数理解析研究所講究録
第 829 巻 1993 年 42-76

OPTIMAL CHAOS WHEN FUTURE UTILITIES ARE DISCOUNTED ARBITRARILY WEAKLY

京都大学 西村和雄 (Kazuo Nishimura)
横浜国立大学 矢野 誠 (Makoto Yano)

1. INTRODUCTION

Think of a dynamic optimization problem of the following form.

$$\mathrm{Max}_{k_0,k_1,\ldots} \Sigma_{t=1}^{\infty}\rho^t v(k_{t-1},k_t) \text{ s.t. } k_0 = x.$$

The intended interpretation is as follows: k_t is the level of capital stock at time t, and $v(k_{t-1},k_t)$ is the maximum utility that the economy can achieve in the period between time t-1 and time t when it has stock k_{t-1} at time t-1 and plans to have stock k_t at time t. The optimization problem stipulates to choose a path of capital accumulation, $k_0,k_1,\ldots$, starting from x that maximizes the discounted sum of utilities over the time horizon. The solution to the optimization problem, in general, can be described by a set valued function H; i.e., if path k_t solves the maximization problem, then $k_t \in H(k_{t-1})$ for every t. In this sense, we call H an optimal transition correspondence.

The question that this study deals with is whether or not an optimal transition correspondence is a chaotic function of any sense, i.e., the possibility of optimal chaos. The present study is not the first to consider this question. In fact, the question was raised as soon as non-linear dynamics was introduced into economics at the end of 1970s (see Nishimura and Yano, 1992a, for a more detailed discussion). Boldrin and Montrucchio (1986) and Deneckere and Pelikan (1986) gave a positive answer to this question for the case in which the discount factor of future utilities, ρ, is in the vicinity of 0.01.

Nishimura and Yano (1992b) strenthens this result by demonstrating the possibility of optimal chaos for the case in which the value of ρ is arbitrarily close to 1. From the view point of economics, the value of discount factor ρ is critical. It is related to the real interest rate, roughly speaking, by $i = 1/\rho - 1$. The case of $\rho = 0.01$ corresponds to an interest rate around 9900%. Because the real world rate of annual interest rarely exceeds 5%, the span of an individual period in the model with $\rho = 0.05$ must exceed 90 years. In other words, chaotic optimal accumulaiton captured in the existing studies is detectable only if economic data over several hundred years are observed. The result of this study, in contrast, demonstrates that, at least theoretically, it is possible to interpret erratic movements of economic variables over any span of periods as a result of intertemporal optimizations.

The purpose of this study is to provide an intuitive explanation for the result of Nishimura and Yano (1992b). In the following part of this study, Section 2 introduces our model. Section 3 discusses the mathematical results on ergodic chaos on which our study is based. Section 4 provides a sufficient condition for an optimal transition function to be ergodically chaotic. Section 5 states our main result. Section 6 provides a proof.

2. THE TWO-SECTOR LEONTIEF MODEL

The model we consider is of two sectors, each of which have Leontief production functions (two-sector Leontief model). Think of two goods C and K. Good C is a pure consumption good. Each sector uses both good K and labor L as input. Good K is not consumed. Good K input must be made one period prior to the period in which output is produced. In this sense, we may call good K a capital good. Labor input is made in the same period as output is produced. Sectors C and K have Leontief production functions as follows:

(2.1) $$c_t = \min\{K_{Ct-1}, L_{Ct}/\alpha\};$$

(2.2) $$y_t = \mu\min\{K_{Kt-1}, L_{Kt}/\beta\}.$$

For i = C,K, $K_{it-1} \geq 0$ is sector i's capital input in period t-1, and $L_{it} \geq 0$ is sector i's labor input in period t. Moreover, $c_t \geq 0$ and $y_t \geq 0$ are outputs in period t. For the sake of simplicity, assume that every period a level of each sector's capital input is freely adjustable. Thus, by denoting k_{t-1}, the aggregate capital input,

(2.3) $$k_{t-1} = K_{Ct-1}+K_{Kt-1} \geq 0.$$

Denote by n_t the aggregate labor input. Then,

(2.4) $$n_t = L_{Ct}+L_{Kt} \geq 0.$$

Assume that labor supply is inelastic and time independent. Goods are normalized so that the labor endowment $\bar{n}$ is equal to 1. Thus,

(2.5) $$n_t \leq \bar{n} = 1 .$$

Assume that the consumers' preference is represented by a linear utility function

(2.6) $$u(c_t) = c_t \geq 0$$

for $c_t \geq 0$ in each period. The set of feasible input/output combinations is

(2.7) $F = \{(k,n,y,c) \in R^4_+ |$ (2.1) through (2.6) are satisfied$\}$.

The optimal growth model is described by

(2.8) $\max_{\{k_{t-1},n_t,k_t,c_t\}} \quad \Sigma_{t=1}^{\infty}\rho^t u(c_t)$

s.t. $k_0 \leq \bar{k}$, $k_t \leq y_t$ and $(k_{t-1},n_t,y_t,c_t) \in F$ for $t = 1,2,\ldots,$

where discount factor ρ satisfies

(2.9) $0 < \rho < 1.$

With respect to parameters ρ, μ, β, and α, assume the following:

(2.10) $\mu > 1/\rho;$

(2.11) $\beta > \alpha > 0.$

As (2.2) indicates, condition (2.10) implies that if labor input is not binding, the marginal product of capital input in the capital good sector is larger than 1 plus the long-run interest rate. Condition (2.11) implies that the consumption good sector is capital intensive.

The above optimal growth model can be expressed in the form presented at the outset of this study by defining v as

(2.12) $v(k,y) = \max_{(n,c)} u(c)$ s.t. $(k,n,y,c) \in F$.

By using this reduced form function, define the value function

(2.13) $V(y) = \max \quad \Sigma_{t=1}^{\infty}\rho^t v(k_{t-1},k_t)$ s.t. $k_0 = y$

and the optimal transition correspondence

(2.14) $H(k) = \{h \in R |\ v(k,h)+V(h) \geq v(k,y)+V(y)$

for any $y \geq 0$ such that $v(k,y)$ is well defined.$\}$.

In the case in which H(k) is a singleton for every k, we call H(k) an optimal transition function.

Let $H^t(k) = H(H^{t-1}(k))$, where $H^0(k) = k$. It is easy to demonstrate that if (k_{t-1}, n_t, y_t, c_t), $t = 1,2,\ldots$, is an optimal path from $\bar{k}$ solving the optimization problem (2.8), then

$$k_t \in H^t(\bar{k}) \tag{2.15}$$

for any t. By a routine method, the following can be established.

Proposition 1: For each $k \in R$, the optimal transition correspondence, $H(k)$, is non-empty.

In order to construct the actual form of v, think of the case in which c_t and y_t are produced without wasting inputs. In that case, by (2.1) and (2.2), $c_t = K_{Ct-1} = L_{Ct}/\alpha$ and $y_t = \mu K_{Kt-1} = \mu L_{Kt}/\beta$. Thus, if $k_t \le y_t$, by (2.5), (2.3) and (2.4) can be written as follows:

$$c_t + k_t/\mu \le k_{t-1} \; ; \tag{2.16a}$$

$$\alpha c_t + \beta k_t/\mu \le 1 \; ; \tag{2.16b}$$

$$c_t \ge 0,\ k_t \ge 0 \text{ and } k_{t-1} \ge 0. \tag{2.17}$$

It is easy to demonstrate that the constraints of (2.8) is equivalent to (2.16a), (2.16b) and (2.17) together with the initial condition, $k_0=\bar{k}$. By using (2.16a), (2.16b) and (2.17), the actual form of v can be shown to be

$$v(k,y) = \begin{cases} k - \frac{1}{\mu}y & \text{if} \quad y \le -\frac{\alpha}{\beta-\alpha}\mu k + \frac{1}{\beta-\alpha}\mu \\ \frac{1}{\alpha} - \frac{\beta/\alpha}{\mu}y & \text{if} \quad y \ge -\frac{\alpha}{\beta-\alpha}\mu k + \frac{1}{\beta-\alpha}\mu . \end{cases} \tag{2.18}$$

Figure 1 illustrates the projection of the graph of $u = v(k,y)$. Segment PQ illustrates the set of (k,y) at which both (2.16) and (2.17) are binding. On that set, the graph of v makes a kink. Segment OP corresponds to the case in which only (2.16a) is binding and in which $c_t = 0$. Half line PZ corresponds to the case in which only (2.16b) is binding and in which $c_t = 0$. The domain of v is below lines OP and PZ, i.e.,

(2.19) $$D = \{(k,y)\in R^2_+ |\ y \leq \mu k \text{ if } k \leq 1/\beta \text{ and } y \leq \mu/\beta \text{ if } k \geq 1/\beta\}.$$

3. CHAOS: DEFINITIONS

This section briefly explains the mathematical result on chaotic dynamics on which this study is based. For the sake of discussion, let I be a non-degenerate interval with endpoints a and b, $a < b$. Call function f a transition function if it maps I into itself. Transition function $f:I\to I$ generates a trajectory of states $f^t(x)$, $t = 0,1,2,\ldots$, for any initial point $x \in I$, where $f^t(x) = f(f^{t-1}(x))$ and $f^0(x) = x$. A transition function, $f:I\to I$, is expansive if it is continuous and piecewise twice continuously differentiable and if there is $g > 0$ such that

(3.1) $$|f'(x)| \geq 1+g$$

for every $x \in I$ at which f has a derivative (Lasota and Yorke, 1973). A transition function, $f:I\to I$, is unimodal if it is continuous and if there is $c \in I$, $a < c < b$, such that f is strictly increasing on the interval between a and c and strictly decreasing on the interval between c and b (Collet and Eckmann, 1980). Point $x \in I$ is a cyclical point of transition function $f:I\to I$ if there is $t \geq 1$ such that $f^t(x) = x$. In particular, if $f^t(x) = x$ and $f^\tau(x) \neq x$ for $\tau = 1,2\ldots,t-1$, x is a cyclical point of period t. Call a trajectory starting from a cyclical point a cyclical trajectory. Denote by χ_B the characteristic function; $\chi_B(x) = 1$ if $x \in B$ and $\chi_B(x) = 0$ if $x \notin B$.

