a simple program on a computer.

We can prove that (1.2) or

$$x(t+1) = (kx(t) + \ell x(t-1)) + aAx(t), \tag{2.2}$$

where $k+\ell=1$, cannot be stable for any choice of k and ℓ. That is, the matrix

$$B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \ell & 0 & k & -a \\ 0 & \ell & a & k \end{bmatrix}$$

has at least one pair of eigenvalues on the unit circle or outside. To prove this we use a theorem due to Cohn[1,p.115].

Cohn's Theorem: Suppose the equation $f(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n = 0$ $(a_0 \neq)$ has all its roots within the unit circle, then the equation $f(z) + \lambda f^*(z) = 0$ with $|\lambda| < 1$ also has all its roots within the unit circle, where

$$f^*(z) = \bar{a}_n z^n + \bar{a}_{n-1} z^{n-1} + \cdots + \bar{a}_0.$$

($\bar{a}$ is the conjugate of a.)

The eigenequation of B is

$$f(z) = z^4 - 2kz^3 + (k^2 - 2(1-k) + a^2)z^2 + 2k(1-k)z + (1-k)^2 = 0.$$

Putting $\lambda = -(1-k)^2$, we form $f(z) + \lambda f^*(z)$ and divide this by x to obtain

$$g(z) = (1 + (1-k)^2)z^3 - 2(k^2 - k + 1)z^2 + (k^2 + 2k - 2 + a^2)z + 2(1-k) = 0$$

Then, divide $g(z)$ by $(1 + (1-k)^2)$, and putting $\lambda = 2(1-k)/(1 + (1-k)^2)$, we form $g(z) + \lambda g^*(z)$ and arrange it to have

$$h(z) = k^2(k-2)^2 z^2 - 2[\ldots]z + \{k^2(k-2)^2 + [(k-1)^2 + 1]a^2\} = 0.$$

Since the coefficient of z^2 is smaller than the constant term, this quadratic equation $h(z)$ cannot have its two roots within the unit circle. Thus, at least one pair of eigenvalues of B cannot lie within the unit circle.

When we consider the case of order three, an interesting phenomenon takes place. Eq.(2.2) is now changed to

$$x(t+1) = (kx(t) + \ell x(t\text{-}1) + mx(t\text{-}2)) + aAx(t), \quad (2.3)$$

where the coefficients k, ℓ, and m are all nonnegative, and their sum is unity.

The associated matrix is

$$B = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ m & 0 & \ell & 0 & k & -a \\ 0 & m & 0 & \ell & a & k \end{bmatrix}$$

Let us fix $m=0.3$, and we have the following eigenequation.

$$x^6 - 2kx^5 + (k^2 + 2k - 1.4 + a^2)x^4 + 2\{k(0.7 - k) - 0.3\}x^3$$
$$+ (k^2 - 0.8k + 0.49)x^2 + 0.6(0.7 - k)x + 0.09 = 0. \quad (2.4)$$

When $a < 1$, the modulus of each solution of (2.4) is less than unity whatever value k may take between 0 and 0.7, implying the asymptotic stability of the difference equation (2.3) of order 3. Once the speed of adjustment a becomes greater than 1.4 or so, at least one pair of solution is outside(or on) the unit circle regardless of the value of k. The 'quickest'(in a loose sense) convergence seems to be realized when a is around 1.26 and k is roughly between 0.5 and 0.7.

3. Stabilization of Scarf's Examples

In this section, we again consider R^2. One of the unstable examples in Scarf[2] is, when discretized and normalized(by choosing commodity 3 as the numeraire), is described as

$$x_1(t+1) = x_1(t) + aE_1(x_1(t), x_2(t)),$$
$$x_2(t+1) = x_2(t) + aE_2(x_1(t), x_2(t)),$$

where $E_1 = (-x_2)/(x_1+x_2) + x_3/(x_3+x_1)$, and $E_2 = x_1/(x_1+x_2) + (-x_3)/(x_2+x_3)$, and a is the speed of adjustment. Several authors have devised out more elaborate methods to calculate equilibrium prices because a simple tatonnement process fails to converge in this Scarf's example.(See Smale[3]). Using our 'conservative' mechanism to employ the past prices and average them, the above system can be stabilized in a simple and 'natural' way. That is,

$$x_i(t+1) = (kx_i(t)+\ell x_i(t-1)+mx_i(t-2)) + aE_i(x_1(t),x_2(t)) \quad \text{for } i=1,2.$$

By selecting k, ℓ, m, a, and an **initial vector** properly, a solution path converges. Through a suitable choice, convergence is quite rapid, and the region of initial vectors which guarantees stability is large enough.

Let us define $b(x;3) \equiv ((kx_1(t)+\ell x_1(t-1)+mx_1(t-2), (kx_2(t)+\ell x_2(t-1)+mx_2(t-2))'$, and call $b(x;3)$ the benchmark vector of order 3. In the framework of competitive adjustment of tatonnement process, the benchmark vector provide a basis to conduct a conservative revision of prices on the side of auctioneer.

The second group of unstable examples in Scarf[2] can be handled in a similar way. In these examples, however, it seems that we need a system of order **four**.

4. Remarks

(1) Our simple method can be applied to compute a fixed point in a simple way.

(2) When the above lagging approach is applied to a system which yields chaotic movements in a certain region of parameters, the region producing chaos shrinks.

(3) Nonlinear difference equations of the type $x(t+1)=x(t)+aF(x(t))$, can be classified according to the minimum order r of the benchmark vector by introducing which they can be made stable. If any orbit which starts within a compact region is bounded, can we say the minimum order is finite ?

(4) In the nonlinear case, even when stabilized, we have to ask how large the area is within which the initial vector is designated and the solution orbit converges to an equilibrium.

References

[1] Cohn, A., Über die Anzahl der Wurzeln einer algebraischen Gleichung in einem Kreise. Math.Zeit. 14(1922).

[2] Scarf, H., Some examples of global instability of the competirive equilibrium. Int.Econ.Rev. 1(1960).

[3] Smale, S., A convergent process of price adjustment and global Newton method. J.Math.Econ. 3(1976).

[*] Bear,D.V.T., J.Conlisk, The effects of unlagging and speed of adjustment. J.Econ.Theory 21(1979).

[*] Schur, I., Über Potenzreihen, die im Innern des Einheitkreises beschränkt sind. J.f.Math. 148(1918).