Definition 1: A transition function $f:I\to I$ is chaotic in the sense of ergodic oscillations if there is a unique probability measure ν that is absolutely continuous and satisfies that for almost every $x \in I$,

$$\lim_{t\to\infty} t^{-1}\Sigma_{\tau=0}^{t-1}\chi_B(f^{\tau}(x)) = \nu(B) \tag{3.2}$$

for any Borel set $B \subset I$.

The work of Lasota and Yorke (1974) and Li and Yorke (1978) implies the following.

Proposition 2: If a transition function $f:I\to I$ is expansive and unimodal, it is chaotic in the sense of ergodic oscillations.

4. CHARACTERIZATION OF CHAOTIC OPTIMAL TRANSITION FUNCTIONS

It is intuitively reasonable to conjecture that given a level of capital stock at the beginning of a period, say time t-1, it is optimal to choose a stock level at the end of the period, i.e., at time t, in such a way that both the capital and the labor constraints, (2.16a) and (2.16b) may be satisfied. Figure 1 indicates, in other words, that if $1/\beta \le k_{t-1} \le 1/\alpha$, it is optimal to choose k_t so that (k_{t-1},k_t) lies on PQ. Moreover, if this is in fact the case, it is reasonable to conjecture that if $0 \le k_{t-1} < 1/\beta$, it is optimal to choose k_t so that (k_{t-1},k_t) lies on OP.

This intuition indicates that there may be a case in which the graph of the optimal transition function lies on the kinked line OPQ or, in other words, the optimal transition function is

$$(4.1) \qquad h(k) = \begin{cases} \mu k & \text{if } 0 \leq k \leq 1/\beta \\ -\frac{\alpha}{\beta-\alpha}\mu k + \frac{1}{\beta-\alpha}\mu & \text{if } 1/\beta \leq k \leq \mu/\beta . \end{cases}$$

In what follows, we will demonstrate that such a case is in fact not void for any value of ρ arbitrarily close to 1.

To this end, we will construct a condition under which h(k) is an optimal transition function, which is unimodal and expansive. Note that, without a loss of generality, we may restrict the domain of transition function h to the interval

$$(4.2) \qquad I \equiv \{k \in R|\ 0 \leq k \leq \mu/\beta\} .$$

Since $\mu > 1$ by (2.9) and (2.10), transition function h is expansive and unimodal under the assumption that

$$(4.3) \qquad \frac{\alpha}{\beta-\alpha}\mu > 1.$$

In order to illustrate our construction of the optimal transition function, denote by $\partial_2 v(k,y)$ and $\partial V(k)$, respectively, the partial subgradient of v(k,y) with respect to y and the subgradient of V;

$$(4.4) \qquad \partial_2 v(k,y) = \{-p \geq 0|\ v(k,y)+py \geq v(\kappa,\eta)+p\eta \text{ for all } (\kappa,\eta) \in D\};$$

$$(4.5) \qquad \partial V(y) = \{p \geq 0|\ V(y)-py \geq V(\eta)-p\eta \text{ for all } \eta \geq 0\}.$$

Since v and V are concave, continuous and of free disposal, sets $\partial_2 v(k,y)$ and $\partial V(y)$ are well-defined for any $(k,y) \in D$ and for any $y \geq 0$, respectively.

For each given k, $-\partial_2 v(k,y)$ may be thought of as a correspondence that associates y with a closed interval in R. In Figure 2, denote the graph of this correspondence by $MC_{|k}$. Similarly, $\partial V(y)$, too, is a correspondence that associates y with a closed interval in R. Denote the graph of this correspondence by MV.

By definition, for each $k \in I$, optimal transition function H(k) is determined at the intersection of curves MV and $MC_{|k}$; i.e.,

$$H(k) = \{y \geq 0 |\ [-\partial_2 v(k,y)] \cap [\rho \partial V(y)] \neq \phi\} \ , \tag{4.6}$$

where $cX = \{z \in R|\ z = cx \text{ and } x \in X\}$ for any subset X of R and any number c.

Figures 2 and 3 illustrate typical relationships between curves MV and $MC_{|k}$. As (2.18) indicates, the shape of $MC_{|k}$ depends on k. If $0 < k \leq 1/\beta$,

$$-\partial v_2(k,y) = \begin{cases} \{1/\mu\} & \text{if} \quad y < \mu k \\ \{p|\ p \geq 1/\mu\} & \text{if} \quad y = \mu k \end{cases} , \tag{4.7}$$

which is illustrated in Figure 2. If $1/\beta < k \leq \mu/\beta$,

$$-\partial_2 v(k,y) = \begin{cases} \{1/\mu\} & \text{if } y < -\frac{\alpha}{\beta-\alpha}\mu k + \frac{1}{\beta-\alpha}\mu \\ \{p|\ 1/\mu \leq p \leq (\beta/\alpha)/\mu\} & \text{if } y = -\frac{\alpha}{\beta-\alpha}\mu k + \frac{1}{\beta-\alpha}\mu \\ \{(\beta/\alpha)/\mu\} & \text{if } -\frac{\alpha}{\beta-\alpha}\mu k + \frac{1}{\beta-\alpha}\mu < y < \mu/\beta \\ \{p|\ p \geq (\beta/\alpha)/\mu\} & \text{if } y = \mu/\beta \end{cases} , \tag{4.8}$$

which is illustrated in Figure 3.

A difficulty lies in the fact that curve MV is determined endogenously by the solution to the optimization problem (2.8). For this reason, its specific shape is generally intractable. By the concavity and monotinicity of V, however, we may prove that curve MV is "downward sloping" (or, more precisely, satisfies that if $p \in \rho\partial V(y)$, $p' \in \rho\partial V(y')$ and $y < y'$, then $p \geq p'$).

Suppose, for the sake of discussion, that function h is the optimal transition function. Take the case of $0 \leq k \leq 1/\beta$. Then, since $h(k) = \mu k$, curves MV and $MC_{|k}$ must intersect each other on the vertical part of $MC_{|k}$, as is shown in Figure 2. Instead, take the case of $1/\beta < k \leq \mu/\beta$. Then, since $h(k) = -\frac{\alpha}{\beta-\alpha}\mu k + \frac{1}{\beta-\alpha}$, curves MV and $MC_{|k}$ must intersect each other on the first vertical segment of $MC_{|k}$, as is shown in Figure 3. Thus, the intersection of curve MV with the kinked line XYZ must be unique and at y = μ/β, and that with the kinked line XABCZ must be unique and at y = $-\frac{\alpha/\beta}{\beta-\alpha}\mu^2+\frac{1}{\beta-\alpha}$.

In what follows, we construct a sufficient condition for this condition to hold because, as is demonstrated below, the converse is also true (see Lemma 1). To this end, we focus on the case in which $1/\beta$ is a cyclical point of transition function h; as is noted above, the position of curve MV in the general case is intractable. We first demonstrate that function h is the optimal transition function if the cyclical trajectory from $1/\beta$, which h generates, is the unique optimal path from $1/\beta$ (Lemma 1). We then construct a condition under which the cyclical trajectory is actually the unique optimal path from $1/\beta$ (Theorem 1).

Lemma 1: If $1/\beta$ is a cyclical point of transition function h (it can be of any periodicity) and if the cyclical path from $1/\beta$ is the unique optimal path from $1/\beta$, then h is the optimal transition function.

Note: The hypothesis of this lemma is stronger than what is needed. The following proof goes through if it is assumed is that $h(1/\beta)$ and $h(\mu/\beta)$ are the unique optimal choices of an end-of-a-period stock levels for beginning-of-a-period stock levels $1/\beta$ and μ/β, respectively.

Proof: It suffices to demonstrate that

$$H(k) = \{h(k)\} \tag{4.9}$$

for every $k \in I$. The hypothesis of the theorem implies

(4.10) $$H(1/\beta) = \{\mu/\beta\}$$

and

(4.11) $$H(\mu/\beta) = \{h(\mu/\beta)\}\ ,$$

the latter of which follows from $\mu/\beta = h(1/\beta)$. By (4.10), (4.6) implies that if the MV and $MC_{|k=1/\beta}$ curves intersect each other only at $y = \mu/\beta$. Since the MV curve is "downward sloping," this implies

(4.12) $$\rho\partial V(y) > 1/\mu \text{ if } 0 \leq y < \mu/\beta\ ,$$

as Figure 2 indicates. Similarly, by (4.11), (4.6) implies that if the MV and $MC_{|k=\mu/\beta}$ curves intersect each other only at $y = h(\mu/\beta)$. Since the MV curve is "downward sloping," this implies

(4.13) $$\rho\partial V(y) < (\beta/\alpha)/\mu \text{ if } y > h(\mu/\beta).$$

In order to complete the proof, take k such that $0 < k < 1/\beta$. Let $y < \mu k$. Since this implies $y < \mu/\beta$, by (4.12), $\rho\partial V(y) > 1/\mu$. By (4.7), moreover, $y < \mu k$ implies $-\partial_2 v(k,y) = \{1/\mu\}$. These facts imply $-\partial_2 v(k,y) \cap \partial V(y) = \phi$. Thus, $y \notin H(k)$ for any $y < h(k)$. Since Proposition 1 implies $H(k) \neq \phi$ and since $(k,y) \in D$ implies $y \leq \mu k$, by (4.1), (4.9) holds for k such that $0 < k < 1/\beta$.

Next, take k such that $1/\beta < k < \mu/\beta$. Let $y < h(k)$. Then, as above, it can be shown that

(4.14) $$y < h(k) \text{ implies } y \notin H(k).$$

Let $h(k) < y \leq \mu/\beta$. Since this implies $h(k) > h(\mu/\beta)$, by (4.13), $\rho\partial V(y) < (\beta/\alpha)/\mu$. By (4.8), moreover, $h(k) < y \leq \mu/\beta$ implies $-\partial_2 v(k,y) \geq (\beta/\alpha)/\mu$. These facts imply $-\partial_2 v(k,y) \cap \rho\partial V(y) = \phi$. Thus,

(4.15) $$h(k) < y \leq \mu/\beta \text{ implies } y \notin H(k).$$

Since $H(k) \neq \phi$ by Proposition 1, (4.14) and (4.15) implies (4.9) for k such that $1/\beta < k < \mu/\beta$.

Either if $k = 1/\beta$ or if $k = \mu/\beta$, (4.9) holds by (4.10) and (4.11). If $k = 0$, $(k,y) \in D$ implies $y = 0$. Thus, $H(k) = \{0\}$; (4.9) holds. <u>Q.E.D</u>.

We establish the optimality of the path from $1/\beta$, $h^t(1/\beta)$, under the assumption that $1/\beta$ is a cyclical point of transition function h. For this purpose, we use the value-loss method (McKenzie, 1986). We say that vector (q,p) is a <u>support price vector of activity</u> (k,y) if

(4.16) $$v(k,y) + py - \rho^{-1}qk \geq v(\zeta,\xi) + p\xi - \rho^{-1}q\zeta$$

for all $(\zeta,\xi) \in D$; recall D is the domain of v. Given an activity $(k,y) \in D$ and a support price vector (q,p) of that activity, we define the <u>value loss</u> of an alternative activity $(\zeta,\xi) \in D$ as follows.

(4.17) $$\Delta(\zeta,\xi;q,p;k,y) = v(k,y) + py - \rho^{-1}qk - [v(\zeta,\xi) + p\xi - \rho^{-1}q\zeta].$$

Then, by (4.16), $\Delta(\zeta,\xi;q,p;k,y) \geq$ for any $(\zeta,\xi) \in D$.

See Figure 1. Along the cyclical path from $1/\beta$, $k_t = h^t(1/\beta)$, activity (k_{t-1},k_t) lies either at point P, on segment OP but not at the endpoints, or on PQ but not at the endpoints. We first characterize support price vectors in these cases. Denote

(4.18) $$\gamma = (\beta - a)/a.$$

The next lemma is concerned with the case in which (h_{t-1},h_t) is at point P $= (1/\beta,\mu/\beta)$.

<u>Lemma 2</u>: Price vector (q_0,q_1) is a support price vector of activity $(1/\beta,\mu/\beta)$ if and only if the following conditions are satisfied.

(4.19) $$q_0 \geq 0 ;$$

(4.20) $$-q_0 + \rho\mu q_1 \geq 0 ;$$

(4.21) $$\gamma q_0 + \rho\mu q_1 \geq \rho(1+\gamma) \ .$$

Suppose, moreover, that (4.19), (4.20) and (4.21) are satisfied with strict inequality. Then, for $(k,y) \in D$,

(4.22) $\Delta(k,y;q_0,q_1;1/\beta,\mu/\beta) > 0$ if and only if $(k,y) \neq (1/\beta,\mu/\beta)$.

Proof: By (2.18), $v(1/\beta,\mu/\beta) = 0$. Thus, point $(v,k,y) = (0,1/\beta,\mu/\beta)$ lies on the graph of function $v = v(k,y)$ at $(k,y) = (1/\beta,\mu/\beta)$. The plane through point $(0,1/\beta,\mu/\beta)$ is expressed as

(4.23) $$v = \rho^{-1}q_0(k-1/\beta) - q_1(y-\mu/\beta) \equiv f(k,y).$$

In Figure 1, $P = (1/\beta,\mu/\beta)$, $Q = (1/\alpha,0)$ and $Z = (1+1/\beta,\mu/\beta)$. The corresponding utility levels are $v(1/\beta,\mu/\beta) = 0$ at P, $v(1/\alpha,0) = 1/\alpha$ at Q, and $v(1+1/\beta,\mu/\beta) = 0$ at Z. Let $O' = (0,0,0)$, $P' = (0,1/\beta,\mu/\beta)$, $Q' = (1/\alpha,1/\alpha,0)$, $Z' = (0,1+1/\beta,\mu/\beta)$ and $K' = (1/\alpha,1+1/\beta,0)$ in the v-k-y space. Then, by (2.18), the graph of $v = v(k,y)$ consists of triangle O'P'Q' and the face surrounded by lines P'Q', P'Z' and Q'K'. By the concavity of $v = v(k,y)$, therefore, the plane defined by (4.23) supports the graph of $v(k,y)$ at point P' if and only if the rays from P' through O', Q' and Z' all lie below the plane defined by (4.23), i.e., if and only if the following inequalities hold.

(4.24) $$-\rho^{-1}q_0/\beta + q_1\mu/\beta = f(0,0) \geq v(0,0) = 0;$$

(4.25) $$\rho^{-1}q_0(1/\alpha-1/\beta) + q_1\mu/\beta = f(1/\alpha,0) \geq v(1/\alpha,0) = 1/\alpha \ ;$$

(4.26) $$\rho^{-1}q_0 = f(1+1/\beta,\mu/\beta) \geq v(1+1/\beta,\mu/\beta) = 0.$$

These expressions are equivalent to (4.19), (4.20) and (4.21).

Moreover, if the inequality conditions in (4.24), (4.25) and (4.26) are satisfied strictly, then $\Delta(k,y;q_0,q_1;1/\beta,\mu/\beta) = 0$ if and only if $(k,y) = (1/\beta,\mu/\beta)$. This implies (4.22). Q.E.D.

The next lemma is concerned with the case in which (h_{t-1},h_t) lies, in Figure 1, on segment PQ but not at the endpoints.

Lemma 3: Suppose that (k_{t-1},k_t) satisfies $k_t = h(k_{t-1})$ and $k_{t-1} > 1/\beta$. Then, (q_{t-1},q_t) is a support price vector of (k_{t-1},k_t) if and only if there is q_{t-1} such that

(4.27) $$0 \leq q_{t-1} \leq \rho$$

and

(4.28) $$q_t = [-\gamma q_{t-1} + \rho(1+\gamma)]/(\rho\mu) \ .$$

Suppose, moreover, that (4.27) is satisfied with strict inequality. Then, for $(k,y) \in D$,

(4.29) $$\Delta(k,y;q_{t-1},q_t;k_{t-1},k_t) > 0 \text{ if and only if } y \neq h(k) \text{ or } k < 1/\beta.$$

Proof: First, we will prove that (q_{t-1},q_t) is a support price vector of (k_{t-1},k_t) if and only if there is λ, $0 \leq \lambda \leq 1$, such that

(4.30) $$(\rho^{-1}q_{t-1},-q_t) = \lambda(1,-\tfrac{1}{\mu}) + (1-\lambda)(0,-\tfrac{\beta/\alpha}{\mu}).$$

Since $(v,k,y) = (v(k_{t-1},k_t),k_{t-1},k_t)$ is on the plane defined by $v = k - \frac{1}{\mu}y$, the concavity of $v(k,y)$ implies that

(4.31) $$v(k_{t-1},k_t) + \tfrac{1}{\mu}k_t - k_{t-1} \geq v(k,y) + \tfrac{1}{\mu}y - k$$

for any $(k,y) \in D$. Since, moreover, $(v,k,y) = (v(k_{t-1},k_t),k_{t-1},k_t)$ is on the plane defined by $v = \frac{1}{\alpha} - \frac{\beta/\alpha}{\mu}y$, the concavity of $v(k,y)$ implies

(4.32) $$v(k_{t-1},k_t) + \tfrac{\beta/\alpha}{\mu}k_t - 0\cdot k_{t-1} \geq v(k,y) + \tfrac{\beta/\alpha}{\mu}y - 0\cdot k$$

for all $(k,y) \in D$. Let $0 \leq \lambda \leq 1$. Multiply λ to expression (4.31) and $(1-\lambda)$ to expression (4.32). By adding the resulting expressions together, the claim at the beginning of the proof follows. Since $q_{t-1} = \rho\lambda$ by

(4.30), there is λ, $0 \leq \lambda \leq 1$, such that (4.32) holds if and only if there is q_{t-1} such that (4.27) and (4.28) hold.

See Figure 1. If (k,y) lies in triangle OPQ, (4.31) holds with equality. If (k,y) lies in the region surrounded by lines PZ, PQ and the horizontal axis of Figure 1, (4.32) holds with equality. Let $0 < \lambda < 1$. Then, by construction, $\Delta(k,y;q_{t-1},q_t;k_{t-1},k_t) = 0$ if and only if (k,y) lies on both triangle OPQ and the region surrounded by lines PZ, PQ and the horizontal axis, i.e., (k,y) lies on segment PQ. Since $q_{t-1} = \rho\lambda$ by (4.30), $0 < \lambda < 1$ means $0 < q_{t-1} < \rho$. Thus, (4.29) holds. Q.E.D.

Finally, the next lemma is concerned with in the case in which (k_{t-1},k_t) lies on segment OP but not at the endpoints.

Lemma 4: Suppose that (k_{t-1},k_t) satisfies $k_t = h(k_{t-1})$ and $0 < k_{t-1} < 1/\beta$. Then, (q_{t-1},q_t) is a support price vector of (k_{t-1},k_t) if there is q_{t-1} such that

(4.33) $$q_{t-1} \geq \rho$$

and

(4.34) $$q_t = q_{t-1}/(\rho\mu) .$$

Suppose, moreover, that (4.33) is satisfied with strict inequality. Then, for $(k,y) \in D$,

(4.35) $$\Delta(k,y;q_{t-1},q_t;k_{t-1},k_t) > 0 \text{ if and only if } y \neq h(k) \text{ or } k > 1/\beta.$$

Proof: We will first prove that (q_{t-1},q_t) is a support price vector of (k_{t-1},k_t) if there is $\varphi \geq 1$ such that

(4.36) $$(\rho^{-1}q_{t-1},-q_t) = \varphi(1,-\frac{1}{\mu}).$$

Since $k_t = \mu k_{t-1}$ by the hypothesis of the lemma, by (2.18),

$$(4.37) \qquad \frac{1}{\mu}k_t - k_{t-1} \geq \frac{1}{\mu}y - k$$

for all $(k,y) \in D$. Since $(v,k,y) = (0,k_{t-1},k_t)$ is on the plane defined by $v = k - \frac{1}{\mu}y$, the concavity of $v(k,y)$ implies that

$$(4.38) \qquad v(k_{t-1},k_t) + \frac{1}{\mu}k_t - k_{t-1} \geq v(k,y) + \frac{1}{\mu}y - k$$

for any $(k,y) \in D$. By (4.37) and (4.38), for any $\delta \geq 0$, it holds

$$(4.39) \qquad v(k_{t-1},k_t) + (1+\delta)\frac{1}{\mu}k_t - (1+\delta)k_{t-1} \geq v(k,y) + (1+\delta)\frac{1}{\mu}y - (1+\delta)k$$

for any $(k,y) \in D$. By defining $\varphi = 1+\delta$, the claim at the beginning of the proof follows. Since $q_{t-1} = \rho\varphi$ by (4.36), there is $\varphi \geq 1$ such that (4.36) holds if and only if there is $q_{t-1} > \rho$ such that (3.34) holds.

If and only if (k,y) lies below OP, (4.37) holds with strict inequality. Thus, if $\delta > 0$, $\Delta(k,y;q_{t-1},q_t;k_{t-1},k_t) = 0$ if and only if $y = \mu k$. Since $q_{t-1} = \rho\varphi = \rho(1+\delta)$, $\delta > 0$ means $q_{t-1} > \rho$; (4.35) holds. Q.E.D.

The following theorem provides a sufficient condition that guarantees that function h is the optimal transition function. The sufficient condition consists of two parts: A and B. Part A is simply that $1/\beta$ is a cyclical point of period $N \geq 3$. Part B implies that the cyclical trajectory from $1/\beta$ is the unique optimal path from $1/\beta$.[10]

Theorem 1: Function $h: I \to I$ is the optimal transition function, which is expansive and unimodal, if the following conditions are satisfied.

A: $1/\beta$ is a cyclical point of period $N \geq 3$.

B: There are prices $q_0, q_1, \ldots, q_N$ such that the following is satisfied:

$$\text{(i)} \qquad q_N = q_0;$$

(ii)
$$\begin{cases} q_0 > 0 \\ -q_0 + \rho\mu q_1 > 0 \\ \gamma q_0 + \rho\mu q_1 > \rho(1+\gamma) \end{cases};$$

(iii) Let $k_t = h^t(1/\beta)$ and $t = 2,3,\ldots,N$.

a: $$q_t = \begin{cases} q_{t-1}/(\rho\mu) & \text{if} \quad k_{t-1} < 1/\beta \\ [-\gamma q_{t-1}+\rho(1+\gamma)]/(\rho\mu) & \text{if} \quad k_{t-1} > 1/\beta . \end{cases}$$

b: $$\begin{cases} q_{t-1} > \rho & \text{if} \quad k_{t-1} < 1/\beta \\ 0 < q_{t-1} < \rho & \text{if} \quad k_{t-1} > 1/\beta . \end{cases}$$

(Note that, given condition A, $k_{t-1} \neq 1/\beta$ for $t = 2,3,\ldots,N$.)

Proof: By Lemma 1, it suffices to prove that path k_t is the unique optimal path from $1/\beta$. By construction, (k_{t-1},k_t) lies either at point P, on segment OP but not at its endpoints, or on segment PQ but not at its endpoints. For each $\tau = 0,1,2,\ldots$, define $(q_{N\cdot\tau},q_{N\cdot\tau+1},\ldots,q_{N\cdot\tau+N-1}) = (q_0,q_1,\ldots,q_{N-1})$. Under conditions B-i, B-ii and B-iii, by Lemmas 2, 3 and 4, (q_{t-1},q_t) is a support price vector of activity (k_{t-1},k_t) for $t = 1,2,\ldots$ Define

(4.40) $$\Delta_t(k,y) = \Delta(k,y;q_{t-1},q_t;k_{t-1},k_t).$$

Then, for any $(k,y) \in D$,

(4.41) $$\Delta_t(k,y) \geq 0, \qquad t = 1,2,\ldots$$

In order to prove the optimality of path k_t, take an arbitrary alternative path h_t, $t = 0,1,\ldots$, such that $(h_{t-1},h_t) \in D$ and $h_0 = 1/\beta$. Then, by the definition of value losses, for any T,

(4.42) $$\Sigma_{t=1}^{T}\rho^t(v(k_{t-1},k_t) - v(h_{t-1},h_t)) = \Sigma_{t=1}^{T}\rho^t\Delta_t(h_{t-1},h_t) - \rho^T q_T(h_T-k_T) .$$

Since $q_T \in \{q_0,q_1,\ldots,q_{N-1}\}$ by definition, by taking $T \to \infty$,

(4.43) $$\Sigma_{t=1}^{\infty}\rho^t v(k_{t-1},k_t) - \Sigma_{t=1}^{\infty}\rho^t v(h_{t-1},h_t) = \Sigma_{t=1}^{\infty}\rho^t\Delta_t(h_{t-1},h_t) \geq 0$$

for any path h_t, $t = 0,1,\ldots$, such that $(h_{t-1},h_t) \in D$ and $h_0 = 1/\beta$. This implies that path k_t is optimal.

In order to prove the uniqueness of path k_t, suppose that there is another optimal path h_t, $t = 0,1,\ldots$, from $1/\beta$. Then, there is t* such that $h_{t*} \neq h(h_{t*-1})$. Under the hypothesis of the theorem, (4.19), (4.20), (4.21), (4.27) and (4.33) are all satisfied with strict inequality. Therefore, by (4.22), (4.29) and (4.35), $h_{t*} \neq h(h_{t*-1})$ implies

(4.44) $$\Delta_t(h_{t*-1},h_{t*}) > 0.$$

This implies, however, by (4.41), (4.43) and (4.44),

(4.45) $$\Sigma_{t=1}^{\infty}\rho^t v(k_{t-1},k_t) - \Sigma_{t=1}^{\infty}\rho^t v(h_{t-1},h_t) \geq \Delta_t(h_{t*-1},h_{t*}) > 0 ,$$

which contradicts the optimality of path h_t. Q.E.D.

5. NISHIMURA AND YANO'S RESULT

Nishimura and Yano (1992b) demonstrates that for any value of discount factor ρ arbitrarily close to 1, it is possible to choose values of parameters μ, α, and β in such a way that transition function h(k) is optimal at the same time as it is unimodal and expansive. That is to say,

Proposition 3: No matter how close to 1 the value of discount factor ρ, $0 < \rho < 1$, is, it is possible to choose values of parameters μ and γ in such a way that h(k) is an ergodically chaotic optimal transition function.

Note that the conditions that we have imposed on parameters so far are equivalent to the following:

(5.1) $$0 < \rho < 1 ;$$

(5.2) $$\rho\mu > 1 \;;$$

(5.3) $$\mu > \gamma > 0 \;;$$

(5.4) $$\alpha > 0 \;.$$

(Recall that $\gamma = (\beta-\alpha)/\alpha$.) In addition, we require

(5.5) $$\rho\mu < \sqrt{\gamma}.$$

6. PROOF OF THE MAIN RESULT

In what follows, we will first construct a condition under which $1/\beta$ is a cyclical point of h(k) with the order of periodicity $N = 3x2^n$ for n = 0,1,... We will then construct a condition under which there is price path $q_0,\ldots,q_N$ that satisfies condition B-i, B-ii and B-iii-a of Theorem 1. The main result follows from Theorem 1. We set

(6.1) $$\alpha = 1.$$

In order to construct a condition under which $1/\beta$ is a cyclical point of h(k) with the order of periodicity $N = 3x2^n$ for n = 0,1,..., define the following expressions.

(6.2) $$m(n) = [2^n+(-1)^{n-1}]/3 \;;$$

(6.3) $$f\left[\mu^{2^n};\gamma\right] = \left[\mu^{2^n}\right]^2 - \gamma^{2m(n)+(-1)^n}\left[\mu^{2^n}\right] - \gamma^{4m(n)+(-1)^n} \;.$$

<u>Lemma 5</u>: Let n = 0,1,2,... Point $k = 1/\beta$ is a cyclical point of h(k) with the order of periodicity $N = 3x2^n$ if and only if $f\left[\mu^{2^n};\gamma\right] = 0$. (Note: This condition gives rise to the general form of conditions (5.8), (5.21), (5.32) and (5.44).)

Proof: We will prove this inductively. For this purpose, it is convenient to shift the origin of the transition function function $y = h(k)$ to $(1/\beta, 1/\beta)$ by defining $\kappa = k - 1/\beta$ and $\zeta = y - 1/\beta$. Then, by (4.1), h(k) is equivalent to

$$\text{(6.4)} \qquad \eta(\kappa) = \begin{cases} \mu\kappa + \frac{1}{\beta}(\mu-1) & \text{if } -1/\beta < \kappa \leq 0 \\ -\frac{\mu}{\gamma}\kappa + \frac{1}{\beta}(\mu-1) & \text{if } \quad 0 < \kappa < \frac{1}{\beta}(\mu-1). \end{cases}$$

If and only if 0 is a cylical point of η, $1/\beta$ is a cyclical point of h.

For n = 1, it is easy to see that the lemma follows. One way to construct a condition under which $\kappa = 0$ is a cyclical point $\eta(\kappa)$ with order $6 = 3x2^1$ is to construct, first, transition function $\eta^2(\kappa)$ and, then, to find a condition under which 0 is a cyclical point of η^2 with the order of periodicity equal to 3. To this end, in Figure 5, we depict function $\eta(\kappa)$ by kinked segment $A_0P_0B_0$. As the diagram indicates, P_0 indicates $(0,\eta(0))$, B_0 indicates $(\eta(0),\eta^2(0))$, and A_0 inidcates $(\eta^2(0),\eta^3(0))$. By construction, $(\eta^{t-1}(0),\eta^t(0))$ must lie on kinked segment $A_0P_0B_0$ for any t = 1,2,... Curve $B_1P_1Q_1R_1$ illustrates function $\eta^2(\kappa) = \eta\circ\eta(\kappa)$. If 0 is a cyclical point of η^2 with the order of periodicity equal to 3, then $(0,\eta^2(0))$ is at P_1 and $(\eta^2(0),\eta^{2x2}(0))$ is at B_1. Moreover, $A_1 = (\eta^{2x2}(0),\eta^{2x3}(0))$ must lie on the open segment between P_1 and S; otherwise, it is impossible to construct a period-3 path from 0. In order to construct a condition under which 0 is a cyclical point of η^2 with order 3, it suffices to focus on function η^2 restricted to the interval between κ_{B_1} and κ_{A_1}, where κ_{A_1} and κ_{B_1} are, respectively, the κ-coorinate of points A_1 and B_1. On this interval, $\eta^2(\kappa)$ can be expressed as follows:

$$\text{(6.5)}\qquad \eta^2(\kappa) = \begin{cases} -\frac{\mu^2}{\gamma}\kappa - \frac{1}{\beta}(\mu-1)(\frac{\mu}{\gamma}-1) & \text{if } \kappa_{B_1} \le \kappa \le 0 \\ \frac{\mu^2}{\gamma^2}\kappa - \frac{1}{\beta}(\mu-1)(\frac{\mu}{\gamma}-1) & \text{if } 0 < \kappa \le \kappa_{A_1} \end{cases} .$$

Repeating similar arguments, we may inductively construct conditions under which 0 is a cyclical point of η^{2^n} with the order of periodicity equal to 3. That is to say, we may find kinked segment $A_nP_nB_n$ such that $(0,\eta^{2^n}(0))$ is at P_n, $(\eta^{2^n}(0),\eta^{2^n x2}(0))$ is at B_n, and $(\eta^{2^n x2}(0),\eta^{2^n x3}(0))$ is at A_n. Moreover, denoting by κ_{A_n} and k_{B_n}, respectively, the κ-coodinates of points A_n and B_n, function $\eta^{2^n}(\kappa)$ restricted to the interval between κ_{A_n} and κ_{B_n} is as follows:

If n is even,

$$\text{(6.6)}\quad \eta^{2^n}(\kappa)=\begin{cases} \frac{\mu^{2^n}}{\gamma^{2m(n)}}\kappa+\frac{1}{\beta}\left[1-\frac{\mu}{\gamma}\right]\left[1-\frac{\mu^2}{\gamma}\right]\left[1-\frac{\mu^4}{\gamma^2}\right]\cdots\left[1-\frac{\mu^{2^{n-1}}}{\gamma^{m(n)}}\right] & \text{if } \kappa_{A_n}\le\kappa\le 0 \\ -\frac{\mu^{2^n}}{\gamma^{2m(n)+(-1)^n}}\kappa+\frac{1}{\beta}\left[1-\frac{\mu}{\gamma}\right]\left[1-\frac{\mu^2}{\gamma}\right]\left[1-\frac{\mu^4}{\gamma^2}\right]\cdots\left[1-\frac{\mu^{2^{n-1}}}{\gamma^{m(n)}}\right] & \text{if } 0\le\kappa\le\kappa_{B_n} \end{cases} .$$

If n is odd,

$$\text{(6.7)}\quad \eta^{2^n}(\kappa)=\begin{cases} -\frac{\mu^{2^n}}{\gamma^{2m(n)+(-1)^n}}\kappa+\frac{1}{\beta}\left[1-\frac{\mu}{\gamma}\right]\left[1-\frac{\mu^2}{\gamma}\right]\left[1-\frac{\mu^4}{\gamma^2}\right]\cdots\left[1-\frac{\mu^{2^{n-1}}}{\gamma^{m(n)}}\right] & \text{if } \kappa_{B_n}\le\kappa\le 0 \\ \frac{\mu^{2^n}}{\gamma^{2m(n)}}\kappa+\frac{1}{\beta}\left[1-\frac{\mu}{\gamma}\right]\left[1-\frac{\mu^2}{\gamma}\right]\left[1-\frac{\mu^4}{\gamma^2}\right]\cdots\left[1-\frac{\mu^{2^{n-1}}}{\gamma^{m(n)}}\right] & \text{if } 0\le\kappa\le\kappa_{A_n} \end{cases} .$$

(Note: Since $1 > \mu^{2^{n-1}}/\gamma^{m(n)}$, the constant term in (6.6) and (6.7) is positive if n is even and negative if n is odd.)

Given the transtion functions defined by (6.6) and (6.7), it is easy to demonstrate that for either odd or even n, point 0 is a cyclical point of $\eta^{2^n}(\kappa)$ with the order of periodicity 3 if

$$(6.8) \qquad 1 + \frac{\mu^{2^n}}{\gamma^{2m(n)}} \left[1 - \frac{\mu^{2^n}}{\gamma^{2m(n)+(-1)^n}}\right] = 0,$$

which is equivalent to $f\left[\mu^{2^n};\gamma\right] = 0$ by (6.3). Q.E.D.

Let $\tilde{k}_1 = \mu/\beta$ and $\tilde{k}_t = h(\tilde{k}_{t-1})$ for $t = 2,3,\ldots$ Note the following:

Lemma 6: Let $N = 3x2^n$. Then,

$$(6.9) \qquad \tilde{k}_N - 1/\beta = \frac{1}{\beta}(\mu-1)\left[1-\frac{\mu}{\gamma}\right]\left[1-\frac{\mu^2}{\gamma}\right]\cdots\left[1-\frac{\mu^{2^{n-1}}}{\gamma^{m(n)}}\right]\left[1+\frac{\mu^{2^n}}{\gamma^{2m(n)}}\left[1-\frac{\mu^{2^n}}{\gamma^{2m(n)+(-1)^n}}\right]\right].$$

Proof: This follows from (6.6) and (6.7). Q.E.D.

Next, we will construct a condition under which there is price path $q_0,\ldots,q_N$ that satisfies condition B-i, B-ii and B-iii-a of Theorem 1. As is shown below, conditions B-i and B-iii-a determines a relationship between q_0 and q_1. This relationship can be illustrated by a line in Figure 4, which we call line L. Because the region in which condition B-ii is satisfied can be indicated by Γ, conditions B-i, B-ii and B-iii-a are satisfied if and only if line L cuts the interior of region Γ.

As Figure 4 indicates, this condition can be characterized by the slope of line L and the position of the point on L at which the q_1-coordinate is $1/\mu$. In order to determine this position, consider the sequence, denoted by $\tilde{q}_1,\ldots,\tilde{q}_N$, that follows condition B-iii-a of Theorem 1 from $q_1 = 1/\mu$. Moreover, set $\tilde{q}_0 = \tilde{q}_N$. Then, by construction, $(\tilde{q}_0,\tilde{q}_1)$ lies on line L, and its q_1-coodinate is $1/\mu$. In order to characterizes the position of the q_0-coodinate of $(\tilde{q}_0,\tilde{q}_1)$, $\tilde{q}_0 = \tilde{q}_N$, we first prove the following.

Lemma 7: Let $N = 3x2^n$. Then,

$$(6.10)\quad \tilde{q}_N - \rho = \rho\left[\frac{1}{\rho\mu} - 1\right]\left[1 - \frac{\gamma}{\rho\mu}\right]\left[1 - \frac{\gamma}{(\rho\mu)^2}\right]\cdots\left[1 - \frac{\gamma^{m(n)}}{(\rho\mu)^{2^{n-1}}}\right]\left[1 + \frac{\gamma^{2m(n)}}{(\rho\mu)^{2^n}}\left[1 - \frac{\gamma^{2m(n)+(-1)^n}}{(\rho\mu)^{2^n}}\right]\right]$$

Proof: Let $t = 2,\ldots,N$. Then, $\tilde{k}_{t-1} \neq 1/\beta$ for any $t = 2,\ldots,N$, since $\tilde{k}_1$ is a cyclical point of order N. Recall that we have set $a = 1$. Thus, whenever $\tilde{k}_{t-1} > 1/\beta$, $\tilde{k}_t = -\frac{\mu}{\gamma}\tilde{k}_{t-1} + \frac{\mu}{\gamma}$ and $\tilde{q}_t = -\frac{\gamma}{\rho\mu}\tilde{q}_{t-1} + \frac{1+\gamma}{\mu}$ by (4.1) and condition B-iii-a. In contrast, whenever $\tilde{k}_{t-1} < 1/\beta$, $\tilde{k}_t = \mu\tilde{k}_{t-1}$ and $\tilde{q}_t = \frac{1}{\rho\mu}\tilde{q}_{t-1}$. This parallel indicates that $\tilde{q}_N - \rho$ is equal to the right-hand side of (6.11), which is parallel to that of $\tilde{k}_N - 1/\beta$, given (6.9). Q.E.D.

Define

$$(6.11)\qquad g\left[(\rho\mu)^{2^n};\gamma\right] = \left[(\rho\mu)^{2^n}\right]^2 + \gamma^{2m(n)}\left[(\rho\mu)^{2^n}\right] - \gamma^{4m(n)+(-1)^n} .$$

Lemma 8: Let $N = 3x2^n$. There is a price sequence $q_0,\ldots,q_N$ that satisfies conditions B-i, B-ii and B-iii-a of Theorem 1 if and only if $g\left[(\rho\mu)^{2^n};\gamma\right] < 0$.

Proof: We first prove, inductively,

$$(6.12)\qquad \gamma^{m(n)}/(\rho\mu)^{2^{n-1}} > 1$$

for $n = 1,2,\ldots$ Since $\sqrt{\gamma} > \rho\mu > 1$ by (5.2) and (5.5), $\gamma > \rho\mu$ and $\gamma > (\rho\mu)^2$. By $\gamma > \rho\mu$, (6.12) holds for $n = 1$. By $\gamma > (\rho\mu)^2$, (6.12) holds for $n = 2$. Suppose that (6.12) holds for $n = i$ and $i+1$. Then, by the

definition of m(n), $[\gamma^{m(i+2)}/(\rho\mu)^{2^{(i+2)-1}}] =$
$[\gamma^{m(i)}/(\rho\mu)^{2^{i-1}}]^2 x[\gamma^{m(i+1)}/(\rho\mu)^{2^{(i+2)-1}}] > 1^2 x1 = 1$. This implies (6.12) for $n = i+2$; (6.12) is proved.

In order to complete the proof, as is noted above, it suffices to demonstrate that line L cuts region Γ in Figure 4. Recall that $(\tilde{q}_0, \tilde{q}_1)$ lies on L and that its q_1-coodinate is $\tilde{q}_1 = 1/\mu$. Therefore, in the case in which line L is upward-sloping, it cuts region Γ if and only if $\tilde{q}_N < \rho$. In the case in which line L is downward-sloping, it cuts region Γ if and only if $\tilde{q}_N > \rho$. Moreover, as the above examples indicate, it may be proved that line L is down-sloping if and only if, given $N = 3x2^n$, n is an even number and that it is upward-sloping if and only if n is an odd number. These facts imply that line L_N cuts region Γ if and only if $(\tilde{q}_N - \rho)(-1)^n > 0$.

Since $\rho\mu > 1$ by (5.2), the term in the first pair of parentheses on the right-hand side of (6.10) is negative. By (6.12), moreover, the terms in the second through (n+1)-th pairs of parentheses are all negative as well. Since, therefore, the terms in the first (n+1) pairs of parentheses are all negative, $(\tilde{q}_N - \rho)(-1)^n > 0$ if and only if the term in the last pair of parentheses is negative, i.e.,

$$(6.13) \qquad \left[1+\frac{\gamma^{2m(n)}}{(\rho\mu)^{2^n}}\left[1-\frac{\gamma^{2m(n)+(-1)^n}}{(\rho\mu)^{2^n}}\right]\right] < 0 .$$

which is equivalent to $g\left[(\rho\mu)^{2^n};\gamma\right] < 0$ by (6.11). <u>Q.E.D</u>.

The next theorem characterizes a condition under which h(k) is an optimal transition function.

Theorem 6: Let n = 0,1,... and ρ^*, μ^* and γ^* be values of ρ, μ and γ satisfying $f\left[\mu^{*2^n};\gamma^*\right] = 0$ and $g\left[(\rho^*\mu^*)^{2^n};\gamma^*\right] < 0$. Denote by $h^*(k)$ the transition function $h(k)$ that corresponds to these values of parameters. If $g\left[(\rho^*\mu^*)^{2^n};\gamma^*\right]$ is sufficiently close to 0, $h^*(k)$ is an optimal transition function, which is ergodically chaotic. In addition, point $1/\beta^*$ is a cyclical point of $h^*(k)$ with the order of periodicity $N = 3x2^n$, where β^* is the value of β corresponding to μ^* and γ^*.

Proof: Since $f\left[\mu^{*2^n};\gamma^*\right] = 0$, by Lemma 5, point $k^* = 1/\beta^*$ is a cyclical point of $h^*(k)$ with the order of periodicity $N = 3x2^n$. Since $g\left[(\rho^*\mu^*)^{2^n};\gamma^*\right] < 0$, by Lemma 8, there is a price sequence $q_0^*, q_1^*, \ldots, q_N^*$ that satisfies conditions B-i, B-ii and B-iii-a of Theorem 1. By Theorem 1, therefore, it suffices to prove that the price sequence can be constructed in such a way that it satisfies condition B-iii-b as well.

Consider ρ, μ and γ that satisfy $f\left[\mu^{2^n};\gamma\right] = 0$ and $g\left[(\rho\mu)^{2^n};\gamma\right] = 0$. The structural similarity between the expression of $\tilde{q}_N - \rho$, given by Lemma 7, and that of $\tilde{k}_N - 1/\beta$, given by Lemma 6, implies the following: If $(\tilde{k}_t, \tilde{k}_{t+1})$ lies on segment OP, but point P, of Figure 1, $(\tilde{q}_t, \tilde{q}_{t+1})$ lies on segment AX, but point A, of Figure 4. If $(\tilde{k}_t, \tilde{k}_{t+1})$ lies on segment PR, but point P, of Figure 1, $(\tilde{q}_t, \tilde{q}_{t+1})$ lies on segment YA, but point A, of Figure 4. (In this case, $\tilde{q}_{t+1}/\tilde{q}_t = 1/(\rho\mu)$ if and only if $\tilde{k}_{t+1}/\tilde{k}_t = \mu$, and $\tilde{q}_{t+1}/\tilde{q}_t = -\gamma/(\rho\mu)$ if and only if $\tilde{k}_{t+1}/\tilde{k}_t = -\mu/\gamma$. The structural similarity between the expression of $\tilde{q}_N - \rho$ and that of $\tilde{k}_N - 1/\beta$ implies that whenever $\tilde{k}_{t+1}/\tilde{k}_t = \mu$, $\tilde{q}_{t+1}/\tilde{q}_t = 1/(\rho\mu)$ must hold and that whenever $\tilde{k}_{t+1}/\tilde{k}_t = -\mu/\gamma$, $\tilde{q}_{t+1}/\tilde{q}_t = -\gamma/(\rho\mu)$ must hold. This implies the above claim.) In short,

$$(6.14) \qquad \begin{cases} \tilde{q}_t > \rho & \text{if } \tilde{k}_t < 1/\beta \\ 0 < \tilde{q}_t < \rho & \text{if } \tilde{k}_t > 1/\beta \ . \end{cases}$$

We now construct a price sequence $q_0^*, q_1^*, \ldots, q_N^*$ that satisfies conditions B-i, B-ii and B-iii of Theorem 1. To this end, set $q_0^* = \rho$, and choose q_1^* in such a way that (q_0^*, q_1^*) is on line L_N. Since $g\left[(\rho^*\mu^*)^{2^n}; \gamma^*\right]$ may be made sufficiently close to 0, it is possible to find parameters ρ, μ and γ satisfying $f\left[(\mu)^{2^n}; \gamma\right] = 0$ and $g\left[(\rho\mu)^{2^n}; \gamma\right] = 0$ in an arbitrarily small neighborhood of ρ^*, μ^* and γ^*. Since, in that case, point (q_0^*, q_1^*) is arbitrarily close to point A in Figure 4, and since line AX lies below the 45o line, it holds $q_1^* < \rho$. Since $\tilde{k}_1 = \mu/\beta > 1/\beta$, $q_1^* < \rho$ implies that condition B-iii-b is satisfied for t = 1. Let $q_2^*, \ldots, q_N^*$ be the sequence that follows condition B-iii-a from the $q_1 = q_1^*$. Then, for t = 1,2,..., T-1, (q_t^*, q_{t+1}^*) is arbitrarily close to point $(\tilde{q}_t, \tilde{q}_{t+1})$. This implies that if $(\tilde{q}_t, \tilde{q}_{t+1})$ lies on open segment YA, (q_t^*, q_{t+1}^*) must lie on open segment Y*A* and that if $(\tilde{q}_t, \tilde{q}_{t+1})$ lies on open segment AX, (q_t^*, q_{t+1}^*) must lie on open segment A*X*, where A*, X* and Y* are the points corresponding to A, X and Y in the case in which the parameter values are ρ^*, μ^* and γ^*. Therefore, by (6.14),

$$(6.15) \qquad \begin{cases} q_t^* > \rho^* & \text{if } k_t^* < 1/\beta^* \\ 0 < q_t^* < \rho^* & \text{if } k_t^* > 1/\beta^* \ . \end{cases}$$

This implies that q_0^*, $q_1^*, \ldots, q_N^*$ satisfies condition B-iii-b. Q.E.D.

We now prove that no matter how close to 1 discount factor ρ is, there are values of parameters μ and γ such that h(k) is an optimal transition function. For this purpose, we demonstrate that if n is sufficiently large, the set of (ρ,μ,γ) that satisfies $f\left[\mu^{2^n};\gamma\right] = 0$ and $g\left[(\rho\mu)^{2^n};\gamma\right] < 0$ together with conditions (5.1) through (5.5) contains (ρ,μ,γ) of which the ρ-coodinate is arbitrarily close to 1.

By (6.11), $g\left[(\rho\mu)^{2^n};\gamma\right] < 0$ is equivalent to

$$\rho^{2^n}\mu^{2^n} < \frac{-\gamma^{2m(n)}+\sqrt{\gamma^{4m(n)}+4\gamma^{4m(n)+(-1)^n}}}{2} \tag{6.16}$$

By (6.11), $f\left[\mu^{2^n};\gamma\right] = 0$ is equivalent to

$$\mu^{2^n} = \frac{\gamma^{2m(n)+(-1)^n}+\sqrt{\gamma^{2[2m(n)+(-1)^n]}+4\gamma^{4m(n)+(-1)^n}}}{2} . \tag{6.17}$$

By (6.17), (6.16) is equivalent to

$$\rho < \left[\frac{-1+\sqrt{1+4\gamma}}{\gamma+\sqrt{\gamma^2+4\gamma}}\right]^{\frac{1}{2^n}} . \tag{6.18}$$

By (6.17) again, condition (5.5), i.e., $\rho\mu < \sqrt{\gamma}$, is equivalent to

$$\rho < \left[\frac{2\gamma^{2^{n-1}}}{\gamma^{2m(n)+(-1)^n}+\sqrt{\gamma^{2[2m(n)+(-1)^n]}+4\gamma^{4m(n)+(-1)^n}}}\right]^{\frac{1}{2^n}} . \tag{6.19}$$

Moreover, condition (5.2), i.e., $\rho\mu > 1$, is equivalent to

$$\rho > \left[\frac{2}{\gamma^{2m(n)+(-1)^n}+\sqrt{\gamma^{2[2m(n)+(-1)^n]}+4\gamma^{4m(n)+(-1)^n}}}\right]^{\frac{1}{2^n}} . \tag{6.20}$$

Finally, condition (5.3), i.e., $\mu > \gamma > 1$, is equivalent to

$$\frac{\gamma^{2m(n)+(-1)^n}+\sqrt{\gamma^{2[2m(n)+(-1)^n]}+4\gamma^{4m(n)+(-1)^n}}}{2} > \gamma^{2^n} . \tag{6.21}$$

We will prove the following theorem.

Theorem 7: For any arbitrary large n, choose ρ and γ in such a way that the following conditions is satisfied.

(6.22) $$1 < \gamma < \left[\frac{1+\sqrt{5}}{2}\right]^{3/(2^n+1)} ;$$

(6.23) $$1/\gamma^{2/3} < \rho < 1/\gamma^{1/6} .$$

Moreover, choose μ in such a way that (6.17) is satisfied. Then, h(k) is an optimal transition function, which is ergodically chaotic.

Proof: Under the hypothesis of the theorem, it suffices to demonstrate that conditions (6.18) through (6.21) together with (5.1), i.e., $0 < \rho < 1$, are satisfied. First, note that

(6.24) $$\lim_{n\to\infty}\left[\frac{-1+\sqrt{1+4\gamma}}{\gamma+\sqrt{\gamma^2+4\gamma}}\right]^{\frac{1}{2^n}} = 1 .$$

Next, note that it is possible to prove

(6.25) $$\frac{2}{\gamma^{2m(n)+(-1)^n}+\sqrt{\gamma^{2[2m(n)+(-1)^n]}+4\gamma^{4m(n)+(-1)^n}}} > \frac{2\gamma^{2^{n-1}}}{(1+\sqrt{5})\gamma^{(2^{n+1}+1)/3}} .$$

Since

(6.26) $$\lim_{n\to\infty}\left[\frac{2\gamma^{2^{n-1}}}{(1+\sqrt{5})\gamma^{(2^{n+1}+1)/3}}\right]^{\frac{1}{2^n}} = 1/\gamma^{1/6} ,$$

it holds that

(6.27) $$\lim_{n\to\infty}\left[\frac{2}{\gamma^{2m(n)+(-1)^n}+\sqrt{\gamma^{2[2m(n)+(-1)^n]}+4\gamma^{4m(n)+(-1)^n}}}\right]^{\frac{1}{2^n}} \geq 1/\gamma^{1/6}.$$

Thus, (6.24) and (6.27) imply that for a sufficiently large n, conditions

(6.18) and (6.19) hold if

(6.28) $$\rho < 1/\gamma^{1/6}.$$

Note that it may be demonstrated that

(6.29) $$\left[\frac{2}{(1+\sqrt{5})\gamma^{(2^{n+1}-2)/3}}\right]^{\frac{1}{2^n}} > \left[\frac{2}{\gamma^{2m(n)+(-1)^n}+\sqrt{\gamma^{2[2m(n)+(-1)^n]}+4\gamma^{4m(n)+(-1)^n}}}\right]^{\frac{1}{2^n}}.$$

Since

(6.30) $$\lim_{n\to\infty}\left[\frac{2}{(1+\sqrt{5})\gamma^{(2^{n+1}-2)/3}}\right]^{\frac{1}{2^n}} = 1/\gamma^{2/3},$$

for a sufficiently large n, condition (6.20) is satisfied if

(6.31) $$\rho > 1/\gamma^{2/3}.$$

In summary, by (6.28) and (6.30), conditions (6.18), (6.19) and (6.20) are satisfied if (6.23) is satisfied.

It is possible to demonstrate

(6.31) $$\frac{\gamma^{2m(n)+(-1)^n}+\sqrt{\gamma^{2[2m(n)+(-1)^n]}+4\gamma^{4m(n)+(-1)^n}}}{2\gamma^{2^n}} \geq \left[\frac{1+\sqrt{5}}{2}\right]\gamma^{-(2^n+1)/3}.$$

Thus, if and only if

(6.32) $$\gamma < \left[\frac{1+\sqrt{5}}{2}\right]^{3/(2^n+1)}$$

condition (6.21) is satisfied. Let $\gamma > 1$. Then, by (6.23), $0 < \rho < 1$; condition (5.1) is satisfied. Thus, under the hypothesis of the theorem, conditions (6.18) through (6.21) together with (5.5) are satisfied. Q.E.D.

Proposition 3 directly follows from Theorem 2. (For a proof, note the following: As $n \uparrow \infty$, $\left[\frac{1+\sqrt{5}}{2}\right]^{3/(2^n+1)} \downarrow 1$. Thus, by (6.22), $\gamma \downarrow 1$. As $\gamma \downarrow 1$, $1/\gamma^{2/3} \uparrow 1$ as well as $1/\gamma^{1/6} \uparrow 1$. Thus, $\rho \uparrow 1$ by (6.23). Thus, the proposition follows from Theorem 7.)

REFERENCES

BOLDRIN, M., AND L. MONTRUCCHIO (1986): "On the Indeterminacy of Capital Accumulation Paths," Journal of Economic Theory 40, 26-39.

DENECKERE, R., AND S. PELIKAN (1986): "Competitive Chaos," Journal of Economic Theory 40, 13-25.

LASOTA, A., AND J. YORKE (1973): "On the Existence of Invariant Measures for Piecewise Monotonic Transformations," Transactions of American Mathematical Society 186, 481-488.

LI, T.-Y., AND J. YORKE (1978): "Ergodic Transformations from an Interval into Itself," Transactions of American Mathematical Society 235, 183-192.

NISHIMURA, K., AND M. YANO (1992a): "Business Cycles and Complex Non-Linear Dynamics," Chaos, Solitons and Fractals 2, 95-102.

__________ (1992b): "Non-Linear Dynamics and Chaos in Optimal Growth," mimeo.

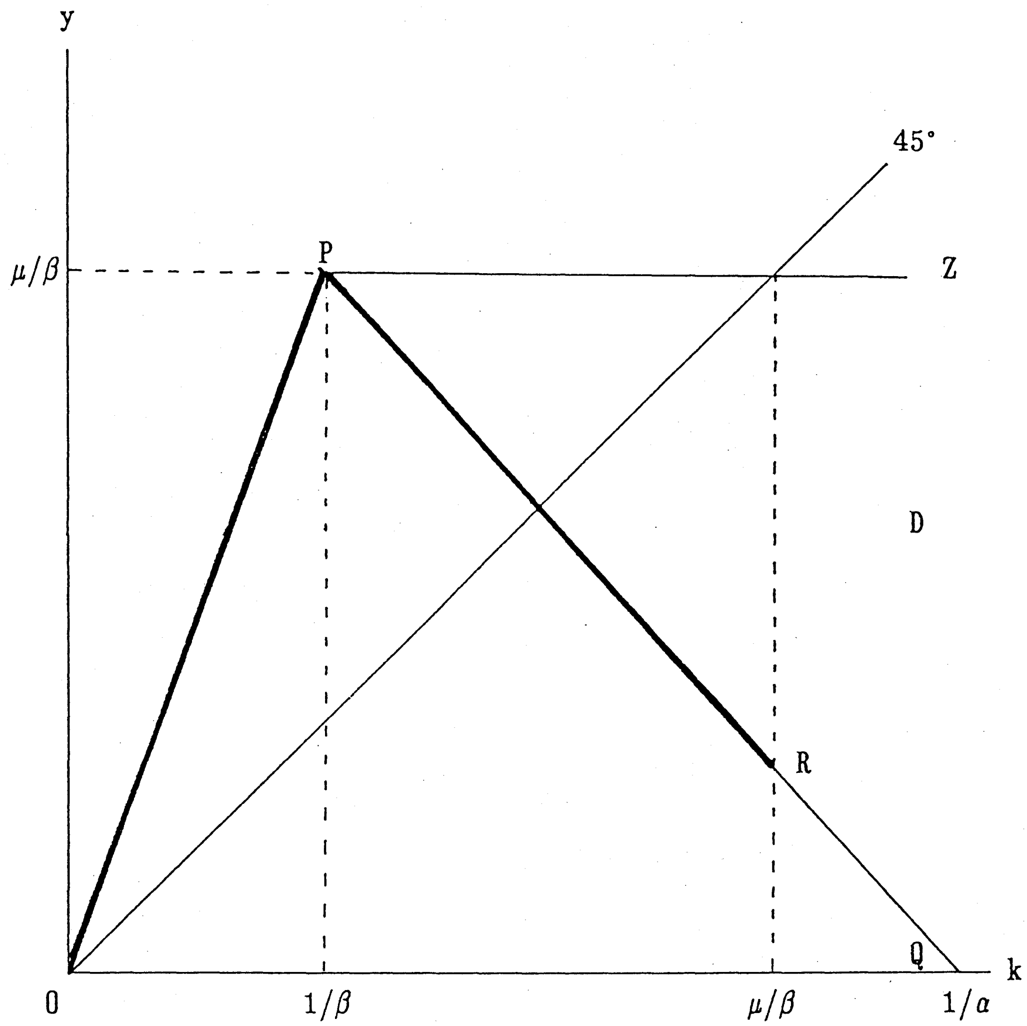

Figure 1

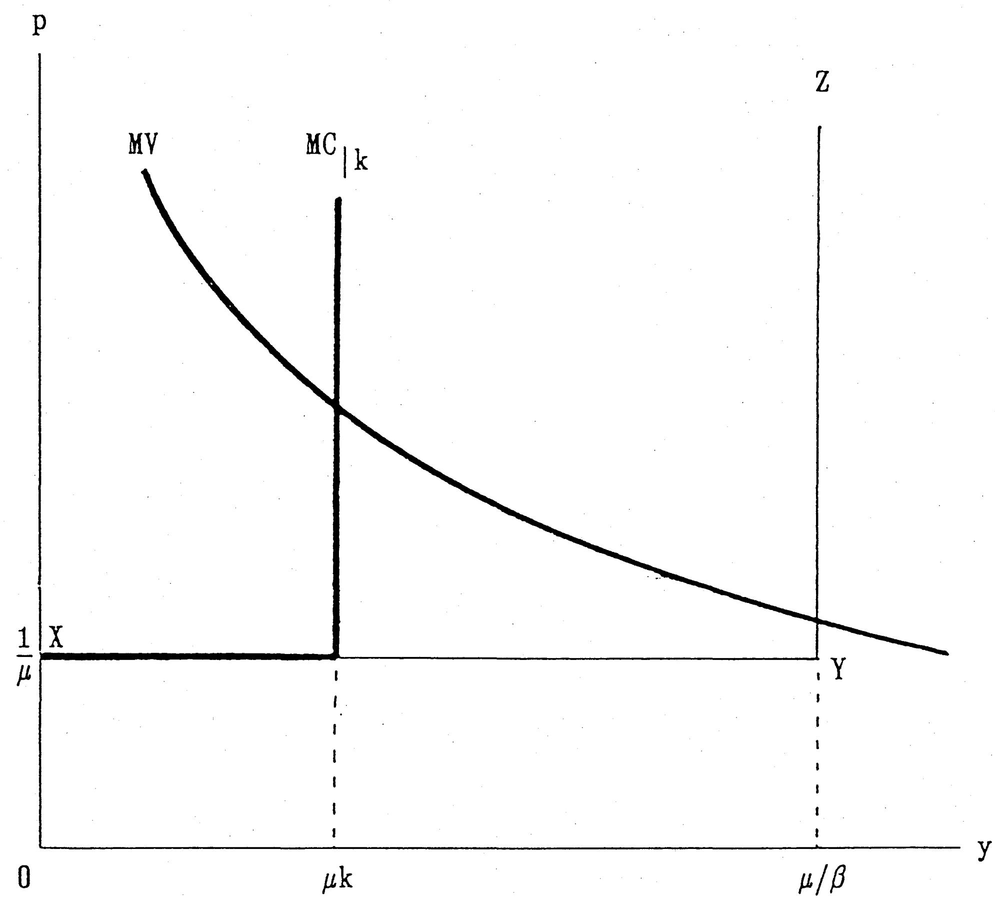

Figure 2

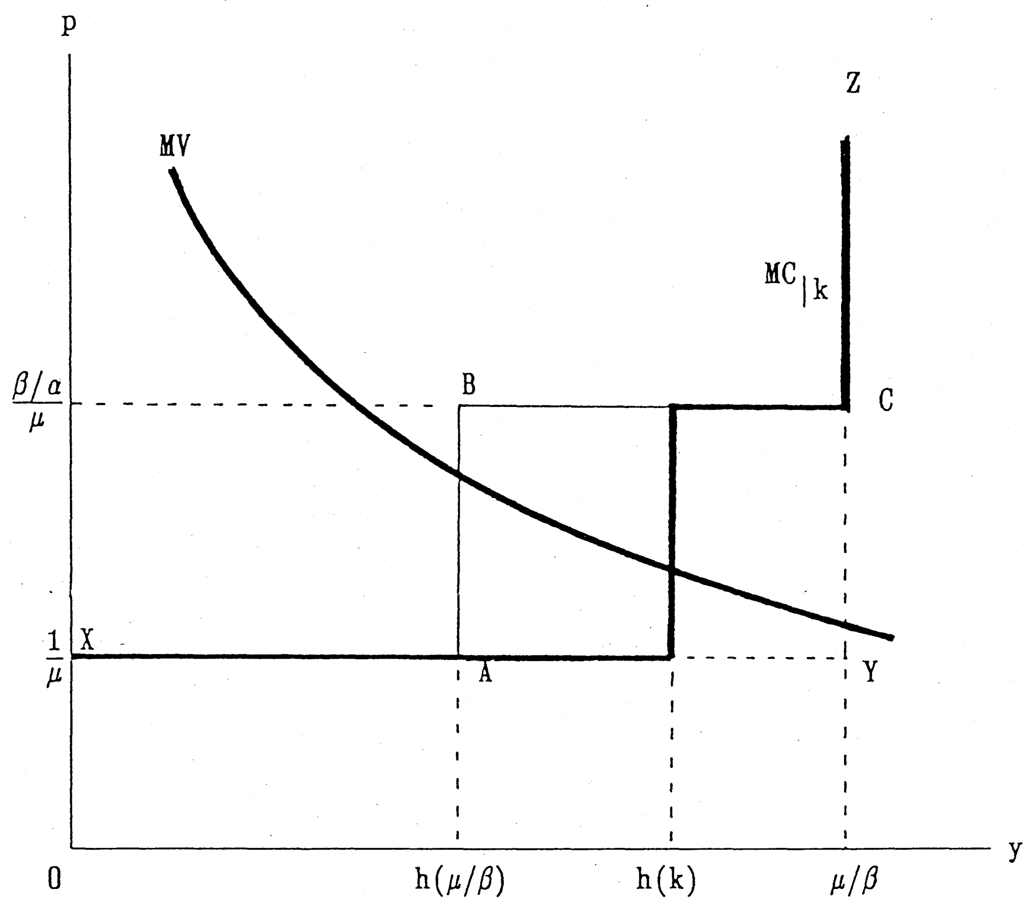

Figure 3

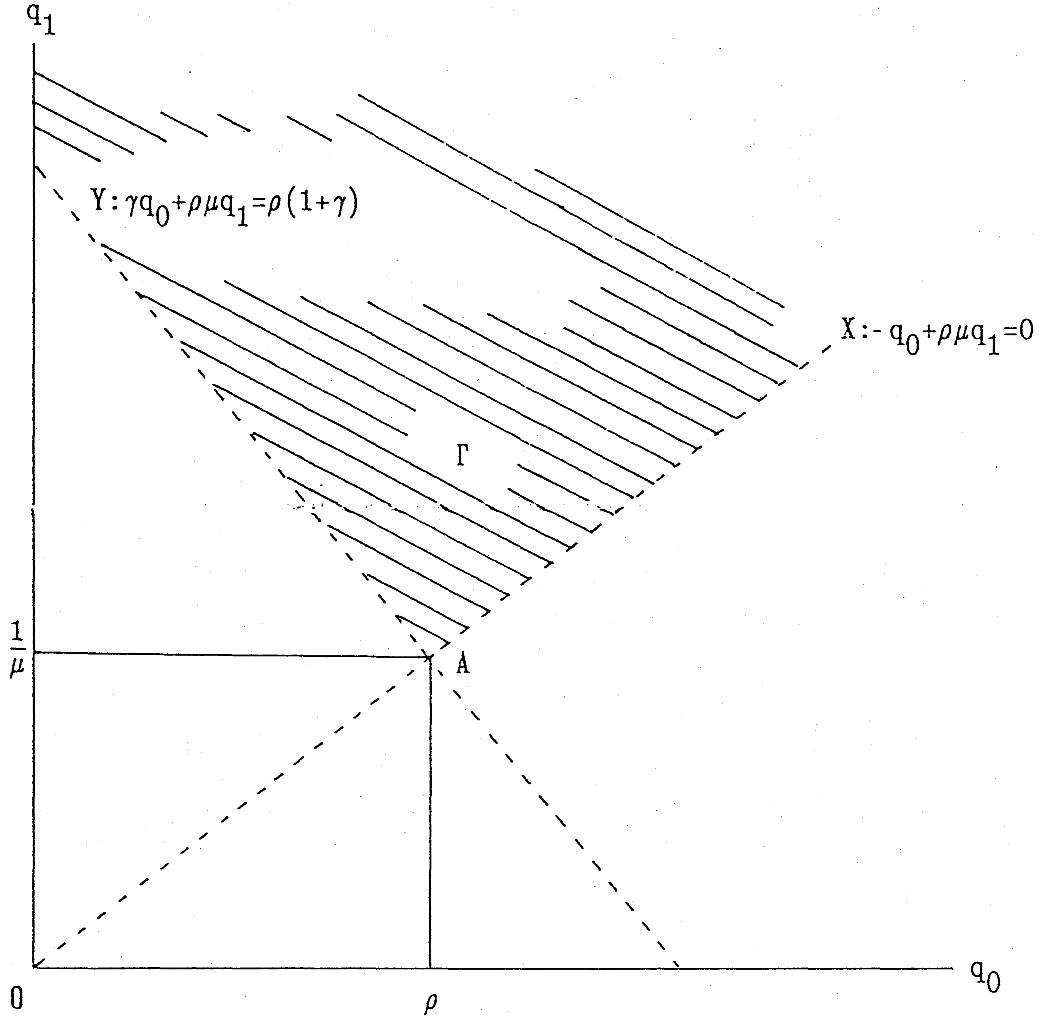
q_1
$Y: \gamma q_0 + \rho\mu q_1 = \rho(1+\gamma)$
$X: -q_0 + \rho\mu q_1 = 0$
Γ
$\frac{1}{\mu}$
A
0
ρ
q_0

Figure 4

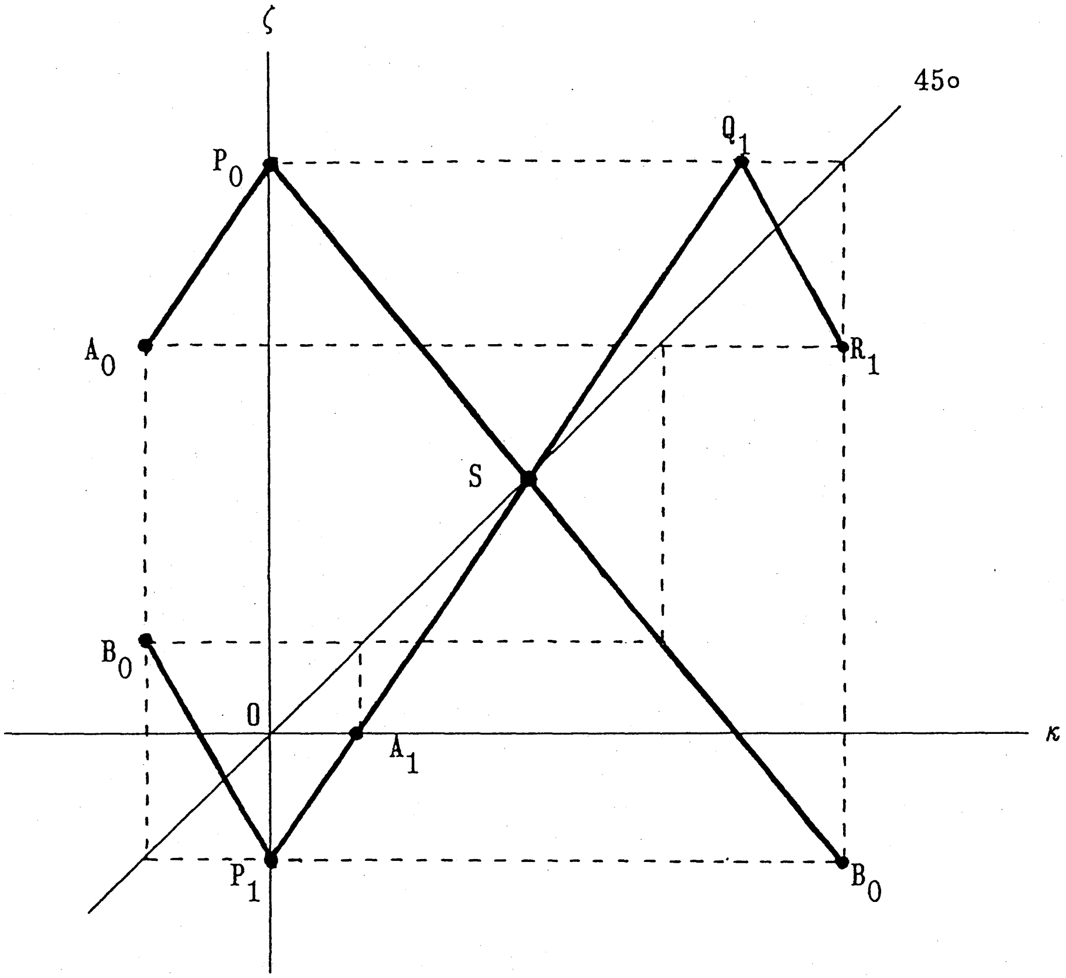

Figure 5